The Cambridge companion to Bertrand Russell, edited by Nicholas Griffin, Cambridge University Press, Cambridge, UK and New York, US, xvii + 550 pp.—therein:

Ivor Grattan-Guinness. *Mathematics in and behind Russell's logicism, and its reception.* Pp. 51–83.

Richard Cartwright. *Russell and Moore: 1895–1905.* Pp. 108–127.

Peter Hylton. *The theory of descriptions.* Pp. 202–240.

Martin Godwyn and Andrew D. Irvine. *Bertrand Russell's logicism.* Pp. 171–201.

Michael Beaney. *Russell and Frege.* Pp. 128–170.

Gregory Landini. *Russell's substitutional theory.* Pp. 241–285.

Alasdair Urquhart. *The theory of types.* Pp. 286–309.

The above papers have been selected from ***The Cambridge companion to Bertrand Russell*** as those likely to be of greatest interest to readers of this journal. Together they cover the logically most productive period of Russell's life, which may be taken to run from 1900 to 1910. The major events of this period include: the discovery of the paradox (May 1901), the publication of ***Principles of mathematics*** (May 1903), the discovery of the theory of descriptions (June 1905), the discovery of the ramified theory of types (1907), and the completion (with Whitehead) of ***Principia mathematica*** (1910). There is a wealth of material from this period that remained unpublished at the time but is becoming available in ***The collected papers of Bertrand Russell***, the relevant volumes being ***Toward the "Principles of mathematics", 1900–02***, edited by Gregory H. Moore, London: Routledge, 1993, ***Foundations of logic, 1903–05***, edited by Alasdair Urquhart with the assistance of Albert C. Lewis, London: Routledge, 1994, and ***Toward "Principia mathematica", 1906–08***, which has yet to be published. The papers under review provide a useful guide to this important chapter of Russell's career, in many cases making use of formerly unpublished material to shed light on Russell's logical development. The papers may be grouped as follows: The first sets the historical stage and provides an overview of Russell's work in logic while the remaining pairs deal respectively with the theory of denoting, the nature of logicism, and the resolution of the antinomies.

Historical. Grattan-Guinness locates the work of Whitehead and Russell at the end of a long tradition of rigorization and reduction in mathematics. This tradition began with a rigorous account of the notions of limit, continuity, and derivative (by Bolzano, Cauchy, Weierstrass and others) and ultimately led to the reduction (via informal set theory) of the reals to the rationals and the rationals to the natural numbers (by Dedekind, Cantor and others). Dedekind and Peano then gave a rigorous account of the natural numbers and Cantor both developed set theory into an independent discipline and extended it into the transfinite. Whitehead and Russell brought all of this material into a single general framework, isolating several axioms of set theory (for example, infinity and choice) and pressed the reduction further with the aim of showing that ultimately even the arithmetical notions could be reduced to set theoretic notions and that these in turn could be reduced to logical notions.

The theory of denoting. Russell's conception of logic is richer than the modern conception both in terms of the principles it embraces and the kinds of entities these principles involve. During the time of the ***Principles*** logic involved the extensional notion of class and the intensional notions of proposition, propositional function and denoting concept. These four notions play a central role in what follows and given that Russell's conception of the three intensional notions is not widely known it will be useful to start with a brief account, following the excellent exposition of Cartwright.

Consider the proposition *Socrates is human*, the propositional function *x is human* and the denoting concept *the master of Plato*. The proposition is taken to be a structured entity that contains as constituents both Socrates and the concept *human*. The occurrence of these constituents, however, differs in that the proposition is *about* the former but not the latter. For this reason Socrates is said to *occur as a term* in the proposition. A *term* simpliciter is then defined to be anything that can occur as a term in some proposition. The propositional function *x is human* may be described as what remains of the proposition when one removes Socrates or as the common component of *Socrates is human*, *Plato is human*, and all such variants. The denoting concept *the master of Plato* is a structured entity which has the characteristic feature that when one substitutes it for Socrates in the above proposition the resulting proposition is not about the concept but rather about what the concept *denotes*, namely, Socrates. In addition to such *definite* denoting concepts (that is, denoting concepts that unambiguously denote a single entity) there are *indefinite* denoting concepts such as *a man* that denote one of many entities. The most general denoting concept, *any*, is referred to by Russell as 'the variable'. (Note that Russell is using both 'term' and 'variable' in a non-linguistic sense.)

