Large Cardinals from Determinacy

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1. Introduction

In this chapter we give an account of Woodin's technique for deriving large cardinal strength from determinacy hypotheses. These results appear here for the first time and for this reason we have gone into somewhat more detail than is customary in a handbook. All unattributed results that follow are either folklore or due to Woodin.

1.1. Determinacy and Large Cardinals

In the era of set theory following the discovery of independence a major concern has been the discovery of new axioms that settle the statements left undecided by the standard axioms (ZFC). One interesting feature that has emerged is that there are often deep connections between axioms that spring from entirely different sources. In this chapter we will be concerned with one instance of this phenomenon, namely, the connection between axioms of definable determinacy and large cardinal axioms.

In this introduction we will give a brief overview of axioms of definable determinacy and large cardinal axioms (in sections A and B), discuss their interconnections (in sections C and D), and give an overview of the chapter (in section E). At some points we will draw on notation and basic notions that are explained in fuller detail in sections 1.2 and 2.1.

A. Determinacy

For a set of reals $A \subseteq \omega^\omega$ consider the game where two players take turns playing natural numbers:

$$\begin{array}{llllllll} \text{I} & x(0) & & x(2) & & x(4) & & \dots \\ \text{II} & & x(1) & & x(3) & & \dots & \end{array}$$

At the end of a round of this game the two players will have produced a real x, obtained through "interleaving" their plays. We say that Player I wins the round if $x \in A$; otherwise Player II wins the round. The set A is said to be *determined* if one of the players has a "winning strategy" in the associated game, that is, a strategy which ensures that the player wins a round regardless of how the other player plays. The *Axiom of Determinacy* (AD) is the statement that *every* set of reals is determined.

It is straightforward to see that very simple sets are determined. For example, if A is the set of all reals then clearly I has a winning strategy; if A is empty then clearly II has a winning strategy; and if A is countable then II has a winning strategy (by "diagonalizing"). A more substantive result is that if A is closed then one player must have a winning strategy. This might lead one to expect that *all* sets of reals are determined. However, it is straightforward to use the Axiom of Choice (AC) to construct a non-determined set (by listing all winning strategies and "diagonalizing" across them). For this reason AD was never really considered as a serious candidate for a new axiom. However, there is an interesting class of related axioms that are consistent with AC, namely, the axioms of *definable determinacy*. These axioms extend the above pattern by asserting that all sets of reals at a given level of complexity are determined, notable examples being, $\utilde{\Delta}^1_1$-determinacy (all Borel sets of reals are determined), PD (all projective sets of reals are determined) and $\mathrm{AD}^{L(\mathbb{R})}$ (all sets of reals in $L(\mathbb{R})$ are determined).

One issue is whether these are really new axioms or whether they follow from ZFC. In the early development of the subject the result on the determinacy of closed sets was extended to higher levels of definability. These developments culminated in Martin's proof of $\utilde{\Delta}^1_1$-determinacy in ZFC. It turns out that this result is close to optimal—as one climbs the hierarchy of definability, shortly after $\underset{\sim}{\Delta}^1_1$ one arrives at axioms that fall outside the provenance of ZFC. For example, this is true of PD and $\mathrm{AD}^{L(\mathbb{R})}$. Thus, we have here a hierarchy of axioms (including PD and $\mathrm{AD}^{L(\mathbb{R})}$) which are genuine candidates for new axioms.

There are actually two hierarchies of axioms of definable determinacy, one involving *lightface* notions of definability (by which we mean notions (such as Δ^1_2) that do not involve real numbers as parameters) and the other involving *boldface* notions of definability (by which we mean notions (such as $\underset{\sim}{\Delta}^1_2$) that do involve real numbers as parameters). (See Jackson's chapter in this Handbook for details concerning the various grades of definability and the relevant notation.) Each hierarchy is, of course, ordered in terms of increasing complexity. Moreover, each hierarchy has a natural limit: the natural limit of the lightface hierarchy is OD-determinacy (all OD sets of reals are determined) and the natural limit of the boldface hierarchy is $\mathrm{OD}(\mathbb{R})$-determinacy (all $\mathrm{OD}(\mathbb{R})$ sets of reals are determined). The reason these are natural limits is that the notions of lightface and boldface ordinal definability are candidates for the richest lightface and boldface notions of definability. To see this (for the lightface case) notice first that any notion of definability which does not render all of the ordinals definable can be transcended (as can be seen by considering the least ordinal which is not definable according to the notion) and second that the notion of ordinal definability cannot be so transcended (since by reflection OD is ordinal definable). It is for this reason that Gödel proposed the notion of ordinal definability as a candidate for an "absolute" notion of definability. Our limiting cases may thus be regarded as two forms of absolute definable determinacy.

So we have two hierarchies of increasingly strong candidates for new axioms and each has a natural limit. There are two fundamental questions concerning such new axioms. First, are they *consistent*? Second, are they *true*? In the most straightforward sense these questions are asked in an *absolute* sense and not relative to a particular theory such as ZFC. But since we are dealing with new axioms, the traditional means of answering such questions—namely, by establishing their consistency or provability relative to the standard axioms (ZFC)—is not available. Nevertheless, one can hope to establish results—such as relative consistency and logical connections with respect to other plausible axioms—that collectively shed light on the original, absolute question. Indeed, there are a number of results that one can bring to bear in favour of PD and $\mathrm{AD}^{L(\mathbb{R})}$. For example, these axioms lift the structure theory that can be established in ZFC to their respective domains, namely, second-order arithmetic and $L(\mathbb{R})$. Moreover, they do so in a fashion which settles a remarkable number of statements that are independent of ZFC. In fact, there is no "natural" statement concerning their respective domains that is known to be independent of these axioms. (For more on the structure theory provided by determinacy and the traditional considerations in their favour see [10] and for more recent work see Jackson's chapter in this Handbook.) The results of this chapter figure in the case for PD and $\mathrm{AD}^{L(\mathbb{R})}$. However, our concern will be with the question of relative consistency; more precisely, we wish to calibrate the consistency strength of axioms of definable determinacy—in particular, the ultimate axioms of lightface and boldface determinacy—in terms of the large cardinal hierarchy.

There are some reductions that we can state at the outset. In terms of consistency strength the two hierarchies collapse at a certain stage: Kechris and Solovay showed that $\mathrm{ZF}+\mathrm{DC}$ implies that in the context of $L[x]$ for $x \in \omega^\omega$, OD-determinacy and Δ^1_2-determinacy are equivalent (see Theorem 6.6). And it is a folklore result that $\mathrm{ZFC}+\mathrm{OD}(\mathbb{R})$-determinacy and $\mathrm{ZFC}+\mathrm{AD}^{L(\mathbb{R})}$ are equiconsistent. Thus, in terms of consistency strength, the lightface hierarchy collapses at Δ^1_2-determinacy and the boldface hierarchy collapses at $\mathrm{AD}^{L(\mathbb{R})}$. So if one wishes to gauge the consistency strength of lightface and boldface determinacy it suffices to concentrate on Δ^1_2-determinacy and $\mathrm{AD}^{L(\mathbb{R})}$.

Now, it is straightforward to see that if Δ^1_2-determinacy holds then it holds in $L[x]$ for some real x and likewise if $\mathrm{AD}^{L(\mathbb{R})}$ (or AD) holds then it holds in $L(\mathbb{R})$. Thus, the natural place to study the consistency strength of lightface definable determinacy is $L[x]$ for some real x and the natural place to study the consistency strength of boldface definable determinacy (or full determinacy) is $L(\mathbb{R})$. For this reason these two models will be central in what follows.

To summarize: We shall be investigating the consistency strength of lightface and boldface determinacy. This reduces to Δ^1_2-determinacy and $\mathrm{AD}^{L(\mathbb{R})}$. The settings $L[x]$ and $L(\mathbb{R})$ will play a central role. Consistency strength will be measured in terms of the large cardinal hierarchy. Before turning to a discussion of the large cardinal hierarchy let us first briefly discuss stronger forms of determinacy.

Our concern in this chapter is with axioms of determinacy of the above form, where the games have length ω and the moves are natural numbers. However, it is worthwhile to note that there are two directions in which one can generalize these axioms.

First, one can consider games of length greater than ω (where the moves are still natural numbers). A simple argument shows that one cannot have the determinacy of all games of length ω_1 but there is a great deal of room below this upper bound and much work has been done in this area. For more on this subject see [11].

Second, one can consider games where the moves are more complex than natural numbers (and where the length of the game is still ω). One alternative is to consider games where the moves are real numbers. The axiom $\mathrm{AD}_{\mathbb{R}}$ states that all such games are determined. One might try to continue in this direction and consider the axiom $\mathrm{AD}_{\mathscr{P}(\mathbb{R})}$ asserting the determinacy of all games where the moves are sets of real numbers. It is straightforward to see that this axiom is inconsistent. In fact, even the definable version asserting that all OD subsets of $\mathscr{P}(\mathbb{R})^\omega$ is inconsistent. Another alternative is to consider games where the moves are ordinal numbers. Again, a simple argument shows that one cannot have the determinacy of all subsets of ${\omega_1}^\omega$. However, a result of Harrington and Kechris shows that in this case if one adds a definability constraint then one can have determinacy at this level. In fact, OD-determinacy implies that every OD set $A \subseteq {\omega_1}^\omega$ is determined. It is natural then to extend this to large ordinals. The ultimate axiom in this direction would simply assert that every OD set $A \subseteq \mathrm{On}^\omega$ is determined. Perhaps surprisingly, at this stage a certain tension arises since recent work in inner model theory provides evidence that this axiom is in fact inconsistent. See [13] for more on this subject.

B. Large Cardinals

Our aim is to calibrate the consistency strength of lightface and boldface determinacy in terms of the large cardinal hierarchy. The importance of the large cardinal hierarchy in this connection is that it provides a canonical means of climbing the hierarchy of consistency strength. To show, for a given hypothesis φ and a given large cardinal axiom L, that the theories $\mathrm{ZFC} + \varphi$ and $\mathrm{ZFC} + L$ are equiconsistent one typically uses the dual methods of *inner model theory* and *outer model theory* (also known as *forcing*). Very roughly, given a model of $\mathrm{ZFC} + L$ one forces to obtain a model of $\mathrm{ZFC} + \varphi$ and given a model of $\mathrm{ZFC} + \varphi$ one uses the method of inner model theory to construct a model of $\mathrm{ZFC} + L$. The large cardinal hierarchy is (for the most part) naturally well-ordered and it is a remarkable phenomenon that given any two "natural" theories extending ZFC one can compare them in terms of consistency strength (equivalently, interpretability) by lining them up with the large cardinal hierarchy.

In a very rough sense large cardinal axioms assert that there are "large" levels of the universe. A template for formulating a broad class of large cardinal axioms is in terms of elementary embeddings. The basic format of the template is as follows: There is a transitive class M and a non-trivial elementary

$$j : V \to M.$$

To say that the embedding is non-trivial is simply to say that it is not the identity, in which case one can show that there is a least ordinal moved. This ordinal is denoted $\mathrm{crit}(j)$ and called the *critical point* of j. A cardinal is said to be a *measurable cardinal* if and only if it is the critical point of such an embedding.

It is easy to see that for any such elementary embedding there is necessarily a certain degree of agreement between V and M. In particular, it follows that $V_{\kappa+1} \subseteq M$, where $\kappa = \mathrm{crit}(j)$. This degree of agreement in conjunction with the elementarity of j can be used to show that κ has strong reflection properties, in particular, κ is strongly inaccessible, Mahlo, weakly compact, etc.

One way to strengthen a large cardinal axiom of the above form is to demand greater agreement between M and V. For example, if one demands that $V_{\kappa+2} \subseteq M$ then the fact that κ is measurable is recognized within M and hence it follows that M satisfies that there is a measurable cardinal below $j(\kappa)$, namely, κ. Thus, by the elementarity of the embedding, V satisfies that there is a measurable cardinal below κ. The same argument shows that there are arbitrarily large measurable cardinals below κ.

This leads to a natural progression of increasingly strong large cardinal axioms. It will be useful to discuss some of the major axioms in this hierarchy: If κ is a cardinal and $\eta > \kappa$ is an ordinal then κ is *η-strong* if there is a transitive class M and a non-trivial elementary embedding $j : V \to M$ such that $\mathrm{crit}(j) = \kappa$, $j(\kappa) > \eta$ and $V_\eta \subseteq M$. A cardinal κ is *strong* iff it is η-strong for all η. As we saw above if κ is $(\kappa+2)$-strong then κ is measurable and there are arbitrarily large measurable cardinals below κ. Next, one can demand that the embedding preserve certain classes: If A is a class, κ is a cardinal, and $\eta > \kappa$ is an ordinal then κ is *η-A-strong* if there exists a $j : V \to M$ which witnesses that κ is η-strong and which has the additional feature that $j(A \cap V_\kappa) \cap V_\eta = A \cap V_\eta$. The following large cardinal notion will play a central role in this chapter.

1.1 Definition. A cardinal κ is a *Woodin cardinal* if κ is strongly inaccessible and for all $A \subseteq V_\kappa$ there is a cardinal $\kappa_A < \kappa$ such that

$$\kappa_A \text{ is } \eta\text{-}A\text{-strong},$$

for each η such that $\kappa_A < \eta < \kappa$.

It should be noted that in contrast to measurable and strong cardinals, Woodin cardinals are not characterized as the critical point of an embedding or collection of embeddings. In fact, a Woodin cardinal need not be measurable. However, if κ is a Woodin cardinal, then V_κ is a model of ZFC and from the point of view of V_κ there is a proper class of strong cardinals.

Going further, a cardinal κ is *superstrong* if there is a transitive class M and a non-trivial elementary embedding $j : V \to M$ such that $\text{crit}(j) = \kappa$ and $V_{j(\kappa)} \subseteq M$. If κ is superstrong then κ is a Woodin cardinal and there are arbitrarily large Woodin cardinals below κ.

One can continue in this vein, demanding greater agreement between M and V. The ultimate axiom in this direction would, of course, demand that $M = V$. This axiom was proposed by Reinhardt. But shortly after its introduction Kunen showed that it is inconsistent with ZFC. In fact, Kunen showed that assuming ZFC, there can be no non-trivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$. (An interesting open question is whether these axioms are inconsistent with ZF or whether there is a hierarchy of "choiceless" large cardinal axioms that climb the hierarchy of consistency strength far beyond what can be reached with ZFC.)

There is a lot of room below the above upper bound. For example, a very strong axiom is the statement that there is a non-trivial elementary embedding $j : V_{\lambda+1} \to V_{\lambda+1}$. The strongest large cardinal axiom in the current literature is the axiom asserting that there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ such that $\text{crit}(j) < \lambda$. Surprisingly, this axiom yields a structure theory of $L(V_{\lambda+1})$ which is closely analogous to the structure theory of $L(\mathbb{R})$ under the axiom $\text{AD}^{L(\mathbb{R})}$. This parallel between axioms of determinacy and large cardinal axioms suggests seeking stronger large cardinal axioms by following the guide of the strong axioms of determinacy discussed at the close of the previous section. In fact, there is evidence that the parallel extends. For example, there is a new large cardinal axiom that is the analogue of $\text{AD}_{\mathbb{R}}$. See [13] for more on these recent developments.

C. Determinacy from Large Cardinals

Let us return to the questions of the truth and the consistency of axioms of definable determinacy, granting that of large cardinal axioms. In the late 1960s Solovay conjectured that $\mathrm{AD}^{L(\mathbb{R})}$ is provable from large cardinal axioms (and hence that ZF+AD is consistent relative to large cardinal axioms). This conjecture was realized in stages.

In 1970 Martin showed that if there is a measurable cardinal then all $\utilde{\Sigma}^1_1$ sets of reals are determined. Later, in 1978, he showed that under the much stronger assumption of a non-trivial iterable elementary embedding $j : V_\lambda \to V_\lambda$ all $\utilde{\Sigma}^1_2$ sets of reals are determined. It appeared that there would be a long march up the hierarchy of axioms of definable determinacy. However, in 1984 Woodin showed that if there is a non-trivial elementary embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with $\mathrm{crit}(j) < \lambda$, then $\mathrm{AD}^{L(\mathbb{R})}$ holds.

The next major advances concerned reducing the large cardinal hypothesis used to obtain $\mathrm{AD}^{L(\mathbb{R})}$. The first step in this direction was made shortly after, in 1985, when Martin and Steel proved the following remarkable result, using a completely different technique:

Theorem 1.2 (Martin and Steel). *Assume* ZFC*. Suppose that there are n Woodin cardinals with a measurable cardinal above them all. Then $\utilde{\Sigma}^1_{n+1}$-determinacy holds.*

It follows that if there is a Woodin cardinal with a measurable cardinal above, then Δ^1_2-determinacy holds and if there are infinitely many Woodin cardinals then PD holds. Finally, the combination of Martin and Steel's work and Woodin's work on the stationary tower (see [9]) led to a significant reduction in the hypothesis required to obtain $\mathrm{AD}^{L(\mathbb{R})}$.

Theorem 1.3. *Assume* ZFC*. Suppose there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then* $\mathrm{AD}^{L(\mathbb{R})}$*.*

A more recent development is that, in addition to being implied by large cardinal axioms, $\mathrm{AD}^{L(\mathbb{R})}$ is implied by a broad array of other strong axioms, which have nothing to do with one another—in fact, there is reason to believe that $\mathrm{AD}^{L(\mathbb{R})}$ is implied by *all* sufficiently strong "natural" theories. For further discussion of this subject and other more recent results that contribute to the case for certain axioms of definable determinacy see [8], [12], and [14].