There is a certain tension in the ***Principles*** concerning terms and denoting concepts. In a famous passage (quoted by both Cartwright and Hylton), Russell argues that if A is a term then the proposition *A is not* must be either false or meaningless: "For if A were nothing, it could not be said not to be; "A is not" implies that there is a term A whose being is denied, and hence A is" (***Principles*** p. 449). If one takes as A the denoting concept *the present king of France* then this argument seems to show that the present king of France has being. Although Russell does not discuss the present king of France in the ***Principles*** he does discuss chimeras and, in the continuation of the above quotation, he does indeed conclude that chimeras have being. Cartwright and Hylton (and others) claim that in the ***Principles*** Russell did not see that his theory of denoting concepts enabled him to escape this conclusion. This, however, is questionable. The ***Principles*** is a patchwork of various papers and drafts written during different periods and the above passage is a relic from a paper written by 21 June 1900, long before Russell came to investigate the notion of denoting. One should therefore turn to more recent passages for a fuller picture. Once the theory of denoting enters, the above argument disappears and in §47 (written May 1901) Russell retreats to the innocuous claim that anything that can be *mentioned* has being. In §73 (written May 1902) Russell discusses denoting concepts that do not denote. He rejects the proposition *chimeras are animals* on the ground that if legitimate it would be about nothing (since the denoting concept *chimeras* does not denote) but he retains the proposition *for all x, if x is a chimera then x is an animal.* On similar grounds it would seem that he would reject the proposition *the present king of France is not* while retaining *there is not an x such that x is the present king of France*. In this way one can trace the emerging influence of the theory of denoting in the ***Principles*** and in the end see that the machinery for escaping the alleged conclusion is fairly explicitly at work.

The theory of denoting was further developed by Russell during the years 1903–1905 and culminated in "On Denoting", ***Mind***, 14 (1905), pp. 479–93, one of the most famous papers of twentieth-century philosophy. This is the paper in which Russell contextually defines away descriptive phrases, thereby eliminating all denoting concepts (with the exception of the variable) from among his intensional primitives. This technique is now standard in logic and is nicely explained in Hylton's paper. But "On Denoting" has remained enigmatic in other respects and commentators have debated such questions as: Is Frege the true target of Russell's criticisms? How is one to make sense of the famous

Gray's Elegy passage? What is the true reason for the new theory? What is the alleged connection with the resolution of the antinomies? Fortunately, with the appearance of ***Foundations of logic, 1903–1905***, we now have much more information about Russell's views at the time. Here one finds well over 100 pages of material—much of it first-rate—in which Russell develops the theory of denoting up to and including the point where he discovers the new theory. It is unfortunate that Hylton does not discuss this material since it sheds a great deal of light on the above and other questions. For example, consider the question of the reasons for the new theory. In the unpublished manuscripts one sees that Russell's old theory of denoting had something to say about each of the three puzzles listed in "On Denoting" as test cases for any theory of denoting. This leads to the question of why the new theory was adopted in place of the old. The manuscripts tell the story. A note on the first leaf of the manuscript "On Fundamentals" reads "Pp. 18ff. contain the reasons for the new theory of denoting". Here one finds that in developing the theory (in light of the puzzles) Russell is forced to distinguish various modes of occurrence and the theory becomes overly complicated. He then gives an argument—an illuminating version of the Gray's Elegy argument—to the effect that by the lights of the theory it is impossible to express the theory. Since the difficulties have to do with making statements *about* denoting concepts, Russell takes the course of eliminating denoting concepts while preserving the function they were introduced to perform. He is thereby led to the new theory. The new theory is then applied to give a clean solution to one of the puzzles and to eliminate classes. Thus, on the face of it, the reasons in favour of the new theory appear to be (i) that it is simpler, (ii) that it provides a cleaner treatment of the puzzles, (iii) that it avoids what Russell saw as the internal collapse of the old theory, and (iv) that it has fruitful consequences, in particular, the elimination of the classes, which were suspect because of the paradox. More, of course, needs to be said on the subject. It is to be regretted that although this material has been available now in published form for over ten years it has barely entered the literature.

The nature of logicism. The paper of Beaney contains a useful comparison of the logicism of Frege with that of Russell. The emphasis, however, is on Frege. Godwyn and Irvine concentrate on Russell's logicism but rely heavily on later, more popular expositions. It would have been nice to see a more mathematically informed treatment of logicism and its limitations. For example, both papers give an oversimplified presentation of Frege's theory of number and neither paper mentions Myhill's result that if the controversial axiom of reducibility is removed from the system of ***Principia*** then the resulting system cannot define the natural numbers in such a way that all instances of induction are provable. The paper of Godwyn and Irvine also contains a number of errors. For example, in the course of discussing the distinction between a law of thought in the psychological and the logical sense the authors say that "Frege seems willing to admit that 'it is impossible to effect any sharp separation between the two'". This is a striking attribution since Frege is well-known to maintain precisely such a sharp separation. When one looks up the cited attribution one finds that by "the two" Frege is referring not to *psychology* and logic but to *mathematics* and logic, as one should expect of a logicist. The paper does, however, have the virtue of correcting a common misconception regarding Russell's logicism by discussing in detail his regressive method for discovering the axioms and the associated view according to which, just as in the natural sciences, the laws of logic derive their justification in part through their consequences.

The resolution of the antinomies. In May 1901 Russell discovered that his conception of classes and propositional functions led to antinomies. His first attempt at a solution was to stratify classes and deny that propositional functions are terms.