Each of the above results concerns the *truth* of axioms of definable determinacy, granting large cardinal axioms. A closely related question concerns the *consistency* of axioms of definable determinacy, granting that of large cardinal axioms. For this one can get by with slightly weaker large cardinal assumptions.

Theorem 1.4. *Assume* ZFC. *Suppose δ is a Woodin cardinal. Suppose $G \subseteq \mathrm{Col}(\omega, \delta)$ is V-generic. Then $V[G] \models \Delta^1_2$-determinacy.*

Theorem 1.5. *Assume* ZFC. *Suppose that λ is a limit of Woodin cardinals. Suppose $G \subseteq \mathrm{Col}(\omega, {<}\lambda)$ is V-generic and let $\mathbb{R}^* = \bigcup\{\mathbb{R}^{V[G \restriction \alpha]} \mid \alpha < \lambda\}$. Then $L(\mathbb{R}^*) \models \mathrm{AD}$.*

For more on the topic of this section see Neeman's chapter in this Handbook.

D. Large Cardinals from Determinacy

The above results lead to the question of whether the large cardinal assumptions are "necessary". Of course, large cardinal assumptions (in the traditional sense of the term) cannot be necessary in the strict sense since axioms of definable determinacy (which concern sets of reals) do not outright imply the existence of large cardinals (which are much larger objects). The issue is whether they are necessary in the sense that one cannot prove the axioms of definable determinacy with weaker large cardinal assumptions. To establish this one must show that the consistency of the axioms of definable determinacy implies that of the large cardinal axioms and one way to do this is to show that axioms of definable determinacy imply that there are *inner models* of the large cardinal axioms.

There are two approaches to inner model theory, each originating in the work of Gödel. These approaches have complementary advantages and disadvantages. The first approach is based on L, the universe of constructible sets. The advantage of this approach is that L is very well understood; in fact, it is fair to say that within ZFC one can carry out a "full analysis" of this model. As a consequence of this one can show, for example, that under ZF + AD, ω_1^V is inaccessible in L. The disadvantage is that L is of limited applicability since it cannot accommodate strong large cardinal axioms such as the statement that there is a measurable cardinal. So if the large cardinal assumptions in Theorems 1.4 and 1.5 are close to optimal then L is of no use in establishing this.

The second approach is based on HOD, the universe of hereditarily ordinal definable sets. This inner model can accommodate virtually all large cardinal axioms that have been investigated. But it has a complementary defect in that one cannot carry out a full analysis of this structure within ZFC.

A major program in set theory—*the inner model program*—aims to combine the advantages of the two approaches by building inner models that resemble L in having a highly ordered inner structure but which resemble HOD in that they can accommodate strong large cardinal axioms.

"L-like" inner models at the level of Woodin cardinals were developed in stages beginning with work of Martin and Steel, and continuing with work of Mitchell and Steel. The Mitchell-Steel inner models are true analogs of L. Martin and Steel used their models to show that the large cardinal hypotheses in their proofs of determinacy were essentially optimal. For example, they showed that if there is a Woodin cardinal then there is a canonical inner model M that contains a Woodin cardinal and has a Δ^1_3 well-ordering of the reals. It follows that one cannot prove $\underset{\sim}{\Sigma}^1_2$-determinacy from the assumption of a Woodin cardinal alone.

However, this still left open a number of questions. First, does the consistency of ZFC + "There is a Woodin cardinal" follow from that of $\mathrm{ZFC} + \Delta^1_2$-determinacy? Second, can one build an inner model of a Woodin cardinal directly from ZFC + Δ^1_2-determinacy? Third, what is the strength of $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$? To approach these questions it would seem that one would need *fine-structural* inner model theory. However, at the time when the central results of this chapter were proved, fine-structural inner model theory had not yet reached the level of Woodin cardinals. One option was to proceed with HOD.

In contrast to L the structure of HOD is closely tied to the universe in which it is constructed. In the general setting, where one works in ZF and constructs HOD in V, the structure theory of HOD is almost as intractable as that of V. Surprisingly if one strengthens the background theory then the structure theory of HOD becomes tractable. For example, Solovay showed that under ZF + AD, HOD satisfies that ω_1^V is a measurable cardinal. It turns out that both lightface and boldface definable determinacy are able to illuminate the structure of HOD (when constructed in the natural inner models of these axioms—$L[x]$ and $L(\mathbb{R})$) to the point where one can recover the large cardinals that are necessary to establish their consistency.

In the case of lightface definable determinacy the result is the following:

Theorem 1.6. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone* *of* x,

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

Thus, the consistency strength of ZFC + OD-determinacy is precisely that of ZFC + "There is a Woodin cardinal". For the case of boldface determinacy let us first state a preliminary result of which the above result is a localization. First we need a definition. Let

$$\Theta = \sup\{\alpha \mid \text{there is a surjection } \pi : \mathbb{R} \to \alpha\}.$$

Theorem 1.7. *Assume* ZF + AD. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

In fact, both of these results are special instances of a general theorem on the generation of Woodin cardinals—the Generation Theorem. In addition to giving the above results, the Generation Theorem will also be used to establish the optimal large cardinal lower bound for boldface determinacy:

Theorem 1.8. *Assume* ZF + AD. *Suppose* Y *is a set. There is a generalized Prikry forcing* $\mathbb{P}_Y$ *through the* Y*-degrees such that if* $G \subseteq \mathbb{P}_Y$ *is* V*-generic and* $\langle [x_i]_Y \mid i < \omega \rangle$ *is the associated sequence, then*

$$\mathrm{HOD}^{V[G]}_{Y, \langle [x_i]_Y \mid i<\omega \rangle, V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$ *is the* Y*-degree of* x.

Thus, the consistency strength of ZFC + OD($\mathbb{R}$)-determinacy and of ZF + AD is precisely that of ZFC + "There are ω-many Woodin cardinals".

The main results of this chapter have applications beyond equiconsistency; in particular, the theorems play an important role in the structure theory of AD^+ (a potential strengthening of AD that we will define and discuss in Section 8). For example, Steel showed that under AD, in $L(\mathbb{R})$ every uncountable regular cardinal below Θ is a measurable cardinal. (See Steel's chapter in this Handbook for a proof.) This theorem generalizes to a theorem of AD^+ and the theorems of this chapter are an important part of the proof. We will discuss some other applications in the final section of this chapter.

E. Overview

The results on the strength of lightface and boldface determinacy were established in the late 1980s. However, the current presentation and many of the results that follow are quite recent. One of the key new ingredients is the following abstract theorem on the generation of Woodin cardinals, which lies at the heart of this chapter:

Theorem 1.9 (Generation Theorem). *Assume* ZF. *Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$,

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$,

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α,

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models$ *Strategic determinacy with respect to* B.

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

Here T_0 is the theory ZF + $\mathrm{AC}_\omega(\mathbb{R})$ − Power Set + “$\mathscr{P}(\omega)$ exists″ and the notion of “strategic determinacy” is a technical notion that we will state precisely later.

The Generation Theorem provides a template for generating models containing Woodin cardinals. One simply has to show that in a particular setting the various conditions can be met, though this is often a non-trivial task. The theorem is also quite flexible in that it is a result of ZF that does not presuppose DC and has applications in both lightface and boldface settings. It will play a central role in the calibration of the strength of both lightface and boldface determinacy.

We shall approach the proof of the Generation Theorem by proving a series of increasingly complex approximations.

In Section 2 we take the initial step by proving Solovay's theorem that under ZF + AD, ω_1^V is a measurable cardinal in HOD and we show that the associated measure is normal. The proof that we give is slightly more complicated than the standard proof but has the virtue of illustrating in a simple setting some of the key components that appear in the more complex variations. We illustrate this at the end of the section by showing that the proof of Solovay's theorem generalizes to show that under ZF + AD, the ordinal $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. Our main aim in this section is to illustrate the manner in which "boundedness" and "coding" combine to yield normal ultrafilters. In subsequent sections stronger forms of boundedness (more precisely, "reflection") and stronger forms of coding will be used to establish stronger forms of normality.

In Section 3 we prove the strong forms of coding that will be central throughout.

In Section 4, as a precursor to the proof of the Generation Theorem, we prove the following theorem:

Theorem 1.10. *Assume* ZF + DC + AD. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

The assumption of DC is merely provisional—it will ultimately be eliminated when we prove the Generation Theorem. Toward the proof of the above theorem, we begin in §4.1 by establishing the reflection phenomenon that will play the role played by boundedness in §2. We will then use this reflection phenomenon in $L(\mathbb{R})$ to define for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$ that is intended to witness that $\underset{\sim}{\delta}^2_1$ is λ-strong. In §4.2 we shall introduce and motivate the notion of *strong normality* by showing that the strong normality of μ_λ ensures that $\underset{\sim}{\delta}^2_1$ is λ-strong. We will then show how reflection and uniform coding combine to secure strong normality. In §4.3 we will prove the above theorem by relativizing the construction of §4.2 to subsets of $\Theta^{L(\mathbb{R})}$.

In Section 5 we extract the essential components of the above construction and prove two abstract theorems on Woodin cardinals in a general setting, one that involves DC and one that does not. The first theorem is proved in §5.1. The importance of this theorem is that it can be used to show that in certain strong determinacy settings HOD can contain many Woodin cardinals. The second theorem is the Generation Theorem, the proof of which will occupy the remainder of the section. The aim of the Generation Theorem is to show that the construction of Section 4 can be driven by lightface determinacy alone. The difficulty is that the construction of Section 4 involves games that are defined in terms of real parameters. To handle this we introduce the notion of "strategic determinacy", a notion that resembles boldface determinacy in that it involves real parameters but which can nonetheless hold in settings where one has AC. To motivate the notion of "strategic determinacy" we shall begin in §5.2 by examining one such setting, namely, $L[S, x]$ where S is a class of ordinals and x is a real. Once we show that "strategic determinacy" can hold in this setting we shall return in §5.3 to the general setting and prove the Generation Theorem. In the final subsection, we prove a number of special cases, many of which are new. Although some of these applications involve lightface settings, they all either involve assuming full AD or explicitly involve "strategic determinacy".

In Section 6 we use two of the special cases of the Generation Theorem to calibrate the consistency strength of lightface and boldface definable determinacy in terms of the large cardinal hierarchy. In the case of the first result the main task is to show that Δ^1_2-determinacy suffices to establish that "strategic determinacy" can hold. In the case of the second result the main task is to show that the Generation Theorem can be iteratively applied to generate ω-many Woodin cardinals.

In Section 7 we show that the Generation Theorem can itself be localized in two respects. In the first localization we show that Δ^1_2-determinacy implies that for a Turing cone of x, $\omega_1^{L[x]}$ is a Woodin cardinal in an inner model of $L[x]$. In the second localization we show that the proof can in fact be carried out in second-order arithmetic.

In Section 8 we survey some further results. First, we discuss results concerning the actual equivalence of axioms of definable determinacy and axioms asserting the existence of inner models with Woodin cardinals. Second, we revisit the analysis of $\mathrm{HOD}^{L(\mathbb{R})}$ and $\mathrm{HOD}^{L[x][g]}$, for certain generic extensions $L[x][g]$, in light of the advances that have been made in fine-structural inner model theory. Remarkably, it turns out that not only are these models well-behaved in the context of definable determinacy—they are actually fine-structural inner models, but of a kind that falls outside of the traditional hierarchy.

We have tried to keep the account self-contained, presupposing only acquaintance with the constructible universe, the basics of forcing, and the basics of large cardinal theory. In particular, we have tried to minimize appeal to fine-structure and descriptive set theory. Fine-structure enters only in Section 8 where we survey more recent developments, but even there one should be able to get a sense of the lay of the land without following the details. For the relevant background and historical development of the subject see [1], [2] and [10].

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1.2. Notation

For the most part our notational conventions are standard. Nevertheless, some comments are in order.

(1) We use $\mu\alpha\,\varphi(\alpha)$ to indicate the least ordinal α such that $\varphi(\alpha)$.

(2) In writing OD_X and HOD_X we always mean that X itself (as opposed to its elements) is allowed as a parameter. The notation $\mathrm{OD}_{\{X\}}$ is sometimes used for this, for example, in contexts where one would like to speak of both $\mathrm{OD}_{\{X\}}$ and OD_X. However, in this chapter we will have no occasion to speak of the latter and so we have dropped the curly brackets on the ground that they would only serve to clutter the text. We also use OD_X as both a name (for the class of sets which are ordinal definable from X) and as an adjective (for example when we say that a particular class is OD_X.) We use $<_{\mathrm{OD}_X}$ for a fixed canonical OD_X-well-ordering of OD_X sets. The notation $\mathrm{OD}(\mathbb{R})$ is used in analogy with $L(\mathbb{R})$.

(3) A *strategy* for Player I is a function $\sigma : \bigcup_{i<\omega} \omega^{2i} \to \omega$. Letting $\sigma * y$ be the real produced when Player I follows σ and Player II plays y, we say that σ is a *winning strategy* for Player I in the game with payoff $A \subseteq \omega^\omega$ if for all $y \in \omega^\omega$, $\sigma * y \in A$. The corresponding notions for Player II

are defined similarly. We typically reserve σ for strategies for Player I and τ for strategies for Player II. The play that results from having II play y against σ is denoted $\sigma * y$ and likewise the play that results from having I play x against τ is denoted $x * \tau$. We write $x * y$ for the real that results from having Player I play x and Player II play y and in this case we let $(x * y)_I = x$ and $(x * y)_{II} = y$. For example, $(\sigma * y)_I$ is the real that Player I plays when following the strategy σ against II's play of y. If σ is a strategy for Player I and τ is a strategy for Player II we write $\sigma * \tau$ for the real produced by playing the strategies against one another. Occasionally, when $z = x * y$ we write z_{even} to indicate x and z_{odd} to indicate y.

(4) If X is a subset of the plane $\omega^\omega \times \omega^\omega$ we use $\text{proj}_1(X)$ for the "projection to the first coordinate" and $\text{proj}_2(X)$ for the "projection to the second coordinate".

(5) For $n_0, \ldots, n_{k-1} \in \omega$, we use $\langle n_0, \ldots, n_{k-1} \rangle$ to denote the natural number encoding $(n_0, \ldots, n_{k-1})$ via a recursive bijection between ω^k and ω (which we fix throughout) and we let $(n)_i$ be the associated projection functions. For $x \in \omega^\omega$ and $i \in \omega$ we also use $(x)_i$ for the projection function associated to a recursive bijection between $(\omega^\omega)^\omega$ and ω^ω. See [10, Chapter 3] for further details on such recursive coding and decoding functions.

There is a slight conflict in notation in that angle brackets are also traditionally used for sequences and n-tuples. We have lapsed into this usage at points but the context resolves the ambiguity; for example, when we write $\langle x_\alpha \mid \alpha < \lambda \rangle$ it is clear that we are referring to a sequence.

(6) In this chapter by the "reals" we mean ω^ω, which, under the standard topology, is homeomorphic to the irrationals as normally construed. However, we continue to use the symbol '$\mathbb{R}$' in contexts where it is traditional, for example, in $L(\mathbb{R})$.

(7) We use $\text{tc}(x)$ for the transitive closure of x.

(8) A base theory that will play a central role throughout is

$$\mathrm{T}_0 = \mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) - \text{Power Set} + \text{“}\mathscr{P}(\omega) \text{ exists''}.$$

2. Basic Results

The central result of this section is Solovay's theorem to the effect that under ZF + AD, ω_1 is a measurable cardinal. The proof that we will give is slightly more involved than the standard proof but it has the advantage of illustrating some of the key components in the more general theorems to be proved in later sections. One thing we would like to illustrate is the manner in which "boundedness" and "coding" combine to yield normal ultrafilters. In subsequent sections stronger forms of boundedness (more precisely, "reflection") and stronger forms of coding will be used to establish stronger forms of normality. This will culminate in the production of Woodin cardinals.

In §2.1 we review some basic consequences of ZF + AD. In §2.2 we prove $\undertilde{\Sigma}^1_1$-boundedness and use it to prove the Basic Coding Lemma, a simple case of the more general coding lemmas to be proved in Section 3. In §2.3 we use $\undertilde{\Sigma}^1_1$-boundedness to show that the club filter on ω_1 witnesses that ω_1 is a measurable cardinal and we use $\undertilde{\Sigma}^1_1$-boundedness and the Basic Coding Lemma to show that this ultrafilter is normal. In §2.4 we introduce $\undertilde{\delta}^2_1$ and establish its basic properties. Finally, in §2.5 we draw on the Coding Lemma of Section 3 to show that the proof of Solovay's theorem generalizes to show that assuming ZF+DC+AD then in the restricted setting of $L(\mathbb{R})$ the ordinal $(\undertilde{\delta}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal. Later, in Section 4, we will dispense with DC and reprove this theorem in ZF + AD.

2.1. Preliminaries

In order to keep this account self-contained, in this subsection we shall collect together some of the basic features of the theory of determinacy. These concern (1) the connection between determinacy and choice, (2) the implications of determinacy for regularity properties, and (3) the implications of determinacy for the Turing degrees. See [2], [10], and Jackson's chapter in this Handbook for further details and references.

Let us begin with the axiom of choice. A straightforward diagonalization argument shows that AD contradicts the full axiom of choice, AC. However, certain fragments of AC are consistent with AD and, in fact, certain fragments of AC follow from AD.

2.1 Definition. The *Countable Axiom of Choice*, AC_ω, is the statement that every countable set consisting of non-empty sets has a choice function. The *Countable Axiom of Choice for Sets of Reals*, $AC_\omega(\mathbb{R})$, is the statement that every countable set consisting of non-empty sets of reals has a choice function.

Theorem 2.2. *Assume* $\mathrm{ZF} + \mathrm{AD}$. *Then* $AC_\omega(\mathbb{R})$.

Proof. Let $\{X_n \mid n < \omega\}$ be a countable collection of non-empty sets of reals. Consider the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

where I wins if and only if $y \notin X_{x(0)}$. (Notice that we are leaving the definition of the payoff set of reals A implicit. In this case the payoff set is $\{x \in \omega^\omega \mid x_{\text{odd}} \notin X_{x(0)}\}$. In the sequel we shall leave such routine transformations to the reader.) Thus, Player I is to be thought of as playing an element X_n of the countable collection and Player II must play a real which is not in this element. Of course, Player I cannot win. So there must be a winning strategy τ for Player II. The function

$$\begin{aligned} f : \omega &\longrightarrow \omega^\omega \\ n &\longmapsto (\langle n, 0, 0, \dots \rangle * \tau)_{II} \end{aligned}$$

is a choice function for $\{X_n \mid n < \omega\}$. $\square$

Corollary 2.3. *Assume* $\mathrm{ZF} + \mathrm{AD}$. *Then* ω_1 *is regular.*

2.4 Definition. The *Principle of Dependent Choices*, DC, is the statement that for every non-empty set X and for every relation $R \subseteq X \times X$ such that for all $x \in X$ there is a $y \in X$ such that $(x, y) \in R$, there is a function $f : \omega \to X$ such that for all $n < \omega$, $(f(n), f(n+1)) \in R$. The *Principle of Dependent Choices for Sets of Reals*, $DC_\mathbb{R}$, is simply the restricted version of DC where X is $\mathbb{R}$.

It is straightforward to show that DC implies AC_ω and Jensen showed that this implication cannot be reversed. Solovay showed that $\mathrm{Con}(\mathrm{ZF} + \mathrm{AD}_\mathbb{R})$ implies $\mathrm{Con}(\mathrm{AD} + \neg\mathrm{DC})$ and this was improved by Woodin.

Theorem 2.5. *Assume* $\mathrm{ZF} + \mathrm{AD} + V = L(\mathbb{R})$. *Then in a forcing extension there is an inner model of* $\mathrm{AD} + \neg AC_\omega$.

Theorem 2.6 (Kechris). *Assume* ZF + AD. *Then* $L(\mathbb{R}) \models$ DC.

1 Open Question. Does AD imply $\mathrm{DC}_{\mathbb{R}}$?

Thus, of the above fragments of AC, $\mathrm{AC}_\omega(\mathbb{R})$ is known to be within the reach of AD, $\mathrm{DC}_{\mathbb{R}}$ could be within the reach of AD, and the stronger principles AC_ω and DC are known to be consistent with but independent of AD (assuming consistency of course). For this reason, to minimize our assumptions, in what follows we shall work with $\mathrm{AC}_\omega(\mathbb{R})$ as far as this is possible. There are two places where we invoke DC, namely, in Kunen's theorem (Theorem 3.11) and in Lemma 4.8 concerning the well-foundedness of certain ultrapowers. However, in our applications, DC will reduce to $\mathrm{DC}_{\mathbb{R}}$ and so if the above open question has a positive answer then these appeals to DC can also be avoided.

We now turn to regularity properties. The axiom of determinacy has profound consequences for the structure theory of sets of real numbers. See [10] and Jackson's chapter in this Handbook for more on this. Here we mention only one central consequence that we shall need below.

Theorem 2.7 (Mycielski-Swierczkowski; Mazur, Banach; Davis). *Assume* ZF + AD. *Then all sets are Lebesgue measurable, have the property of Baire, and have the perfect set property.*

Proof. See [2, Section 27]. □

Another important consequence we shall need is the following:

Theorem 2.8. *Assume* ZF + AD. *Then every ultrafilter is* ω_1*-complete.*

Proof. Suppose $\mathscr{U} \subseteq \mathscr{P}(X)$ is an ultrafilter. If $\mathscr{U}$ is not ω_1-complete then there exists $\{X_i \mid i < \omega\}$ such that

(1) for all $i < \omega$, $X_i \in \mathscr{U}$ and

(2) $\bigcap_{i<\omega} X_i \notin \mathscr{U}$.

Without loss of generality we can suppose that $\bigcap_{i<\omega} X_i = \varnothing$. So this gives a partition $\{Y_i \mid i < \omega\}$ of X into disjoint non-empty sets each of which is not in $\mathscr{U}$. Define $\mathscr{U}^* \subseteq \mathscr{P}(\omega)$ as follows:

$$\sigma \in \mathscr{U}^* \quad \text{iff} \quad \bigcup\{Y_i \mid i \in \sigma\} \in \mathscr{U}.$$

This is an ultrafilter on ω which is not principal since by assumption $Y_i \notin \mathscr{U}$ for each $i < \omega$. However, as Sierpiński showed, a non-principal ultrafilter over ω (construed as a set of reals) is not Lebesgue measurable. □

Finally, we turn to the implications of determinacy for the Turing degrees. For $x, y \in \omega^\omega$, we say that x is *Turing reducible* to y, $x \leqslant_T y$, if x is recursive in y and we say that x is *Turing equivalent* to y, $x \equiv_T y$, if $x \leqslant_T y$ and $y \leqslant_T x$. The *Turing degrees* are the corresponding equivalence classes $[x]_T = \{y \in \omega^\omega \mid y \equiv_T x\}$. Letting

$$\mathscr{D}_T = \left\{[x]_T \mid x \in \omega^\omega\right\}$$

the relation $\leqslant_T$ lifts to a partial ordering on $\mathscr{D}_T$. A *cone of Turing degrees* is a set of the form

$$\left\{[y]_T \mid y \geqslant_T x_0\right\}$$

for some $x_0 \in \omega^\omega$. A *Turing cone of reals* is a set of the form

$$\left\{y \in \omega^\omega \mid y \geqslant_T x_0\right\}$$

for some $x_0 \in \omega^\omega$. In each case x_0 is said to be the *base* of the cone. In later sections we will discuss different degree notions. However, when we speak of a "cone of x" without qualification we always mean a "Turing cone of x". The *cone filter on $\mathscr{D}_T$* is the filter consisting of sets of Turing degrees that contain a cone of Turing degrees.

Theorem 2.9 (Cone Theorem) (Martin). *Assume* ZF + AD. *The cone filter on $\mathscr{D}_T$ is an ultrafilter.*

Proof. For $A \subseteq \mathscr{D}_T$ consider the game

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

where I wins iff $[x*y]_T \in A$. If I has a winning strategy σ_0 then σ_0 witnesses that A is in the cone filter since for $y \geqslant_T \sigma_0$, $[y]_T = [\sigma_0 * y]_T \in A$. If II has a winning strategy τ_0 then τ_0 witnesses that $\mathscr{D}_T \smallsetminus A$ is in the cone filter since for $x \geqslant_T \tau_0$, $[x]_T = [x * \tau_0]_T \in \mathscr{D}_T \smallsetminus A$. □

It follows that under ZF + AD each statement $\varphi(x)$ either holds for a Turing cone or reals x or fails for a Turing cone of reals x.

The proof of the Cone Theorem easily relativizes to fragments of definable determinacy. For example, assuming Σ^1_2-determinacy every Σ^1_2 set which is invariant under Turing equivalence either contains or is disjoint from a Turing cone of reals.

It is of interest to note that when Martin proved the Cone Theorem he thought that he would be able to refute AD by finding a property that "toggles". He started with Borel sets and, when no counterexample arose, moved on to more complicated sets. We now know (assuming there are infinitely many Woodin cardinals with a measurable above) that no counterexamples are to be found in $L(\mathbb{R})$. Moreover, the statement that there are no counterexamples in $L(\mathbb{R})$ (i.e. the statement that *Turing determinacy* holds in $L(\mathbb{R})$) actually implies $\mathrm{AD}^{L(\mathbb{R})}$ (over ZF + DC). Thus, the basic intuition that the Cone Theorem is strong is correct—it is just not as strong as 0=1.

2.2. Boundedness and Basic Coding

We begin with some definitions. For $x \in \omega^\omega$, let E_x be the binary relation on ω such that mE_xn iff $x(\langle m,n\rangle) = 0$, where recall that $\langle\cdot,\cdot\rangle : \omega\times\omega\to\omega$ is a recursive bijection. The real x is said to *code* the relation E_x. Let $\mathrm{WO} = \{x \in \omega^\omega \mid E_x \text{ is a well-ordering}\}$. For $x \in \mathrm{WO}$, let α_x be the ordertype of E_x and, for $\alpha < \beta < \omega_1$ let $\mathrm{WO}_\alpha = \{x \in \mathrm{WO} \mid \alpha_x = \alpha\}$, $\mathrm{WO}_{<\alpha} = \{x \in \mathrm{WO} \mid \alpha_x < \alpha\}$, $\mathrm{WO}_{(\alpha,\beta]} = \{x \in \mathrm{WO} \mid \alpha < \alpha_x \leqslant \beta\}$ and likewise for other intervals of countable ordinals. For $x \in \mathrm{WO}$, let $\mathrm{WO}_x = \mathrm{WO}_{\alpha_x}$. It is straightforward to see that these sets are Borel and that WO is a complete $\utilde{\Pi}^1_1$ set. (See [1, Chapter 25] for details.)

In addition to the topological and recursion-theoretic characterizations of $\utilde{\Sigma}^1_1$ there is a model-theoretic characterization which is helpful in simplifying complexity calculations. A model (M,E) satisfying T_0 (or some sufficiently strong fragment of ZF) is an *ω-model* if $(\omega^M, E{\restriction}\omega^M) \cong (\omega, \in{\restriction}\omega)$, where recall that T_0 is the theory $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) - \text{Power Set} + \text{``}\mathscr{P}(\omega)\text{ exists''}$. Notice that ω-models are correct about arithmetical statements and hence Π^1_1-statement are downward absolute to ω-models. Moreover, the statement "There exists a real coding an ω-model of T_0" is Σ^1_1, in contrast to the statement "There exists a real coding a well-founded model of T_0", which is Σ^1_2. Thus we have the following characterization of the pointclass $\utilde{\Sigma}^1_1$: $A \subseteq \omega^\omega$ is $\utilde{\Sigma}^1_1$ iff there is a formula φ and there exists a $z \in \omega^\omega$ such that

$$A = \{y \in \omega^\omega \mid \text{there is a real coding an } \omega\text{-model } M \text{ with } z \in M \text{ such that } y \in M \text{ and } M \models \mathrm{T}_0 + \varphi[y,z]\}.$$

The lightface version Σ^1_1 is defined similarly by omitting the parameter z, as are the Σ^1_1 subsets of $(\omega^\omega)^n$ and the Σ^1_1-statements, etc. Theories much weaker than T_0 yield an equivalent definition. For example, one can use the finite theory ZF_N for some sufficiently large N.

As an illustration of the utility of this model-theoretic characterization of $\utilde{\Sigma}^1_1$ we shall use it to show that for each $x \in \mathrm{WO}$, $\mathrm{WO}_{<x}$ is $\utilde{\Delta}^1_1$: Notice that ω-models of T_0 correctly compute "$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$" in the following sense: If $x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$ and M is an ω-model of T_0 which contains x and y, then $M \models$ "$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$". (By downward absoluteness M satisfies that $x, y \in \mathrm{WO}$ and hence that α_y and α_x are defined. Furthermore, since M is an ω-model it correctly computes the ordering of α_x and α_y.) If $x \in \mathrm{WO}$ and M is an ω-model of T_0 which satisfies "$x, y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$" then $y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$. (The point is that M satisfies that there is an order-preserving map $f : E_y \to E_x$ and, since ω-models are correct about such maps and since E_x is truly well-founded, it follows that $y \in \mathrm{WO}$ and $\alpha_y < \alpha_x$). So, for $x \in \mathrm{WO}$,

$$\begin{aligned}\mathrm{WO}_{<x} = \{y \in \omega^\omega \mid\ & \text{there is a real coding an } \omega\text{-model } M \text{ such that} \\ & x, y \in M \text{ and } M \models T_0 + \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}\} \\ \\ = \{y \in \omega^\omega \mid\ & \text{for all reals coding } \omega\text{-models } M \text{ if } x, y \in M \\ & \text{and } M \models T_0 \text{ then } M \models \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}\}.\end{aligned}$$

Thus, for $x \in \mathrm{WO}$, $\mathrm{WO}_{<x}$ is $\utilde{\Delta}^1_1$ and hence Borel.

Lemma 2.10 ($\utilde{\Sigma}^1_1$-Boundedness)(Luzin-Sierpiński). *Assume* ZF+$\mathrm{AC}_\omega(\mathbb{R})$. *Suppose* $X \subseteq \mathrm{WO}$ *and* X *is* $\utilde{\Sigma}^1_1$. *Then there exists an* $\alpha < \omega_1$ *such that* $X \subseteq \mathrm{WO}_{<\alpha}$.

Proof. Assume toward a contradiction that X is unbounded. Then

$$y \in \mathrm{WO} \text{ iff there is a } x \in X \text{ such that } \alpha_y < \alpha_x$$

since for $x \in X \subseteq \mathrm{WO}$, ω-models of T_0 correctly compute $\alpha_y < \alpha_x$. By the above remark, we can rewrite this as

$$\begin{aligned}y \in \mathrm{WO} \text{ iff }& \text{there is a } x \in X \text{ and there is an } \omega\text{-model } M \text{ such that} \\ & x, y \in M \text{ and } M \models T_0 + \text{“}x, y \in \mathrm{WO} \text{ and } \alpha_y < \alpha_x\text{”}.\end{aligned}$$

Thus, WO is $\utilde{\Sigma}^1_1$, which contradicts the fact that WO is a complete $\utilde{\Pi}^1_1$ set. (Without appealing to the fact that WO is a complete $\utilde{\Pi}^1_1$ set we can arrive at a contradiction (making free use of AC) as follows: Let $z \in \omega^\omega$ be such that both X and WO are $\Sigma^1_1(z)$. Let α be such that $V_\alpha \models \mathrm{T}_0$ and choose $Y \prec V_\alpha$ such that Y is countable and $z \in Y$. Let N be the transitive collapse of Y. By correctness, $X \cap N = X^N$. Choose a uniform ultrafilter $U \subseteq \mathscr{P}(\omega_1)^N$ such that if

$$j : N \to \mathrm{Ult}(N, U)$$

is the associated embedding then $\mathrm{crit}(j) = \omega_1^N$ and $j(\omega_1^N)$ is not well-founded. (To obtain such an ultrafilter build a generic for $(\mathscr{P}(\omega_1)/\text{countable})^N$. See Lemma 22.20 of [1].) Since $\mathrm{Ult}(N, U)$ is an ω-model of T_0 it correctly computes WO. It follows that $(\mathrm{WO})^{\mathrm{Ult}(N,U)} \subseteq \mathrm{WO}$, which in turn contradicts the fact that $\omega_1^{\mathrm{Ult}(N,U)}$ is not well-founded.) □

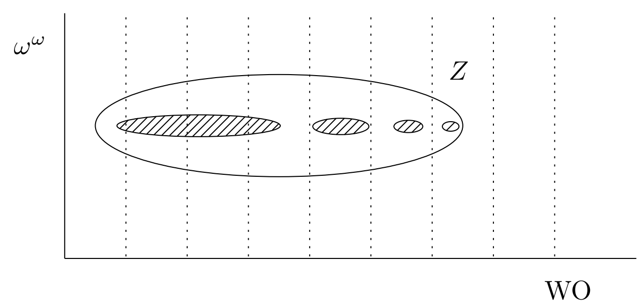
Lemma 2.11 (Basic Coding) (Solovay). *Assume* ZF + AD. *Suppose* $Z \subseteq \mathrm{WO} \times \omega^\omega$. *Then there exists a* $\underset{\sim}{\Sigma}^1_2$ *set* Z^* *such that*

(1) $Z^* \subseteq Z$ *and*

(2) *for all* $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ *iff* $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

Moreover, there is such a Z^* *which is of the form* $X \cap (\mathrm{WO} \times \omega^\omega)$ *where* $X \subseteq \omega^\omega \times \omega^\omega$ *is* $\underset{\sim}{\Sigma}^1_1$.

Proof. Here is the picture:



The space $\mathrm{WO} \times \omega^\omega$ is sliced into sections $\mathrm{WO}_\alpha \times \omega^\omega$ for $\alpha < \omega_1$. Z is represented by the unshaded ellipse and Z^* is represented by the shaded region. Basic Coding says that whenever Z meets one of the sections so does Z^*. In such a situation we say that Z^* is a *selector* for Z.

To see that Z^* exists, consider the game

I	$x(0)$		$x(1)$		$x(2)$		...
II		$y(0)$		$y(1)$		...	

where II wins iff whenever $x \in \mathrm{WO}$ then y codes a countable set Y such that

(1) $Y \subseteq Z$ and

(2) for all $\alpha \leqslant \alpha_x$, $Y \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

The idea is that Player I challenges by playing a countable ordinal α_x and Player II meets this challenge provided he can play (a code for a) a selector Y for $Z \cap (\mathrm{WO}_{\leqslant \alpha_x} \times \omega^\omega)$.

Claim. There can be no winning strategy for Player I in this game.

Proof. Suppose σ is a winning strategy for I. As the play unfolds, Player I can attempt to increase α_x as Player II's play is revealed. However, Player II can anticipate all such attempts as follows: The set

$$X = \{(\sigma * y)_I \mid y \in \omega^\omega\}$$

is $\Sigma^1_1(\sigma)$ and, since σ is winning for I, $X \subseteq \mathrm{WO}$. So, by $\underset{\sim}{\Sigma}^1_1$-boundedness, there is an $\beta < \omega_1$ such that $X \subseteq \mathrm{WO}_{<\beta}$. Since we have $\mathrm{AC}_\omega(\mathbb{R})$ (by Theorem 2.2), we can choose a countable set $Y \subseteq Z$ such that for all $\alpha < \beta$, $Y \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$. Let y code Y and play y against σ. The resulting play $\sigma * y$ is a win for II, which is a contradiction. $\square$

Thus II has a winning strategy τ. For $x \in \mathrm{WO}$, let Y^x be the countable subset of Z coded by $(x * \tau)_{II}$. Then

$$Z^* = \bigcup \{Y^x \mid x \in \mathrm{WO}\}$$

is $\Sigma^1_2(\tau)$ and such that

(1) $Z^* \subseteq Z$,

(2) for all $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$.

Hence Z^* is as desired.

To see that we can choose Z^* to be of the form $X \cap (\mathrm{WO} \times \omega^\omega)$ where $X \subseteq \omega^\omega \times \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1$, let

$$X = \{(a,b) \mid \text{there is an } \omega\text{-model } M \text{ such that } a, b, \tau \in M \text{ and } M \models \mathrm{T}_0 + (a,b) \in Z^{**}\}$$

where

$$Z^{**} = \bigcup \{Y^x \cap (\mathrm{WO}_{\alpha_x} \times \omega^\omega) \mid x \in \mathrm{WO}\}.$$

This set is $\Sigma^1_1(\tau)$. The trouble is that although for $a \in \mathrm{WO}$ such models M are correct about $(a,b) \in Z^{**}$, M might think $a \in \mathrm{WO}$ when $a \notin \mathrm{WO}$. To overcome this difficulty we pare down, letting $Z^* = X \cap (\mathrm{WO} \times \omega^\omega)$. $\square$

2.3. Measurability

Theorem 2.12 (Solovay). *Assume* ZF + AD. *Then the club filter is an* ω_1*-complete ultrafilter on* ω_1.

Proof. The ultrafilter on ω_1 will be extracted from a game. As motivation, for the moment work in ZFC. For $S \subseteq \omega_1$, consider the game

$$\begin{array}{lccccccc} \mathrm{I} & \alpha_0 & & \alpha_1 & & \alpha_2 & & \dots \\ \mathrm{II} & & \beta_0 & & \beta_1 & & \dots & \end{array}$$

where we demand that $\alpha_0 < \beta_0 < \alpha_1 < \cdots < \omega_1$ and where the first player that fails to meet this demand loses and if both players meet the demand then I wins provided $\sup_{i<\omega} \alpha_i \in S$.

We claim that I wins this game for S if and only if S contains a club in ω_1. Suppose first that S contains a club C. Let σ be a strategy for I which chooses an element of C larger than the last ordinal played by II. This is a winning strategy for I. For if II meets the first condition then the ordinals played form an increasing sequence. The even elements of this sequence are in C and hence the supremum of the sequence is in C (since C is club) and hence in S. Thus σ is a winning strategy for I. Suppose next that I have a winning strategy σ. Let C be the set of limit ordinals $\gamma < \omega_1$ with the feature that for every $i < \omega$ and for every increasing sequence $\xi_0, \dots, \xi_{2i}$ of ordinals less than γ, the response $\sigma(\langle \xi_0, \dots, \xi_{2i} \rangle)$ is also less than γ. Let C' be the limit points of C. Since ω_1 is regular it follows that C and C' are club in ω_1. Now suppose $\gamma \in C'$. Let $\langle \gamma_i \mid i < \omega \rangle$ be an increasing sequences of ordinals in C which is cofinal in γ and such that γ_0 is greater than I's first move via σ. The key point is that this sequence is a legal play for II. Player II has "taken control" of the game. Now, since σ is a winning strategy for I it follows that the supremum, γ, is in S. Thus, S contains the club C'. So, if we had determinacy (which of course is impossible in ZFC) we would have an ultrafilter on ω_1.

Now return to ZF + AD. We want to mimic the above game via a game where each player plays natural numbers. This can be done since in an integer game each player ultimately plays a real x that can be regarded as coding ω-many reals $(x)_i$ each of which (potentially) codes a countable ordinal. More precisely, for $S \subseteq \omega_1$, let $G(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

with the following rules: Rule 1: For all $i < \omega$, $(x)_i, (y)_i \in \mathrm{WO}$. If Rule 1 is violated then, letting i be least such that either $(x)_i \notin \mathrm{WO}$ or $(y)_i \notin \mathrm{WO}$, I wins if $(x)_i \in \mathrm{WO}$; otherwise II wins. Now suppose Rule 1 is satisfied. Rule 2: $\alpha_{(x)_0} < \alpha_{(y)_0} < \alpha_{(x)_1} < \alpha_{(y)_1} \cdots$. The first failure defines who wins as above. If both rules are satisfied then I wins if and only if $\sup_{i<\omega} \alpha_{(x)_i} \in S$.

Now let

$$\mu = \{S \subseteq \omega_1 \mid \text{I wins } G(S)\}.$$

We claim that if I has a winning strategy in $G(S)$ then S contains a club: Let σ be a winning strategy for I. For $\alpha < \omega_1$, let

$$X_\alpha = \big\{((\sigma * y)_I)_n \mid n < \omega,\ y \in \omega^\omega,\ \text{and} \\ \forall i < n\,\big((y)_i \in \mathrm{WO} \text{ and } \alpha_{(y)_i} < \alpha\big)\big\}.$$

Notice that each $X_\alpha \subseteq \mathrm{WO}$ (since X_α is $\undertilde{\Sigma}^1_1$ (in σ and the code for α) and σ is a winning strategy) and so by $\undertilde{\Sigma}^1_1$-boundedness, there exists an α' such that $X_\alpha \subseteq \mathrm{WO}_{<\alpha'}$. Let $f : \omega_1 \to \omega_1$ be the function which given α chooses the least α' such that $X_\alpha \subseteq \mathrm{WO}_{<\alpha'}$. As before let C be the set of limit ordinals $\gamma < \omega_1$ with the feature that for every $\xi < \gamma$, $f(\xi) < \gamma$. Let C' be the limit points of C. Since ω_1 is regular (by Corollary 2.3) it follows that C and C' are club in ω_1. Now suppose $\gamma \in C'$. Let $\langle \gamma_i \mid i < \omega \rangle$ be an increasing sequences of ordinals in C which is cofinal in γ. Let $y \in \omega^\omega$ be such that for all $i < \omega$, $\alpha_{(y)_i} = \gamma_i$. We claim that playing y against σ witnesses that $\gamma \in S$. It suffices to show that y is legal with respect to the two rules. For then the supremum, γ, must be in S since σ is a winning strategy for I. Now the first rule is trivially satisfied since we chose y such that for all $i < \omega$, $(y)_i \in \mathrm{WO}$. To see that the second rule is satisfied we need to see that for each $i < \omega$, $\alpha_{((\sigma * y)_I)_i} < \gamma_i$. This follows from the fact that $X_{\gamma_i} \subseteq \mathrm{WO}_{<\gamma_i}$. Again, Player II has "taken control" of the game.

A similar argument shows that if II has a winning strategy in $G(S)$ then $\omega_1 \smallsetminus S$ contains a club. Thus the club filter on ω_1 is an ultrafilter and so μ is that ultrafilter. Finally, the fact that μ is ω_1-complete follows from Theorem 2.8. □

We now wish to show that under AD the club filter is normal. This was proved by Solovay, using DC. We shall give a proof that avoids appeal to DC and generalizes to larger ordinals.

Theorem 2.13. *Assume* ZF + AD. *Then the club filter is an ω_1-complete normal ultrafilter on ω_1.*

Proof. For $S \subseteq \omega_1$ let $G(S)$ be the game from the previous proof and let μ be as defined there. We know that μ is the club filter. To motivate the proof of normality we give a proof of ω_1-completeness that will generalize to produce normal ultrafilters on ordinals larger than ω_1. This is merely for illustration—the proof uses DC but this will be eliminated in Claim 2.

Claim 1. μ is ω_1-complete.

Proof. Suppose $S_j \in \mu$ for $j < \omega$. We have to show that $S = \bigcap_{j<\omega} S_j \in \mu$. Let σ_j be a winning strategy for I in $G(S_j)$. Assume toward a contradiction that $S \notin \mu$—that is, that I does not win $G(S)$—and let σ be a winning strategy for I in $G(\omega_1 \smallsetminus S)$. Our strategy is to build a play y that is legal for II against each σ_j and against σ. This will give us our contradiction by implying that $\sup_{i<\omega} \alpha_{(y)_i}$ is in each S_j but not in S.

We build $z_n = (y)_n$ by recursion on n using the foresight provided by $\utilde{\Sigma}^1_1$-boundedness. For the initial step we use $\utilde{\Sigma}^1_1$-boundedness to get $\beta_0 < \omega_1$ such that for all $j < \omega$ and for all $y \in \omega^\omega$

$$\alpha_{((\sigma_j * y)_I)_0} < \beta_0 \ \text{ and } \ \alpha_{((\sigma * y)_I)_0} < \beta_0.$$

Choose $z_0 \in \mathrm{WO}_{\beta_0}$. For the $(n+1)^{\text{st}}$ step we use $\utilde{\Sigma}^1_1$-boundedness to get $\beta_{n+1} < \omega_1$ such that $\beta_n < \beta_{n+1}$ and for all $j < \omega$ and for all $y \in \omega^\omega$, if $(y)_i = z_i$ for all $i \leqslant n$, then

$$\alpha_{((\sigma_j * y)_I)_{n+1}} < \beta_{n+1} \ \text{ and } \ \alpha_{((\sigma * y)_I)_{n+1}} < \beta_{n+1}.$$

Choose $z_{n+1} \in \mathrm{WO}_{\beta_{n+1}}$. Finally, let y be such that for all $n < \omega$, $(y)_n = z_n$. The play y is legal for II against each σ_j and σ, which is a contradiction. □

Claim 2. μ is normal.

Proof. Assume toward a contradiction that $f : \omega_1 \to \omega_1$ is regressive and that there is no $\alpha < \omega_1$ such that $\{\xi < \omega_1 \mid f(\xi) = \alpha\} \in \mu$ or, equivalently (by AD), that for all $\alpha < \omega_1$,

$$S_\alpha = \{\xi < \omega_1 \mid f(\xi) \neq \alpha\} \in \mu.$$

Our strategy is to recursively define

(1.1) an increasing sequence $\langle \eta_i \mid i < \omega \rangle$ of countable ordinals with supremum η, (1.2) a sequence of collections of strategies $\langle X_i \mid i < \omega\rangle$ where X_i contains winning strategies for I in games $G(S_\alpha)$ for $\alpha \in [\eta_{i-1}, \eta_i)$, where $\eta_{-1} = 0$, and

(1.3) a sequence $\langle y_i \mid i < \omega\rangle$ of plays such that y_i is legal for II against any $\sigma \in X_i$ and $\sup_{j<\omega} \alpha_{(y_i)_j} = \eta$.

Since each $\sigma \in X_i$ is a winning strategy for I, y_i will witness that $\eta \in S_\alpha$ for each $\alpha \in [\eta_{i-1}, \eta_i)$, i.e. y_i will witness that $f(\eta) \neq \alpha$ for each $\alpha \in [\eta_{i-1}, \eta_i)$. Thus collectively the y_i will guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts our assumption that $f(\eta) < \eta$.

We begin by letting

$$Z = \{(x, \sigma) \mid x \in \mathrm{WO} \text{ and } \sigma \text{ is a winning strategy for I in } G(S_{\alpha_x})\}.$$

By the Basic Coding Lemma, there is a $Z^* \subseteq Z$ such that

(2.1) for all $\alpha < \omega_1$, $Z^* \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (\mathrm{WO}_\alpha \times \omega^\omega) \neq \varnothing$

(2.2) $Z^* = X \cap (\mathrm{WO} \times \omega^\omega)$ where X is $\utilde{\Sigma}^1_1$.

The key point is that for each $\alpha < \omega_1$,

$$X \cap (\mathrm{WO}_{\leqslant\alpha} \times \omega^\omega)$$

is $\utilde{\Sigma}^1_1$ since $\mathrm{WO}_{\leqslant\alpha}$ is Borel. Thus, we can apply $\utilde{\Sigma}^1_1$-boundedness to these sets.

The difficulty is that to construct the sequence $\langle y_i \mid i < \omega\rangle$ we shall need DC. For this reason we drop down to a context where we have DC and run the argument there.

Let t be a real such that X is $\Sigma^1_1(t)$. By absoluteness, for each $\alpha < \omega_1^{L[t,f]}$, there exists an $(x, \sigma) \in Z^* \cap L[t, f]$ such that $\alpha = \alpha_x$ and σ is a winning strategy for Player I in $G(S_\alpha^{L[t,f]})$ where

$$S_\alpha^{L[t,f]} = \{\eta < \omega_1^{L[t,f]} \mid f(\eta) \neq \alpha\}.$$

For the remainder of the proof we work in $L[t, f]$ and interpret S_α and X via their definitions, simply writing S_α and X.

For the first step let

$$\begin{aligned}
\eta_0 &= \text{some ordinal } \eta \text{ such that } \eta < \omega_1\\
X_0 &= \text{proj}_2\big(X \cap (\text{WO}_{[0,\eta_0)} \times \omega^\omega)\big)\\
Y_0 &= \big\{((\sigma * y)_I)_0 \mid \sigma \in X_0 \wedge y \in \omega^\omega\big\}\\
z_0 &= \text{some real } z \text{ such that } Y_0 \subseteq \text{WO}_{<\alpha_z}.
\end{aligned}$$

So X_0 is a collection of strategies for games $G(S_\alpha)$ where $\alpha < \eta_0$. Since these strategies are winning for I the set Y_0 is contained in WO. Furthermore, Y_0 is $\underset{\sim}{\Sigma}^1_1$ and hence has a bound α_{z_0}. For the next step let

$$\begin{aligned}
\eta_1 &= \text{some ordinal } \eta \text{ such that } \eta_0, \alpha_{z_0} < \eta < \omega_1\\
X_1 &= \text{proj}_2\big(X \cap (\text{WO}_{[\eta_0,\eta_1)} \times \omega^\omega)\big)\\
Y_1 &= \big\{((\sigma * y)_I)_1 \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } (y)_0 = z_0\big\}\\
&\quad \cup \big\{((\sigma * y)_I)_0 \mid \sigma \in X_1,\ y \in \omega^\omega\big\}\\
z_1 &= \text{some real } z \text{ such that } Y_1 \subseteq \text{WO}_{<\alpha_z}.
\end{aligned}$$

For the $(n+1)^{\text{st}}$ step let

$$\begin{aligned}
\eta_{n+1} &= \text{some ordinal } \eta \text{ such that } \eta_n, \alpha_{z_n} < \eta < \omega_1\\
X_{n+1} &= \text{proj}_2\big(X \cap (\text{WO}_{[\eta_n,\eta_{n+1})} \times \omega^\omega)\big)\\
Y_{n+1} &= \big\{((\sigma * y)_I)_{n+1} \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n\ (y)_i = z_i\big\}\\
&\quad \cup \big\{((\sigma * y)_I)_n \mid \sigma \in X_1,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n-1\ (y)_i = z_{i+1}\big\}\\
&\qquad \vdots\\
&\quad \cup \big\{((\sigma * y)_I)_0 \mid \sigma \in X_{n+1},\ y \in \omega^\omega\big\}\\
z_{n+1} &= \text{some real } z \text{ such that } Y_{n+1} \subseteq \text{WO}_{<\alpha_z}.
\end{aligned}$$

Finally, for each $k < \omega$, let y_k be such that $(y_k)_i = z_{i+k}$ for all $i < \omega$. Since each of these reals contains a tail of the z_i's, if $\eta = \sup_{n<\omega} \eta_n$, then

$$\sup_{i<\omega}(\alpha_{(y_k)_i}) = \eta$$

for all $k < \omega$. Furthermore, y_k is a legal play for II against any $\sigma \in X_k$, as witnessed by the $(k+1)^{\text{st}}$ components of Y_n for $n \geqslant k$. Since each $\sigma \in X_k$ is a winning strategy for I, y_k witnesses that $\eta \in S_\alpha$ for $\alpha \in [\eta_{k-1}, \eta_k)$, i.e. that $f(\eta) \neq \alpha$ for any $\alpha \in [\eta_{k-1}, \eta_k)$. Thus, collectively the y_k guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts the fact that $f(\eta) < \eta$. □

This completes the proof of the theorem. $\square$

It should be noted that using DC normality can be proved without using Basic Coding since in place of the sequence $\langle X_i \mid i \in \omega \rangle$ one can use DC to construct a sequence $\langle \sigma_\alpha \mid \alpha < \eta \rangle$ of strategies. This, however, relies on the fact that η is countable. Our reason for giving the proof in terms of Basic Coding is that it illustrates in miniature how we will obtain normal ultrafilters on ordinals much larger than ω_1.

Corollary 2.14 (Solovay). *Assume* ZF + AD. *Then*

$$\mathrm{HOD} \models \omega_1^V \textit{ is a measurable cardinal.}$$

Proof. We have that

$$\mathrm{HOD} \models \mu \cap \mathrm{HOD} \text{ is a normal ultrafilter on } \omega_1^V,$$

since $\mu \cap \mathrm{HOD} \in \mathrm{HOD}$ (as μ is OD and OD is OD). $\square$

Thus, if ZF + AD is consistent, then ZFC + "There is a measurable cardinal" is consistent.

There is also an *effective* version of Solovay's theorem, which we shall need.

Theorem 2.15. *Assume* ZFC + OD-*determinacy. Then*

$$\mathrm{HOD} \models \omega_1^V \textit{ is a measurable cardinal.}$$

Proof. If S is OD then the game $G(S)$ is OD and hence determined. It follows (by the above proof) that if I has a winning strategy in $G(S)$ then S contains a club and if II has a winning strategy in $G(S)$ then $\omega_1 \setminus S$ contains a club. Thus,

$$V \models \mu \cap \mathrm{HOD} \text{ is an ultrafilter on HOD}$$

and so

$$\mathrm{HOD} \models \mu \cap \mathrm{HOD} \text{ is an ultrafilter.}$$

Similarly the proof of Claim 1 in Theorem 2.13 shows that

$$V \models \mu \cap \mathrm{HOD} \text{ is } \omega_1\text{-complete}$$

and so

$$\mathrm{HOD} \models \mu \cap \mathrm{HOD} \text{ is } \omega_1\text{-complete,}$$

which completes the proof. $\square$

2.4. The Least Stable

We now take the next step in generalizing the above result. For this purpose it is useful to think of ω_1 in slightly different terms: Recall the following definition:

$$\underset{\sim}{\delta}^1_1 = \sup\{\alpha \mid \text{there is a } \underset{\sim}{\Delta}^1_1\text{-surjection } \pi : \omega^\omega \to \alpha\}.$$

It is a classical result that $\omega_1 = \underset{\sim}{\delta}^1_1$. Now consider the following higher-order analogue of $\underset{\sim}{\delta}^1_1$:

$$\underset{\sim}{\delta}^2_1 = \sup\{\alpha \mid \text{there is a } \underset{\sim}{\Delta}^2_1\text{-surjection } \pi : \omega^\omega \to \alpha\}.$$

In this section we will work without determinacy and establish the basic features of this ordinal in the context of $L(\mathbb{R})$. In the next section we will solve for U in the equation

$$\frac{\underset{\sim}{\delta}^1_1}{\mathrm{WO}} = \frac{\underset{\sim}{\delta}^2_1}{U}$$

in such a way that U is accompanied by the appropriate boundedness and coding theorems required to generalize Solovay's proof to show that $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}^{L(\mathbb{R})}$ implies that $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ is a measurable cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$.

The following model-theoretic characterization of the pointclass $\underset{\sim}{\Sigma}^2_1$ will be useful in complexity calculations: $A \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}^2_1$ iff for some formula φ and some real $z \in \omega^\omega$,

$$\begin{aligned} A = \{y \in \omega^\omega \mid\ & \text{there is a transitive set } M \text{ such that} \\ & \text{(a)}\ \ \omega^\omega \subseteq M, \\ & \text{(b)}\ \ \text{there is a surjection } \pi : \omega^\omega \to M, \text{ and} \\ & \text{(c)}\ \ M \models \mathrm{T}_0 + \varphi[y, z]\} \end{aligned}$$

As before, theories much weaker than T_0 yield an equivalent definition and our choice of T_0 is simply one of convenience. The lightface version Σ^2_1 is defined similarly by omitting the parameter z.

We wish to study $\underset{\sim}{\delta}^2_1$ in the context of $L(\mathbb{R})$. In the interest of keeping our account self-contained and free of fine-structure we will give a brief introduction to the basic features of $L(\mathbb{R})$ under the stratification $L_\alpha(\mathbb{R})$ for $\alpha \in \mathrm{On}$. For credits and references see [2].

Definability issues will be central. Officially our language is the language of set theory with an additional constant $\dot{\mathbb{R}}$ which is always to be interpreted as $\mathbb{R}$. For a set M such that $X \cup \{\mathbb{R}\} \subseteq M$, let $\Sigma_n(M, X)$ be the collection of sets definable over M via a Σ_n-formula with parameters in $X \cup \{\mathbb{R}\}$. For example, x is $\Sigma_1(L(\mathbb{R}), X)$ iff x is Σ_1-definable over $L(\mathbb{R})$ with parameters from $X \cup \{\mathbb{R}\}$. It is important to note that the parameter $\mathbb{R}$ is always allowed in our definability calculations. To emphasize this we will usually make it explicit.

The basic features of L carry over to $L(\mathbb{R})$, one minor difference being that $\mathbb{R}$ is allowed as a parameter in all definability calculations. For example, for each limit ordinal λ, the sequence $\langle L_\alpha(\mathbb{R}) \mid \alpha < \lambda \rangle$ is $\Sigma_1(L_\lambda(\mathbb{R}), \{\mathbb{R}\})$.

For $X \cup \{\mathbb{R}\} \subseteq M \subseteq N$, let $M \prec_n^X N$ mean that for all parameter sequences $\vec{a} \in (X \cup \{\mathbb{R}\})^{<\omega}$ and for all Σ_n-formulas φ, $M \models \varphi[\vec{a}]$ iff $N \models \varphi[\vec{a}]$. Let $M \prec_n N$ be short for $M \prec_n^M N$.

2.16 Definition. The *least stable in* $L(\mathbb{R})$, $\delta_{\mathbb{R}}$, is the least ordinal δ such that

$$L_\delta(\mathbb{R}) \prec_1^{\mathbb{R} \cup \{\mathbb{R}\}} L(\mathbb{R}).$$

A related ordinal of particular importance is δ_F, the least ordinal δ such that

$$L_\delta(\mathbb{R}) \prec_1 L(\mathbb{R}).$$

We aim to show that $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} = \delta_{\mathbb{R}} = \delta_F$. For notational convenience we write $\underset{\sim}{\delta}^2_1$ for $(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$ and Θ for $\Theta^{L(\mathbb{R})}$.

The definability notions involved in the previous definition also have useful model-theoretic characterizations, which we will routinely employ. For example, $A \subseteq \omega^\omega$ is Σ_1-definable over $L(\mathbb{R})$ with parameters from $\mathbb{R} \cup \{\mathbb{R}\}$ iff there is a formula φ and a $z \in \omega^\omega$,

$$\begin{aligned} A = \{y \in \omega^\omega \mid \exists \alpha \in \mathrm{On} \text{ such that } & \\ & \text{(a)} \;\; L_\alpha(\mathbb{R}) \models \mathrm{T}_0 \text{ and} \\ & \text{(b)} \;\; L_\alpha(\mathbb{R}) \models \varphi[y, z, \mathbb{R}]\}. \end{aligned}$$

Again, theories weaker than T_0 (such as $\mathrm{ZF_N}$ for sufficiently large N) suffice. The existence of arbitrarily large levels $L_\alpha(\mathbb{R})$ satisfying T_0 will be proved below in Lemma 2.22.

Lemma 2.17. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$*. Suppose*

$$\begin{aligned} X = \big\{x \in L_\lambda(\mathbb{R}) \mid {} & x \textit{ is definable over } L_\lambda(\mathbb{R}) \\ & \textit{from parameters in } \mathbb{R} \cup \{\mathbb{R}\}\big\}, \end{aligned}$$

where λ *is a limit ordinal. Then* $X \prec L_\lambda(\mathbb{R})$.

Proof. It suffices (by the Tarski-Vaught criterion) to show that if A is a non-empty set which is definable over $L_\lambda(\mathbb{R})$ from parameters in $\mathbb{R} \cup \{\mathbb{R}\}$, then $A \cap X \neq \varnothing$. Let A be such a non-empty set and choose $x_0 \in A$. Since every set in $L_\lambda(\mathbb{R})$ is definable over $L_\lambda(\mathbb{R})$ from a real and an ordinal parameter,

$$\{x_0\} = \{x \in L_\lambda(\mathbb{R}) \mid L_\lambda(\mathbb{R}) \models \varphi_0[x, c_0, \alpha_0, \mathbb{R}]\}$$

for some formula φ_0, and parameters $c_0 \in \omega^\omega$ and $\alpha_0 \in \mathrm{On}$. Let α_1 be least such that there is exactly one element x such that $L_\lambda(\mathbb{R}) \models \varphi_0[x, c_0, \alpha_1, \mathbb{R}]$ and $x \in A$. Notice that α_1 is definable in $L_\lambda(\mathbb{R})$ from c_0 and the real parameter used in the definition of A. Thus, letting x_1 be the unique element such that $L_\lambda(\mathbb{R}) \models \varphi_0[x_1, c_0, \alpha_1, \mathbb{R}]$ we have a set which is in A (by the definition of x_1) and in X (since it is definable in $L_\lambda(\mathbb{R})$ from c_0 and the real parameter used in the definition of A.) ☐

Lemma 2.18. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *For each* $\alpha < \Theta$, *there is an* OD *surjection* $\pi : \omega^\omega \to \alpha$.

Proof. Fix $\alpha < \Theta$. Since every set in $L(\mathbb{R})$ is OD_x for some $x \in \omega^\omega$ there is an OD_x surjection $\pi : \omega^\omega \to \alpha$. For each $x \in \omega^\omega$, let π_x be the $<_{\mathrm{OD}_x}$-least such surjection if one exists and let it be undefined otherwise. We can now "average over the reals" to eliminate the dependence on real parameters, letting

$$\begin{aligned} \pi : \omega^\omega &\to \alpha \\ x &\mapsto \begin{cases} \pi_{(x)_0}((x)_1) & \text{if } \pi_{(x)_0} \text{ is defined} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is an OD surjection. ☐

Lemma 2.19 (Solovay). *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then* Θ *is regular in* $L(\mathbb{R})$.

Proof. By the proof of the previous lemma, there is an OD sequence

$$\langle \pi_\alpha \mid \alpha < \Theta \rangle$$

such that each $\pi_\alpha : \omega^\omega \to \alpha$ is an OD surjection. Assume for contradiction that Θ is singular. Let

$$f : \alpha \to \Theta$$

be a cofinal map witnessing the singularity of Θ. Let $g : \omega^\omega \to \alpha$ be a surjection. It follows that the map

$$\begin{aligned} \pi : \omega^\omega &\to \Theta \\ x &\mapsto \pi_{f \circ g((x)_0)}((x)_1) \end{aligned}$$

is a surjection, which contradicts the definition of Θ. □

Lemma 2.20. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then*

$$L_\Theta(\mathbb{R}) = \{x \in L(\mathbb{R}) \mid \textit{there is a surjection } \pi : \omega^\omega \to \mathrm{tc}(x)\}.$$

Thus, $\mathscr{P}(\mathbb{R}) \subseteq L_\Theta(\mathbb{R})$.

Proof. For the first direction suppose $x \in L_\Theta(\mathbb{R})$. Let $\lambda < \Theta$ be a limit ordinal such that $x \in L_\lambda(\mathbb{R})$. Thus $\mathrm{tc}(x) \subseteq L_\lambda(\mathbb{R})$. Moreover, there is a surjection $\pi : \omega^\omega \to L_\lambda(\mathbb{R})$, since every element of $L_\lambda(\mathbb{R})$ is definable from an ordinal and real parameters.

For the second direction suppose $x \in L(\mathbb{R})$ and that there is a surjection $\pi : \omega^\omega \to \mathrm{tc}(x)$. We wish to show that $x \in L_\Theta(\mathbb{R})$. Let λ be a limit ordinal such that $x \in L_\lambda(\mathbb{R})$. Thus $\mathrm{tc}(x) \subseteq L_\lambda(\mathbb{R})$. Let

$$\begin{aligned} X = \big\{ y \in L_\lambda(\mathbb{R}) \mid y \text{ is definable over } L_\lambda(\mathbb{R}) \\ \text{from parameters in } \mathrm{tc}(x) \cup \mathbb{R} \cup \{\mathbb{R}\} \big\}, \end{aligned}$$

By the proof of Lemma 2.17, $X \prec L_\lambda(\mathbb{R})$ and $\mathrm{tc}(x) \subseteq X$. By Condensation, the transitive collapse of X is $L_{\bar{\lambda}}(\mathbb{R})$ for some $\bar{\lambda}$. Since there is a surjection $\pi : \omega^\omega \to \mathrm{tc}(x)$ and since all members of $L_{\bar{\lambda}}(\mathbb{R})$ are definable from parameters in $\mathrm{tc}(x) \cup \mathbb{R} \cup \{\mathbb{R}\}$, there is a surjection $\rho : \omega^\omega \to L_{\bar{\lambda}}(\mathbb{R})$. So $\bar{\lambda} < \Theta$ and since $x \in L_{\bar{\lambda}}(\mathbb{R})$ this completes the proof. □

Lemma 2.21. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then*

$$L_\Theta(\mathbb{R}) \models \mathrm{T}_0.$$

Proof. It is straightforward to see that $L_\Theta(\mathbb{R})$ satisfies T_0 − Separation − Replacement.

To see that $L_\Theta(\mathbb{R}) \models$ Separation note that if $S \subseteq x \in L_\Theta(\mathbb{R})$ then $S \in L_\Theta(\mathbb{R})$, by Lemma 2.20. To see that $L_\Theta(\mathbb{R}) \models$ Replacement we verify Collection, which is equivalent to Replacement, over the other axioms. Suppose

$$L_\Theta(\mathbb{R}) \models \forall x \in a \, \exists y \, \varphi(x, y),$$

where $a \in L_\Theta(\mathbb{R})$. Let

$$\begin{aligned} f : a &\mapsto \Theta \\ x &\to \mu\alpha\,(\exists y \in L_\alpha(\mathbb{R}) \text{ such that } L_\Theta(\mathbb{R}) \models \varphi(x,y)). \end{aligned}$$

The ordertype of $\operatorname{ran}(f)$ is less that Θ since otherwise there would be a surjection $\pi : \omega^\omega \to \Theta$ (since there is a surjection $\pi : \omega^\omega \to a$). Moreover, since Θ is regular, it follows that $\operatorname{ran}(f)$ is bounded by some $\lambda < \Theta$. Thus,

$$L_\Theta(\mathbb{R}) \models \forall x \in a\, \exists y \in L_\lambda(\mathbb{R})\, \varphi(x,y),$$

which completes the proof. □

Lemma 2.22. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *There are arbitrarily large* α *such that* $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$.

Proof. The proof is similar to the previous proof. Let us say that α is an $\mathbb{R}$-*cardinal* if for every $\gamma < \alpha$ there does not exist a surjection $\pi : \mathbb{R} \times \gamma \to \alpha$. For each limit ordinal $\gamma \in \mathrm{On}$, letting

$$\Theta(\gamma) = \sup\{\alpha \mid \text{there is a surjection } \pi : \mathbb{R} \times \gamma \to \alpha\}$$

we have that $\Theta(\gamma)$ is an $\mathbb{R}$-cardinal. For each γ which is closed under the Gödel pairing function, the argument of Lemma 2.19 shows that $\Theta(\gamma)$ is regular. The proof of the previous lemma generalizes to show that for every regular $\Theta(\gamma)$, $L_{\Theta(\gamma)}(\mathbb{R}) \models \mathrm{T}_0$. □

Lemma 2.23 (Solovay). *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. $L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R})$.

Proof. Suppose

$$L(\mathbb{R}) \models \varphi[a],$$

where $a \in L_\Theta(\mathbb{R})$ and φ is Σ_1. By Reflection, let λ be a limit ordinal such that

$$L_\lambda(\mathbb{R}) \models \varphi[a].$$

Let

$$X = \big\{ y \in L_\lambda(\mathbb{R}) \mid y \text{ is definable over } L_\lambda(\mathbb{R}) \text{ from parameters in } \operatorname{tc}(a) \cup \mathbb{R} \cup \{\mathbb{R}\} \big\},$$

By Condensation and Lemma 2.20, the transitive collapse of X is $L_{\bar\lambda}(\mathbb{R})$ for some $\bar\lambda < \Theta$. Thus, by upward absoluteness,

$$L_\Theta(\mathbb{R}) \models \varphi[a].$$

□

Lemma 2.24. *Assume* $\mathrm{ZF}+\mathrm{AC}_\omega(\mathbb{R})+V=L(\mathbb{R})$. *There are arbitrarily large* $\alpha<\delta_F$ *such that* $L_\alpha(\mathbb{R})\models \mathrm{T}_0$

Proof. Suppose $\xi<\delta_F$. Since $L_\Theta(\mathbb{R})\models \mathrm{T}_0$,

$$L(\mathbb{R})\models \exists\alpha>\xi\,(L_\alpha(\mathbb{R})\models \mathrm{T}_0).$$

The formula is readily seen to be Σ_1 with parameters in $\{\mathbb{R},\xi\}$ by our model-theoretic characterization. Thus, by the definition of δ_F,

$$L_{\delta_F}(\mathbb{R})\models \exists\alpha>\xi\,(L_\alpha(\mathbb{R})\models \mathrm{T}_0),$$

which completes the proof. □

Lemma 2.25. *Assume* $\mathrm{ZF}+\mathrm{AC}_\omega(\mathbb{R})+V=L(\mathbb{R})$. *Suppose* φ *is a formula and* $a\in\omega^\omega$. *Suppose* λ *is least such that* $L_\lambda(\mathbb{R})\models \mathrm{T}_0+\varphi[a]$. *Let*

$$X=\big\{x\in L_\lambda(\mathbb{R})\mid y \textit{ is definable over } L_\lambda(\mathbb{R}) \textit{ from parameters in } \mathbb{R}\cup\{\mathbb{R}\}\big\},$$

Then $X=L_\lambda(\mathbb{R})$. *Moreover, there is a surjection* $\pi:\omega^\omega\to L_\lambda(\mathbb{R})$ *such that* π *is definable over* $L_{\lambda+1}(\mathbb{R})$ *from* $\mathbb{R}$ *and* a.

Proof. By Lemma 2.17 we have that $X\prec L_\lambda(\mathbb{R})$. By condensation the transitive collapse of X is some $L_{\bar\lambda}(\mathbb{R})$. So $L_{\bar\lambda}(\mathbb{R})\models \mathrm{T}_0+\varphi[a]$ and thus by the minimality of λ we have $\bar\lambda=\lambda$. Since every $x\in X$ is definable from a real parameter and since $L_\lambda(\mathbb{R})\cong X$, we have that every $x\in L_\lambda(\mathbb{R})$ is definable from a real parameter, in other words, $X=L_\lambda(\mathbb{R})$. The desired map $\pi:\omega^\omega\to L_\lambda(\mathbb{R})$ is the map which takes a real coding the Gödel number of φ and a real parameter a to the set $\{x\in L_\lambda(\mathbb{R})\mid L_\lambda(\mathbb{R})\models\varphi[x,a]\}$. This map is definable over $L_{\lambda+1}(\mathbb{R})$. □

Lemma 2.26. *Assume* $\mathrm{ZF}+\mathrm{AC}_\omega(\mathbb{R})+V=L(\mathbb{R})$. *Suppose* $0<\alpha<\delta_{\mathbb{R}}$. *Then there is a surjection* $\pi:\omega^\omega\to L_\alpha(\mathbb{R})$ *such that* $\{(x,y)\mid \pi(x)\in\pi(y)\}$ *is* $\underset{\sim}{\Delta}^2_1$. *Thus,* $\delta_{\mathbb{R}}\leqslant\underset{\sim}{\delta}^2_1$.

Proof. Fix α such that $0<\alpha<\delta_{\mathbb{R}}$. By the minimality of $\delta_{\mathbb{R}}$,

$$L_\alpha(\mathbb{R})\not\prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}).$$

So there is an $a\in\omega^\omega$ and a Σ_1-formula φ such that if β is the least ordinal such that $L_\beta(\mathbb{R})\models\varphi[a]$ then $\beta>\alpha$. Let γ be least such that $\gamma>\beta>\alpha$

and $L_\gamma(\mathbb{R}) \models \mathrm{T}_0$ (which exists by Lemma 2.22). So γ is least such that $L_\gamma(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a]$ and, by Lemma 2.25, there is a surjection $\pi : \omega^\omega \to L_\gamma(\mathbb{R})$ which is definable over $L_{\gamma+1}(\mathbb{R})$ with the parameters $\mathbb{R}$ and a. Let $A = \{(x, y) \mid \pi(x) \in \pi(y)\}$. Let ψ_1 and ψ_2 be the formulas defining A and $(\omega^\omega)^2 \setminus A$ over $L_{\gamma+1}(\mathbb{R})$, respectively. By absoluteness,

$$\begin{aligned}(x,y) \in A \text{ iff there is a transitive set } M \text{ such that}\\ &\omega^\omega \subseteq M,\\ &\text{there is a surjection } \pi : \omega^\omega \to M, \text{ and}\\ &M \models \mathrm{T}_0 + \exists\gamma\,(L_\gamma(\mathbb{R}) \models \mathrm{T}_0 + \varphi[a] \text{ and}\\ &\qquad L_{\gamma+1}(\mathbb{R}) \models \psi_1[x,y]).\end{aligned}$$

This shows, by our model-theoretic characterization of $\underset{\sim}{\Sigma}^2_1$ that A is $\underset{\sim}{\Sigma}^2_1$. A similar argument shows that $(\omega^\omega)^2 \setminus A$ is $\underset{\sim}{\Sigma}^2_1$. Finally, the desired map can be extracted from π. □

We now use a universal $\underset{\sim}{\Sigma}^2_1$ set to knit together all of these "$\underset{\sim}{\Delta}^2_1$ projection maps".

Lemma 2.27. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. *Then there is a partial surjection* $\rho : \omega^\omega \to L_{\delta_\mathbb{R}}(\mathbb{R})$ *such that* $dom(\rho)$ *and* ρ *are both* Σ_1*-definable over* $L_{\delta_\mathbb{R}}(\mathbb{R})$ *with the parameter* $\mathbb{R}$. *Thus,* $L_{\delta_\mathbb{R}}(\mathbb{R}) \prec_1 L(\mathbb{R})$ *and hence* $\delta_F \leqslant \delta_R$.

Proof. Let U be a Σ^2_1 subset of $\omega^\omega \times \omega^\omega \times \omega^\omega$ that is universal for $\underset{\sim}{\Sigma}^2_1$ subsets of $\omega^\omega \times \omega^\omega$, that is, such that for each $\underset{\sim}{\Sigma}^2_1$ subset $A \subseteq \omega^\omega \times \omega^\omega$ there is an $x \in \omega^\omega$ such that $A = U_x$ where by definition

$$U_x = \{(y, z) \in \omega^\omega \times \omega^\omega \mid (x, y, z) \in U\}.$$

We define ρ using U. For the domain of ρ we take

$$\begin{aligned}\mathrm{dom}(\rho) = \{x \in \omega^\omega \mid \exists\alpha \in \mathrm{On}\,(&L_\alpha(\mathbb{R}) \models \mathrm{T}_0 \text{ and}\\ &L_\alpha(\mathbb{R}) \models U_{(x)_0} = (\omega^\omega \times \omega^\omega) \setminus U_{(x)_1})\}.\end{aligned}$$

Notice that $\mathrm{dom}(\rho)$ is $\Sigma_1(L(\mathbb{R}), \{\mathbb{R}\})$ and hence $\Sigma_1(L_{\delta_\mathbb{R}}(\mathbb{R}), \{\mathbb{R}\})$. Notice also that in general if $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ then

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} \subseteq U_{(x)_0}$$

and thus, if in addition,

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = (\omega^\omega \times \omega^\omega) \setminus (U_{(x)_1})^{L_\alpha(\mathbb{R})},$$

then,

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = U_{(x)_0}.$$

We can now define ρ as follows: Suppose $x \in \mathrm{dom}(\rho)$. Let $\alpha(x)$ be the least α as in the definition of $\mathrm{dom}(\rho)$. If there is an ordinal η and a surjection $\pi : \omega^\omega \to L_\eta(\mathbb{R})$ such that

$$\{(t_1, t_2) \mid \pi(t_1) \in \pi(t_2)\} = (U_{(x)_0})^{L_{\alpha(x)}(\mathbb{R})}$$

then let $\rho(x) = \pi((x)_2)$; otherwise let $\rho(x) = \varnothing$. Notice that the map ρ is $\Sigma_1(L(\mathbb{R}), \{\mathbb{R}\})$ and hence $\Sigma_1(L_{\delta_\mathbb{R}}(\mathbb{R}), \{\mathbb{R}\})$. By Lemma 2.26, $\rho : \mathrm{dom}(\rho) \to L_{\delta_\mathbb{R}}(\mathbb{R})$ is a surjection.

For the last part of the proof recall that by definition $L_{\delta_\mathbb{R}}(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R})$. The partial surjection $\rho : \omega^\omega \to L_{\delta_\mathbb{R}}(\mathbb{R})$ allows us to reduce arbitrary parameters in $L_{\delta_\mathbb{R}}(\mathbb{R})$ to parameters in ω^ω. □

Theorem 2.28. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. $\underset{\sim}{\delta}^2_1 = \delta_\mathbb{R} = \delta_F$.

Proof. We have $\delta_\mathbb{R} \leqslant \underset{\sim}{\delta}^2_1$ (by Lemma 2.26), $\delta_\mathbb{R} \leqslant \delta_F$ (by definition), and $\delta_F \leqslant \delta_\mathbb{R}$ (by Lemma 2.27). It remains to show $\underset{\sim}{\delta}^2_1 \leqslant \delta_\mathbb{R}$.

Suppose $\gamma < \underset{\sim}{\delta}^2_1$. We wish to show that $\gamma < \delta_\mathbb{R}$. Let $\pi : \omega^\omega \to \alpha$ be a surjection such that $A = \{(x, y) \mid \pi(x) < \pi(y)\}$ is $\underset{\sim}{\Delta}^2_1$. Using the notation from the previous proof let x be such that

$$U_{(x)_0} = A \text{ and } U_{(x)_1} = (\omega^\omega \times \omega^\omega) \smallsetminus A.$$

There is an ordinal α such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and

$$(U_{(x)_0})^{L_\alpha(\mathbb{R})} = (\omega^\omega \times \omega^\omega) \smallsetminus (U_{(x)_1})^{L_\alpha(\mathbb{R})}.$$

Since

$$L_{\delta_\mathbb{R}}(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}),$$

the least such ordinal, $\alpha(x)$, is less than $\delta_\mathbb{R}$. Thus,

$$(U_{(x)_0})^{L_{\alpha(x)}(\mathbb{R})} = A.$$

Finally, since $L_{\alpha(x)}(\mathbb{R}) \models \mathrm{T}_0$, this model can compute the ordertype, γ, of A. Thus, $\gamma < \alpha(x) < \delta_\mathbb{R}$. □

2.29 Remark. Although we will not need these facts it is worthwhile to note that the above proofs show

(1) $(\underset{\sim}{\Sigma}^2_1)^{L(\mathbb{R})} = \underset{\sim}{\Sigma}_1(L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})) \cap \mathscr{P}(\omega^\omega)$,

(2) $(\underset{\sim}{\Delta}^2_1)^{L(\mathbb{R})} = L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \cap \mathscr{P}(\omega^\omega)$, and

(3) (Solovay's Basis Theorem) if $L(\mathbb{R}) \models \exists X\, \varphi(X)$ where φ is $\underset{\sim}{\Sigma}^2_1$ then $L(\mathbb{R}) \models \exists X \in \underset{\sim}{\Delta}^2_1\, \varphi(X)$.

2.5. Measurability of the Least Stable

We are now in a position to show that under ZF + DC + AD,

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \text{ is a measurable cardinal.}$$

This serves as a warm-up to Section 4, where we will show that under ZF + DC + AD,

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \text{ is } \lambda\text{-strong,}$$

for each $\lambda < \Theta^{L(\mathbb{R})}$, and, in fact, that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \text{ is a Woodin cardinal.}$$

The proof that we give in Section 4 will show that DC can be eliminated from the result of the present section.

First we need an analogue U of WO that enables us to encode (unboundedly many) ordinals below $\underset{\sim}{\delta}^2_1$ and is accompanied by the boundedness and coding theorems required to push the above proof through for $\underset{\sim}{\delta}^2_1$. The following works: Let U be a Σ^2_1 subset of $\omega^\omega \times \omega^\omega$ that is universal for $\underset{\sim}{\Sigma}^2_1$ subsets of ω^ω. For $y \in \omega^\omega$ we let $U_y = \{z \in \omega^\omega \mid (y,z) \in U\}$. For $(y,z) \in U$, let $\Theta_{(y,z)}$ be least such that

$$L_{\Theta_{(y,z)}}(\mathbb{R}) \models \mathrm{T}_0 \;\text{ and }\; (y,z) \in U^{L_{\Theta_{(y,z)}}(\mathbb{R})}.$$

Let $\delta_{(y,z)} = (\underset{\sim}{\delta}^2_1)^{L_{\Theta_{(y,z)}}(\mathbb{R})}$. These ordinals are the analogues of α_x from the proof that ω_1 is measurable. For notational convenience we will routinely use our recursive bijection from $\omega^\omega \times \omega^\omega$ to ω^ω to identify pairs of reals (y,z) with single reals $x = \langle y, z\rangle$. Thus we will write Θ_x and δ_x instead of $\Theta_{(y,z)}$ and $\delta_{(y,z)}$.

Lemma 2.30. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R}) + V = L(\mathbb{R})$. $\{\delta_x \mid x \in U\}$ *is unbounded in* $\underset{\sim}{\delta}^2_1$.

Proof. Let $\alpha < \underset{\sim}{\delta}^2_1$. Let A be (the set of reals coding) a $\underset{\sim}{\Delta}^2_1$ prewellordering of length greater than α. Let $y, z \in \omega^\omega$ be such that $U_y = A$ and $U_z = \omega^\omega \setminus A$. So $L(\mathbb{R}) \models$ "$U_y = \omega^\omega \setminus U_z$". Since $\underset{\sim}{\delta}^2_1$ is the least stable, there is a $\beta < \underset{\sim}{\delta}^2_1$ such that $L_\beta(\mathbb{R}) \models$ "$U_y = \omega^\omega \setminus U_z$" and since $(U_y)^{L_\beta(\mathbb{R})} \subseteq A$ and $(U_z)^{L_\beta(\mathbb{R})} \subseteq \omega^\omega \setminus A$ we have that $A = (U_y)^{L_\beta(\mathbb{R})}$. Now, letting $x \in U \setminus U^{L_\beta(\mathbb{R})}$ and $\gamma < \underset{\sim}{\delta}^2_1$ be such that $L_\gamma(\mathbb{R}) \models$ "$\mathrm{T}_0 + x \in U$", we have that $\alpha < \delta_x$ since $A \in L_\gamma(\mathbb{R})$ and $L_\gamma(\mathbb{R})$ can compute the ordertype of A. □

In analogy with WO, for $x \in U$ let $U_{\delta_x} = \{y \in U \mid \delta_y = \delta_x\}$, $U_{<\delta_x} = \{y \in U \mid \delta_y < \delta_x\}$ and so on.

Lemma 2.31 ($\underset{\sim}{\Delta}^2_1$-Boundedness) (Moschovakis). *Assume* ZF + AC$_\omega(\mathbb{R})$ + $V = L(\mathbb{R})$. *Suppose $X \subseteq U$ and X is $\underset{\sim}{\Delta}^2_1$. Then there exists an $x \in U$ such that such that $X \subseteq U_{<\delta_x}$.*

Proof. Let $y, z \in \omega^\omega$ be such that $U_y = X$ and $U_z = \omega^\omega \setminus X$. (Notice that we are identifying X with the set of reals that recursively encodes it.) As above, there is a $\beta_0 < \underset{\sim}{\delta}^2_1$ such that $X = (U_y)^{L_{\beta_0}(\mathbb{R})}$. Choose γ such that $\beta_0 < \gamma < \underset{\sim}{\delta}^2_1$ and $L_\gamma(\mathbb{R})$ satisfies T_0. Then for all $z \in X$, $\delta_z < \gamma$. Now choose $x \in U$ such that $\delta_x > \gamma$. □

Lemma 2.32 (Coding) (Moschovakis). *Assume* ZF + AD. *Suppose $Z \subseteq U \times \omega^\omega$. Then there exists a $Z^* \subseteq Z$ such that for all $x \in U$*

(*i*) $Z^* \cap (U_{\delta_x} \times \omega^\omega) \neq \varnothing$ *iff* $Z \cap (U_{\delta_x} \times \omega^\omega) \neq \varnothing$

(*ii*) $Z^* \cap (U_{\leqslant\delta_x} \times \omega^\omega)$ *is* $\underset{\sim}{\Delta}^2_1$.

This lemma will follow from the more general coding lemmas of the next section. See Remark 3.6.

Theorem 2.33 (Moschovakis). *Assume* ZF + DC + AD. *Then*

$$L(\mathbb{R}) \models \textit{There is a normal ultrafilter on } \underset{\sim}{\delta}^2_1.$$

Proof. Work in $L(\mathbb{R})$. The proof is virtually a carbon copy of the proof for ω_1. One just replaces $\underset{\sim}{\delta}^1_1$, WO, α_x, and $\underset{\sim}{\Sigma}^1_1$ with $\underset{\sim}{\delta}^2_1$, U, δ_x, and $\underset{\sim}{\Delta}^2_1$, respectively. For completeness we include some of the details, noting the main changes.

For $S \subseteq \underset{\sim}{\delta}^2_1$, let $G(S)$ be the game

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

with the following rules: Rule 1: For all $i < \omega$, $(x)_i, (y)_i \in U$. If Rule 1 is violated then, letting i be least such that either $(x)_i \notin U$ or $(y)_i \notin U$, I wins if $(x)_i \in U$; otherwise II wins. Now suppose Rule 1 is satisfied. Rule 2: $\delta_{(x)_0} < \delta_{(y)_0} < \delta_{(x)_1} < \delta_{(y)_1} \cdots$. The first failure defines who wins as above. If both rules are satisfied then I wins iff $\sup_{i \in \omega} \delta_{(x)_i} \in S$.

Now let

$$\mu = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \text{I wins } G(S)\}.$$

Notice that as before (using $\underset{\sim}{\Delta}^2_1$-boundedness) if I has a winning strategy in $G(S)$ then S contains a set C which is unbounded and closed under ω-sequences. The proof that U is an ultrafilter is exactly as before. To see that it is $\underset{\sim}{\delta}^2_1$-complete and normal one uses the new versions of Boundedness and Coding. We note the minor changes in the proof of normality.

Assume for contradiction that $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ and that there is no $\alpha < \underset{\sim}{\delta}^2_1$ such that $\{\xi \mid f(\xi) = \alpha\} \in \mu$ or, equivalently (by AD) that for all $\alpha < \underset{\sim}{\delta}^2_1$,

$$S_\alpha = \{\xi \mid f(\xi) \neq \alpha\} \in \mu.$$

Let $\langle \delta_\alpha \mid \alpha < \underset{\sim}{\delta}^2_1 \rangle$ enumerate $\langle \delta_x \mid x \in U \rangle$. Here we are appealing to the fact that $\underset{\sim}{\delta}^2_1$ is regular, which can be shown using the Coding Lemma (See [10, p.433]). In analogy with WO, for $\alpha < \beta < \underset{\sim}{\delta}^2_1$, let $U_\alpha = \{x \in U \mid \delta_x = \delta_\alpha\}$, $U_{(\alpha,\beta]} = \{x \in U \mid \delta_\alpha < \delta_x \leqslant \delta_\beta\}$ and likewise for other intervals. Let $\leqslant_U$ be the associated prewellordering.

As before, our strategy is to inductively define

(1.1) an increasing sequence $\langle \eta_i \mid i < \omega \rangle$ of ordinals with supremum η,

(1.2) a sequence of collections of strategies $\langle X_i \mid i < \omega \rangle$ where X_i contains winning strategies for I in games $G(S_\alpha)$ for $\alpha \in [\eta_{i-1}, \eta_i)$, where $\eta_{-1} = 0$, and

(1.3) a sequence $\langle y_i \mid i < \omega \rangle$ of plays such that y_i is legal for II against any $\sigma \in X_i$ and $\sup_{j<\omega} \delta_{(y_i)_j} = \eta$.

Thus the y_i will collectively witness that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts our assumption that $f(\eta) < \eta$. The key difference is that in our present case we need the Coding Lemma since there are too many games. Let

$$Z = \{(x, \sigma) \mid x \in U \text{ and } \sigma \text{ is a winning strategy for I in } G(S_\alpha) \text{ where } \alpha \text{ is such that } \delta_\alpha = \delta_x\}$$

and, by our new Coding Lemma, let $Z^* \subseteq Z$ be such that for all $\alpha < \underset{\sim}{\delta}^2_1$, (2.1) $Z^* \cap (U_\alpha \times \omega^\omega) \neq \varnothing$ iff $Z \cap (U_\alpha \times \omega^\omega) \neq \varnothing$

(2.2) $Z^* \cap (U_{\leqslant\alpha} \times \omega^\omega)$ is $\undertilde{\Delta}^2_1$.

This puts us in a position to apply $\undertilde{\Delta}^2_1$-boundedness.

For the first step let

$$\begin{aligned}
\eta_0 &= \text{some ordinal } \eta \text{ such that } \eta < \undertilde{\delta}^2_1 \\
X_0 &= \operatorname{proj}_2\big(Z^* \cap (U_{[0,\eta_0)} \times \omega^\omega)\big) \\
Y_0 &= \big\{((\sigma{*}y)_I)_0 \mid \sigma \in X_0 \wedge y \in \omega^\omega\big\} \\
z_0 &= \text{some real } z \text{ such that } Y_0 \subseteq U_{<\delta_z}.
\end{aligned}$$

So X_0 is a collection of strategies for games $G(S_\alpha)$ where $\alpha < \eta_0$. Since these strategies are winning for I the set Y_0 is contained in U. Furthermore, Y_0 is $\undertilde{\Delta}^2_1$ and hence has a bound δ_{z_0}. For the induction step let

$$\begin{aligned}
\eta_{n+1} &= \text{some ordinal } \eta \text{ such that } \eta_n, \delta_{z_n} < \eta < \undertilde{\delta}^2_1 \\
X_{n+1} &= \operatorname{proj}_2\big(Z^* \cap (U_{[\eta_n,\eta_{n+1})} \times \omega^\omega)\big) \\
Y_{n+1} &= \big\{((\sigma{*}y)_I)_{n+1} \mid \sigma \in X_0,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n\ (y)_i = z_i\big\} \\
&\quad \cup \big\{((\sigma{*}y)_I)_n \mid \sigma \in X_1,\ y \in \omega^\omega \text{ such that } \forall i \leqslant n-1\ (y)_i = z_{i+1}\big\} \\
&\quad\ \vdots \\
&\quad \cup \big\{((\sigma{*}y)_I)_0 \mid \sigma \in X_{n+1},\ y \in \omega^\omega\big\} \\
z_{n+1} &= \text{some real } z \text{ such that } Y_{n+1} \subseteq U_{<\delta_z}.
\end{aligned}$$

Finally, for $k < \omega$, let y_k be such that $(y_k)_i = z_{i+k}$ for all $i < \omega$. Since each of these reals contains a tail of the z_i's, if $\eta = \sup_{n<\omega} \eta_n$, then

$$\sup_{i<\omega} \big(\delta_{(y_k)_i}\big) = \eta$$

for all $k < \omega$. Furthermore, y_k is a legal play for II against any $\sigma \in X_k$, as witnessed by the $(k+1)^{\text{st}}$ components of Y_n with $n \geqslant k$. Since each $\sigma \in X_k$ is a winning strategy for I, y_k witnesses that $\eta \in S_\alpha$ for $\alpha \in [\eta_{k-1}, \eta_k)$, i.e. that $f(\eta) \neq \alpha$ for any $\alpha \in [\eta_{k-1}, \eta_k)$. So collectively the y_k guarantee that $f(\eta) \neq \alpha$ for any $\alpha < \eta$, which contradicts the fact that $f(\eta) < \eta$. □

Corollary 2.34. *Assume* ZF + DC + AD. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models (\undertilde{\delta}^2_1)^{L(\mathbb{R})} \textit{ is a measurable cardinal.}$$

The above proof uses DC. However, as we shall see in Section 4.1 the theorem can be proved in ZF + AD. See Lemma 4.7.

The coding lemma was used to enable II to "collect together" the relevant strategies and then the $\underset{\sim}{\Delta}^2_1$-boundedness lemma was used to enable II to "take control of the ordinal played" in all such games by devising a play that is legal against all of the relevant strategies and (in each case) has the same fixed ordinal as output. This technique is central in what follows. It is important to note, however, that the above ultrafilter (and, more generally, ultrafilters obtained by such a "sup" game) concentrates on points of cofinality ω. Later we will use a slightly different game, where the role of the $\underset{\sim}{\Delta}^2_1$-boundedness lemma will be played by a certain reflection phenomenon. Before turning to this we prove the coding lemmas we shall need.

3. Coding

In the Basic Coding Lemma we constructed selectors relative to WO; we now do so relative to more general prewellorderings.

3.1. Coding Lemma

We begin by fixing some notation. For $P \subseteq \omega^\omega$, the notion of a $\underset{\sim}{\Sigma}^1_1(P)$ set is defined exactly like that of a $\underset{\sim}{\Sigma}^1_1$ set only now we allow reference to P and to $\omega^\omega \smallsetminus P$. In model-theoretic terms, $X \subseteq \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1(P)$ iff there is a formula φ and a real z such that

$$X = \big\{ y \in \omega^\omega \mid \text{there is an } \omega\text{-model } M \text{ such that} \\ y, z, P \cap M \in M \text{ and } M \models \mathrm{T}_0 + \varphi[y, z, P \cap M] \big\}.$$

The notion of a $\underset{\sim}{\Sigma}^1_1(P, P')$ set is defined in the same way, only now reference to both P and P' and their complements is allowed. The lightface versions of these notions and the versions involving $P \subseteq (\omega^\omega)^n$ are all defined in the obvious way.

Let $U^{(n)}(P)$ be a $\Sigma^1_1(P)$ subset of $(\omega^\omega)^{n+1}$ that is universal for $\underset{\sim}{\Sigma}^1_1(P)$ subsets of $(\omega^\omega)^n$, that is, such that for each $\underset{\sim}{\Sigma}^1_1(P)$ set $A \subseteq (\omega^\omega)^n$ there is an $e \in \omega^\omega$ such that $A = U^{(n)}_e(P) = \{y \in (\omega^\omega)^n \mid (e, y) \in U^{(n)}(P)\}$. We do this in such a way that the same formula is used, so that the definition is uniform in P. Likewise, for $U^{(n)}(P, P')$ etc. (The existence of such a universal set

$U^{(n)}(P)$ is guaranteed by the fact that the pointclass in question, namely, $\underset{\sim}{\Sigma}^1_1(P)$, is ω-parameterized and closed under recursive substitution. See [10], 3E.4 on p. 160 and especially 3H.1 on p. 183. We further assume that the universal sets are "good" in the sense of [10], p. 185 and we are justified in doing so by [10], 3H.1. A particular component of this assumption is that our universal sets satisfy the s-m-n-theorem (uniformly in P (or P and P')). See Jackson's chapter in this Handbook for further details.)

Theorem 3.1 (Recursion Theorem) (Kleene). *Suppose $f : \omega^\omega \to \omega^\omega$ is $\underset{\sim}{\Sigma}^1_1(P)$. Then there is an $e \in \omega^\omega$ such that*

$$U^{(2)}_e(P) = U^{(2)}_{f(e)}(P).$$

Proof. For $a \in \omega^\omega$, let

$$T_a = \{(b, c) \mid (a, a, b, c) \in U^{(3)}(P)\}.$$

Let $d : \omega^\omega \to \omega^\omega$ be Σ^1_1 such that $T_a = U^{(2)}_{d(a)}(P)$. (The function d comes from the s-m-n-theorem. In fact, $d(a) = s(a, a)$ (in the notation of Jackson's chapter) and d is continuous.) Let

$$Y = \{(a, b, c) \mid (b, c) \in U^{(2)}_{f(d(a))}(P)\}$$

and let a_0 be such that $Y = U^{(3)}_{a_0}(P)$. Notice that Y is $\underset{\sim}{\Sigma}^1_1(P)$ using the parameter for Y (as can readily be checked using the model-theoretic characterization of $\underset{\sim}{\Sigma}^1_1(P)$). We have

$$\begin{aligned}(b, c) \in U^{(2)}_{d(a_0)}(P) \text{ iff } & (a_0, a_0, b, c) \in U^{(3)}(P) \\ \text{iff } & (a_0, b, c) \in U^{(3)}_{a_0}(P) = Y \\ \text{iff } & (b, c) \in U^{(2)}_{f(d(a_0))}(P)\end{aligned}$$

and so $d(a_0)$ is as desired. □

Theorem 3.2 (Coding lemma) (Moschovakis). *Assume* ZF+AD. *Suppose $X \subseteq \omega^\omega$ and $\pi : X \to$* On. *Suppose $Z \subseteq X \times \omega^\omega$. Then there is an $e \in \omega^\omega$ such that*

(1) *$U^{(2)}_e(Q) \subseteq Z$ and*

(2) *for all $a \in X$, $U^{(2)}_e(Q) \cap (Q_a \times \omega^\omega) \neq \varnothing$ iff $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$,*

where $Q = \{\langle a, b\rangle \mid \pi(a) \leqslant \pi(b)\}$.

Proof. Assume toward a contradiction that there is no such e. Consider the set G of reals e for which (1) in the statement of the theorem is satisfied:

$$G = \big\{e \in \omega^\omega \mid U^{(2)}_e(Q) \subseteq Z\big\}.$$

So, for each $e \in G$, (2) in the statement of the theorem fails for some $a \in X$. Let α_e be the least α such that (2) fails at the α^{th}-section:

$$\alpha_e = \min\big\{\alpha \mid \exists a \in X \; (\pi(a) = \alpha \wedge U^{(2)}_e(Q) \cap (Q_a \times \omega^\omega) = \varnothing \wedge \\ Z \cap (Q_a \times \omega^\omega) \neq \varnothing)\big\}.$$

Now play the game

$$\begin{array}{llllllll} \text{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \text{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where I wins if $x \in G$ and either $y \notin G$ or $\alpha_x \geqslant \alpha_y$. Notice that by our assumption that there is no index e as in the statement of the theorem, neither I nor II can win a round of this game by playing a selector. The best they can do is play "partial" selectors. For a play $e \in G$, let us call $U^{(2)}_e(Q) \cap (Q_{<\alpha_e} \times \omega^\omega)$ the *partial* selector played. Using this terminology we can restate the winning conditions by saying that II wins either by ensuring that I does not play a subset of Z or, failing this, by playing a partial selector which is *longer* than that played by I.

We will arrive at a contradiction by showing that neither player can win this game.

Claim 1. Player I does not have a winning strategy.

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As in the proof of the Basic Coding Lemma our strategy will be to "bound" all of I's plays and then use this bound to construct a play e^* which defeats σ.

Since σ is a winning strategy,

$$U^{(2)}_{(\sigma * y)_I}(Q) \subseteq Z$$

for all $y \in \omega^\omega$. Let e_σ be such that

$$U^{(2)}_{e_\sigma}(Q) = \textstyle\bigcup_{y\in\omega^\omega} U^{(2)}_{(\sigma * y)_I}(Q).$$

By assumption, $U^{(2)}_{e_\sigma}(Q)$ is not a selector. So α_{e_σ} exists. Since for all $y \in \omega^\omega$, $\alpha_{e_\sigma} \geqslant \alpha_{(\sigma * y)_I}$, we can take α_{e_σ} as our bound. Choose $a \in X$ such that $\pi(a) = \alpha_{e_\sigma}$. Pick $(x_1, x_2) \in Z \cap (Q_a \times \omega^\omega)$. Let e^* be such that

$$U^{(2)}_{e^*}(Q) = U^{(2)}_{e_\sigma} \cup \{(x_1, x_2)\}.$$

So $e^* \in G$. But $\alpha_{e_\sigma} < \alpha_{e^*}$. In summary, we have that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma} < \alpha_{e^*}$. Thus, by playing e^*, II defeats σ. □

Claim 2. Player II does not have a winning strategy.

Proof. Assume toward a contradiction that τ is a winning strategy for II. We shall show that τ yields a selector for Z; in other words, it yields an e^* such that

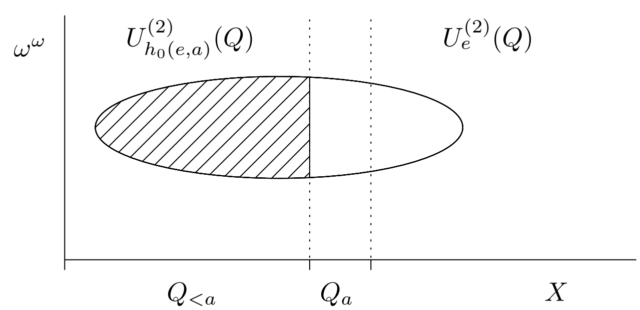
(1) $U^{(2)}_{e^*}(Q) \subseteq Z$ and

(2) for all $a \in X$, $U^{(2)}_{e^*}(Q) \cap (Q_a \times \omega^\omega) \neq \varnothing$ iff $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$.

Choose $h_0 : \omega^\omega \times X \to \omega^\omega$ such that h_0 is $\Sigma^1_1(Q)$ and for all $e, a \in \omega^\omega$,

$$U^{(2)}_{h_0(e,a)}(Q) = U^{(2)}_e(Q) \cap (Q_{<a} \times \omega^\omega).$$

Thus, the set coded by $h_0(e, a)$ is the result of taking the initial segment given by a of the set coded by e.



Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is $\underset{\sim}{\Sigma}^1_1(Q)$ and for all $e \in \omega^\omega$,

$$U^{(2)}_{h_1(e)}(Q) = \bigcup_{a \in X} \left(U^{(2)}_{(h_0(e,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega) \right).$$

Thus, the set coded by $h_1(e)$ is the union of all "a-sections" of sets played by II in response to "$<a$-initial segments" of the set coded by e.

By the recursion theorem there is a fixed point for h_1; that is, there is an e^* such that

$$U^{(2)}_{e^*}(Q) = U^{(2)}_{h_1(e^*)}(Q).$$

This set has the following closure property: if I plays an initial segment of it then II responds with a subset of it. We shall see that $e^* \in G$. Moreover, if $U^{(2)}_{e^*}(Q)$ is *not* a selector then having I play the largest initial segment which is a partial selector, II responds with a *larger* selector, which is a contradiction. Thus, e^* codes a selector. Here are the details.

Subclaim 1. $e^* \in G$.

Proof. Suppose for contradiction that $U^{(2)}_{e^*}(Q) \smallsetminus Z \neq \varnothing$. Choose $(x_1, x_2) \in U^{(2)}_{e^*}(Q) \smallsetminus Z$ with $\pi(x_1)$ minimal. So

$$\begin{aligned}(x_1, x_2) \in U^{(2)}_{e^*}(Q) &= U^{(2)}_{h_1(e^*)}(Q) \\ &= \bigcup_{a \in X} \big(U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega) \big).\end{aligned}$$

Fix $a \in X$ such that

$$(x_1, x_2) \in U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \cap (Q_a \times \omega^\omega).$$

The key point is that $h_0(e^*, a) \in G$ since we chose (x_1, x_2) with $\pi(x_1) = \pi(a)$ minimal. Thus, since τ is a winning strategy, $(h_0(e^*, a)*\tau)_{II} \in G$, and so $(x_1, x_2) \in Z$, which is a contradiction. □

Subclaim 2. α_{e^*} does not exist.

Proof. Suppose for contradiction that α_{e^*} exists. Let $a \in X$ be such that $\pi(a) = \alpha_{e^*}$. Thus $h_0(e^*, a) \in G$ and $\alpha_{h_0(e^*,a)} = \alpha_{e^*}$. Since τ is a winning strategy for II,

$$\alpha_{(h_0(e^*,a)*\tau)_{II}} > \alpha_{h_0(e^*,a)} = \alpha_{e^*},$$

which is impossible since

$$U^{(2)}_{(h_0(e^*,a)*\tau)_{II}}(Q) \subseteq U^{(2)}_{e^*}(Q).$$

Thus α_{e^*} does not exist. □

Hence e^* is the code for a selector. □

This completes the proof of the Coding Lemma. □

3.2. Uniform Coding Lemma

We shall need a uniform version of the above theorem. The version we prove is different than that which appears in the literature ([5]). We shall need the following uniform version of the recursion theorem.

Theorem 3.3 (Uniform Recursion Theorem) (Kleene). *Suppose* $f : \omega^\omega \to \omega^\omega$ *is* $\underset{\sim}{\Sigma}^1_1$. *Then there is an* $e \in \omega^\omega$ *such that for all* $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_e(P, P') = U^{(2)}_{f(e)}(P, P').$$

Proof. The proof is the same as before. The key point is that the definition of the fixed point $d(a_0)$ depends only on f and, of course, d, which is uniform in P, P'. □

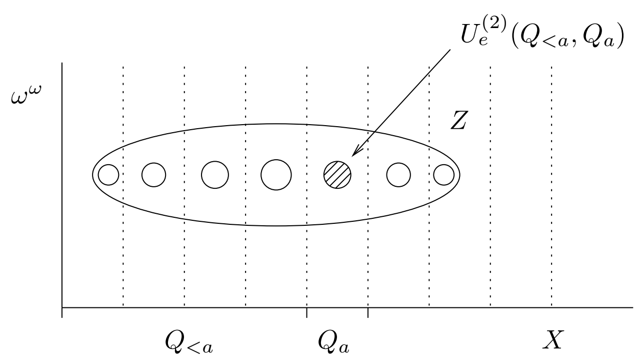
Theorem 3.4 (Uniform Coding Lemma)). *Assume* ZF + AD. *Suppose* $X \subseteq \omega^\omega$ *and* $\pi : X \to \mathrm{On}$. *Suppose* $Z \subseteq X \times \omega^\omega$. *Then there exists an* $e \in \omega^\omega$ *such that for all* $a \in X$,

(1) $U^{(2)}_e(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega)$ *and*

(2) $U^{(2)}_e(Q_{<a}, Q_a) \neq \varnothing$ *iff* $Z \cap (Q_a \times \omega^\omega) \neq \varnothing$,

where $Q_{<a} = \{b \in X \mid \pi(b) < \pi(a)\}$ *and* $Q_a = \{b \in X \mid \pi(b) = \pi(a)\}$.

Proof. Here is the picture:



Think of e as providing a "rolling selector". The unshaded ellipse, Z, is sliced into sections $Z \cap (Q_a \times \omega^\omega)$. The Uniform Coding Lemma tells us that there is a simple selector $U^{(2)}_e(Q_{<a}, Q_a)$ for each of these sections which is uniform

in the parameters $Q_{<a}$, Q_a; that is, there is a fixed e such that $U^{(2)}_e(Q_{<a}, Q_a)$ selects from $Z \cap (Q_a \times \omega^\omega)$, for all parameters $Q_{<a}$, Q_a.

Assume toward a contradiction that there is no such e. Consider the set G of reals e for which (1) in the statement of the theorem is satisfied:

$$G = \big\{e \in \omega^\omega \mid \forall a \in X \; \big(U^{(2)}_e(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega)\big)\big\}.$$

So, for each $e \in G$, (2) in the statement of the theorem fails for some $a \in X$. Let α_e be least such that (2) fails at the ${\alpha_e}^{\text{th}}$-section:

$$\alpha_e = \min\big\{\alpha \mid \exists a \in X \; (\pi(a) = \alpha \wedge U^{(2)}_e(Q_{<a}, Q_a) = \varnothing \wedge \\ Z \cap (Q_a \times \omega^\omega) \neq \varnothing)\big\}.$$

Now play the game

I	$x(0)$		$x(1)$		$x(2)$		$\dots$
II		$y(0)$		$y(1)$		$\dots$	

where I wins if $x \in G$ and either $y \notin G$ or $\alpha_x \geqslant \alpha_y$.

Claim 1. Player I does not have a winning strategy.

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As before our strategy is to "bound" all of I's plays and then use this bound to construct a play e^* for II which defeats σ.

The proof is as before except that we have to take care to choose a parameter e_σ that works uniformly for all parameters $Q_{<a}$, Q_a: Choose e_σ such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e_\sigma}(P, P') = \textstyle\bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(P, P').$$

In particular, e_σ is such that for all $a \in X$,

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) = \textstyle\bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(Q_{<a}, Q_a).$$

Since σ is a winning strategy for I, $(\sigma * y)_I \in G$ for all $y \in \omega^\omega$. Thus,

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega),$$

that is, $e_\sigma \in G$. Notice that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma}$. We have thus "bounded" all of I's plays. It remains to construct a defeating play e^* for II.

Choose $a \in X$ such that $\pi(a) = \alpha_{e_\sigma}$. So

$$U^{(2)}_{e_\sigma}(Q_{<a}, Q_a) = \varnothing$$

and

$$Z \cap (Q_a \times \omega^\omega) \neq \varnothing.$$

Pick $(x_1, x_2) \in Z \cap (Q_a \times \omega^\omega)$. Choose e^* such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e^*}(P, P') = \begin{cases} U^{(2)}_{e_\sigma}(P, P') & \text{if } x_1 \notin P' \\ U^{(2)}_{e_\sigma}(P, P') \cup \{(x_1, x_2)\} & \text{if } x_1 \in P'. \end{cases}$$

In particular, e^* is such that for all $a' \in X$,

$$U^{(2)}_{e^*}(Q_{<a'}, Q_{a'}) = \begin{cases} U^{(2)}_{e_\sigma}(Q_{<a'}, Q_{a'}) & \text{if } x_1 \notin Q_{a'} \\ U^{(2)}_{e_\sigma}(Q_{<a'}, Q_{a'}) \cup \{(x_1, x_2)\} & \text{if } x_1 \in Q_{a'}. \end{cases}$$

So $e^* \in G$. But $\alpha_{e_\sigma} < \alpha_{e^*}$. In summary, we have that for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant \alpha_{e_\sigma} < \alpha_{e^*}$. Thus, by playing e^*, II defeats σ. □

Claim 2. Player II does not have a winning strategy.

Proof. Assume toward a contradiction that τ is a winning strategy for II. We seek e^* such that

$$U^{(2)}_{e^*}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega) \text{ and}$$
$$U^{(2)}_{e^*}(Q_{<a}, Q_a) \neq \varnothing \text{ iff } Z \cap (Q_a \times \omega^\omega) \neq \varnothing.$$

Choose $h_0 : \omega^\omega \times \omega^\omega \to \omega^\omega$ such that h_0 is Σ^1_1 and for all $e, z \in \omega^\omega$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_0(e,z)}(P, P') = \begin{cases} U^{(2)}_e(P, P') & \text{if } z \notin P \cup P' \\ \varnothing & \text{if } z \in P \cup P'. \end{cases}$$

In particular, for all $a \in X$,

$$U^{(2)}_{h_0(e,z)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_e(Q_{<a}, Q_a) & \text{if } z \notin Q_{<a} \cup Q_a \\ \varnothing & \text{if } z \in Q_{<a} \cup Q_a. \end{cases}$$

Notice that for $e \in \omega^\omega$ and $z \in X$, the set $U^{(2)}_{h_0(e,z)}(\cdot,\cdot)$ is such that it agrees with $U^{(2)}_e(\cdot,\cdot)$ for parameters $Q_{<a}, Q_a$ where $\pi(a) < \pi(z)$ and is empty for parameters $Q_{<a}, Q_a$ where $\pi(a) \geqslant \pi(z)$.

Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is $\Sigma^1_1(\tau)$ and for all $e \in \omega^\omega$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_1(e)}(P,P') = \bigcup_{z \in P'} U^{(2)}_{(h_0(e,z)*\tau)_{II}}(P,P').$$

In particular, for all $a \in X$,

$$U^{(2)}_{h_1(e)}(Q_{<a},Q_a) = \bigcup_{z \in Q_a} U^{(2)}_{(h_0(e,z)*\tau)_{II}}(Q_{<a},Q_a).$$

The idea is roughly this: Fix $e \in \omega^\omega$ and $z \in Q_a$. $U^{(2)}_{h_0(e,z)}(\cdot,\cdot)$ is such that it agrees with $U^{(2)}_e(\cdot,\cdot)$ for parameters $Q_{<\bar{a}}, Q_{\bar{a}}$ where $\pi(\bar{a}) < \pi(a)$ and is empty for parameters $Q_{<\bar{a}}, Q_{\bar{a}}$ where $\pi(\bar{a}) \geqslant \pi(a)$. Think of this as a play for I. In the case of interest, this play will be in G. And since τ is a winning strategy, II's response will be in G and when provided with parameters $Q_{<a}$,Q_a it will select from the a-component. $U_{h_1(e)}(Q_{<a}, Q_a)$ is the union of these over $z \in Q_a$.

Let e^* be a fixed point for h_1, by Theorem 3.3.

Subclaim 1. $e^* \in G$.

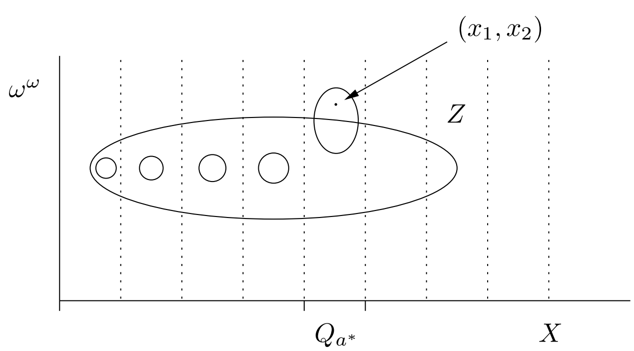
Proof. Suppose for contradiction that for some $a \in X$,

$$U^{(2)}_{e^*}(Q_{<a},Q_a) \setminus (Z \cap (Q_a \times \omega^\omega)) \neq \varnothing.$$

Let a^* be an a where $\pi(a)$ is least such that

$$U^{(2)}_{e^*}(Q_{<a},Q_a) \setminus (Z \cap (Q_a \times \omega^\omega)) \neq \varnothing.$$

Choose $(x_1, x_2) \in U^{(2)}_{e^*}(Q_{<a^*}, Q_{a^*}) \setminus (Z \cap (Q_{a^*} \times \omega^\omega))$.



So

$$(x_1, x_2) \in U^{(2)}_{e^*}(Q_{<a^*}, Q_{a^*}) = U^{(2)}_{h_1(e^*)}(Q_{<a^*}, Q_{a^*}) \\ = \bigcup_{z \in Q_{a^*}} U^{(2)}_{(h_0(e^*,z)*\tau)_{II}}(Q_{<a^*}, Q_{a^*}).$$

Fix $z^* \in Q_{a^*}$ such that

$$(x_1, x_2) \in U^{(2)}_{(h_0(e^*,z^*)*\tau)_{II}}(Q_{<a^*}, Q_{a^*}).$$

The key point is that $h_0(e^*, z^*) \in G$: By the definition of h_0, for all $a \in X$ and for all $z \in \omega^\omega$,

$$U^{(2)}_{h_0(e^*,z)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_{e^*}(Q_{<a}, Q_a) & \text{if } z \notin Q_{<a} \cup Q_a \\ \varnothing & \text{if } z \in Q_{<a} \cup Q_a. \end{cases}$$

We have fixed $z^* \in Q_{a^*}$. For this fixed value, allowing a to vary, we have (i) $z^* \notin Q_{<a} \cup Q_a$ iff $\pi(a) < \pi(a^*)$ and (ii) $z^* \in Q_{<a} \cup Q_a$ iff $\pi(a) \geqslant \pi(a^*)$. So

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = U^{(2)}_{e^*}(Q_{<a}, Q_a),$$

for all a such that $\pi(a) < \pi(a^*)$ and

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = \varnothing,$$

for all a such that $\pi(a) \geqslant \pi(a^*)$. Thus,

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) \subseteq Z \cap (Q_a \times \omega^\omega),$$

for *all* $a \in X$, i.e. $h_0(e^*, z^*) \in G$.

Now since τ is a winning strategy for II, $(h_0(e^*, z^*)*\tau)_{II} \in G$, which means that $(x_1, x_2) \in Z$, a contradiction. □

Subclaim 2. α_{e^*} does not exist.

Proof. Suppose not. Let $a^* \in X$ be such that $\pi(a^*) = \alpha_{e^*}$, and choose $z^* \in Q_{a^*}$. Thus, $h_0(e^*, z^*) \in G$, since $e^* \in G$ by Subclaim 1, and $h_0(e^*, z^*)$ is defined such that for all $a \in X$,

$$U^{(2)}_{h_0(e^*,z^*)}(Q_{<a}, Q_a) = \begin{cases} U^{(2)}_{e^*}(Q_{<a}, Q_a) & \text{if } \pi(a) < \pi(a^*) \\ \varnothing & \text{if } \pi(a) \geqslant \pi(a^*). \end{cases}$$

So, $\alpha_{h_0(e^*,z^*)} = \alpha_{e^*}$. Since τ is a winning strategy for II,

$$\alpha_{(h_0(e^*,z^*)*\tau)_{II}} > \alpha_{h_0(e^*,z^*)} = \alpha_{e^*},$$

which is impossible since

$$U^{(2)}_{(h_0(e^*,z^*)*\tau)_{II}}(Q_{<a}, Q_a) \subseteq U^{(2)}_{e^*}(Q_{<a}, Q_a)$$

for all $a \in X$. □

Thus, e^* is the code for a uniform selector. □

This completes the proof of the Uniform Coding Lemma. □

3.5 Remark. The game in the above proof is definable from X, π, and Z and no choice is required to show that it works. Thus, if these parameters are OD, then ZF + OD-determinacy suffices for the proof.

3.6 Remark. The version of the Coding Lemma stated in Lemma 2.32 follows from the Uniform Coding Lemma: Take $X = U$ and $\pi : U \to \mathrm{On}$ given by $\pi(x) = \delta_x$. Then

$$Z^* = \bigcup_{x \in U} U^{(2)}_e(Q_{<\delta_x}, Q_{\delta_x}).$$

This gives (i). For (ii) note that

$$Z^* \cap (U_{\leqslant \delta_x} \times \omega^\omega) = \bigcup_{y \in U_{\leqslant \delta_x}} U^{(2)}_e(Q_{<\delta_y}, Q_{\delta_y}),$$

which is $\underset{\sim}{\Delta}^2_1$.

2 Open Question (Strong Coding Lemma). Suppose $X \subseteq \omega^\omega$ and $\pi : X \to \mathrm{On}$. Let $\leqslant_X$ be the prewellordering associated with π. Suppose $Z \subseteq X^{<\omega}$ is a tree. Then there exists a subtree $Z^* \subseteq Z$ such that

(1) Z^* is $\Sigma^1_1(\leqslant_X)$ and

(2) for all $\vec{s} \in (Z^*)^{<\omega}$ and for all $a \in X$, if there exists a $t \in Q_a$ such that $\vec{s}^\frown t \in Z$ then there exists a $t \in Q_a$ such that $\vec{s}^\frown t \in Z^*$,

where $Q_a = \{b \in X \mid \pi(b) = \pi(a)\}$.

3.3. Applications

In this section we will bring together some basic results and key applications of the above coding lemmas that will be of use later. It will be useful to do things in a slightly more general fashion than is customary.

For a set X, let

$$\Theta_X = \sup\{\alpha \mid \text{there is an } \mathrm{OD}_X \text{ surjection } \pi : \omega^\omega \to \alpha\}.$$

Lemma 3.7. *Assume* ZF *and suppose* X *is a set. Then there is an* OD_X *sequence* $A = \langle A_\alpha \mid \alpha < \Theta_X \rangle$ *such that* A_α *is a prewellordering of the reals of length* α*.*

Proof. Let A_α be the $<_{\mathrm{OD}_X}$-least prewellordering of the reals of length α, where $<_{\mathrm{OD}_X}$ is the canonical OD_X well-ordering of the OD_X sets. □

Lemma 3.8. *Assume* ZF *and suppose* X *is a set. Suppose that every set is* $\mathrm{OD}_{X,y}$ *for some real* y*. Then* $\Theta = \Theta_X$*.*

Proof. Fix $\alpha < \Theta$. We have to show that there is an OD_X surjection $\pi : \omega^\omega \to \alpha$. There is certainly an $\mathrm{OD}_{X,y}$ surjection for some y. For each $y \in \omega^\omega$, let π_y be the $<_{\mathrm{OD}_{X,y}}$-least such surjection if one exists and let it be undefined otherwise. We can now "average over the reals" to eliminate the dependence on real parameters, letting

$$\begin{aligned} \pi : \omega^\omega &\to \alpha \\ y &\mapsto \begin{cases} \pi_{(y)_0}((y)_1) & \text{if } \pi_{(y)_0} \text{ is defined} \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

This is an OD_X surjection. □

The following theorem is essentially due to Moschovakis. We are just replacing AD with OD_X-determinacy and the changes are straightforward.

Theorem 3.9. *Assume* $\mathrm{ZF} + \mathrm{OD}_X$*-determinacy, where* X *is a set. Then*

$$\mathrm{HOD}_X \models \Theta_X \textit{ is strongly inaccessible.}$$

Proof. First we show that Θ_X is regular in HOD_X. By Lemma 3.7 there is an OD_X sequence

$$\langle \pi_\alpha \mid \alpha < \Theta_X \rangle$$

where each $\pi_\alpha : \omega^\omega \to \alpha$ is a surjection. Assume for contradiction that Θ_X is singular in HOD_X and let

$$f