In an appendix to the ***Principles*** he set forth a theory of types which included, in addition to a hierarchy of types of classes, a separate type for propositions and a type including all types. Both features were designed to preserve the universality of logic, the latter by preserving the meaningfulness of quantification over all objects. However, Russell shows that the system leads to an antinomy concerning propositions (involving classes). He considers the natural course of stratifying propositions but rejects it as "harsh and highly artificial", concluding that "the totality of all logical objects, or of all propositions, involves, it would seem, a fundamental logical difficulty" (p. 528 and cf. Urquhart p. 289).

For the next four years Russell developed a surprising variety of type-free theories. His goal was to discover an intrinsically motivated theory that gave a unified solution to all of the antinomies. The first hope of such a solution came with the theory of descriptions since it enabled Russell to reduce classes to propositional functions. Indeed, as noted earlier, in "On Fundamentals", immediately upon discovering the theory of descriptions Russell uses it to effect such a reduction. This reduction to a minimal framework held the hope of a simple, unified solution to the antinomies, the idea being that natural considerations in the minimal framework would bring (through the reduction) order to the higher reaches. The more minimal the framework the greater the hope. For this reason (and because propositional functions were also problematic) Russell turned to the ultra-minimal framework of the substitution theory where propositional functions were reduced to a system concerning propositions, the variable and a single operation of substitution. The original theory and its descendants are carefully reconstructed in Landini's paper by making extensive use of unpublished manuscripts.

The expression '$p/a;b!q$' is taken to mean that q is the result of substituting b for a in p. For example, if p is the proposition *Socrates is human* and a is Socrates and b is Plato, then q is the proposition *Plato is human.* This primitive is governed by a number of axioms which Landini isolates. The expression 'p/a' is treated as an incomplete symbol for a class, namely, the class of b such that the q such that $p/a;b!q$ is true. In this way, a class is proxied by a pair consisting of a proposition and a constituent. For example, the class of humans is proxied by p/a, where p and a are as above. One can then derive the comprehension principle and show that the resulting classes inherit a simple type structure. The system is sufficiently strong to generate arithmetic. Unfortunately, it is too strong. For, as Russell discovered by April 1906, a clever choice of substitution yields a new antinomy. Russell attempted to remedy the situation by restricting the formulas in the substitution axioms to quantifier-free formulas. But the resulting system is too weak to generate arithmetic and when one adds the necessary comprehension axioms inconsistency returns.

Eventually Russell abandoned the type-free approach and stratified propositions and propositional functions according to the kind of generality they involve. (Landini claims that despite appearances Russell never gave up on the unrestricted variable. This is a difficult thesis to sustain in light of Russell's many apparent statements to the contrary. The reader is referred to Landini's book (JSL LXIV 1370) for his defense of this claim.) The resulting theory—the ramified theory of types—was developed in 1907. This period is covered in Urquhart's excellent paper, a very much needed contribution to the literature. Urquhart gives a rigorous version of the theory based on an earlier exposition by Church. The heuristic principle motivating the theory is the so-called vicious circle principle according to which "Whatever involves an apparent variable must not be among the possible values of that variable". The true motivation behind this principle and the ramified hierarchy has remained something of a mystery. It may be that just as the substitution theory leads to a simple theory of types so too Russell's views at the time on the nature of propositions, propositional functions, and the variable, lead to a ramified theory of types. This is defended by Goldfarb in *Russell's reasons for ramification*, ***Rereading Russell: essays in Bertrand Russell's metaphysics and epistemology***, edited by C. Wade Savage and C. Anthony Anderson, Minneapolis: University of Minnesota Press, 1989. Another possibility is that the theory was adopted because of its fruitful consequences. For Whitehead and Russell say of the theory that "it suggested itself to us in the first instance by its ability to solve certain contradictions", though they add that the theory is "not wholly dependent on this indirect recommendation: it has also a certain consonance with common sense which makes it inherently credible" (***Principia mathematica***, p. 37). Furthermore, it is precisely in 1907 that Russell began underscoring his view that axioms are largely justified by their consequences. It is hoped that this question will be illuminated by the appearance of ***Toward "Principia mathematica", 1906–08***, volume 5 of the ***Collected Papers***.

The main fruitful consequence of the ramified theory is that it provides a unified approach to the resolution of the antinomies. The manner in which it handles the set theoretic antinomies is familiar. That it also handles the semantic antinomies was first shown in detail by Church (JSL XLI 747) and Myhill (*A refutation of an unjustified attack on the axiom of reducibility*, ***Bertrand Russell memorial volume***, edited by George W. Roberts, London: Allen & Unwin, 1979): The system of ***Principia mathematica*** is intended as a general framework to which one can add predicates and principles pertaining to any given special science. When one adds the predicates of an intensional semantics—that is, the predicates expressing when a given open formula expresses a given propositional function—and the natural principles governing them, one can define the extensional notions of truth and satisfaction with the consequence that these are precisely the Tarskian notions. The ramified theory of types thus implicitly contains both the standard resolution to the set theoretic antinomies and the standard resolution to the semantic antinomies. Unfortunately, as noted earlier, without the axiom of reducibility the system is too weak to generate arithmetic. Thus, in his search for an intrinsic solution to the antinomies Russell was led once again to a system too weak for the purposes of logicism.

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Entry for the Table of Contents: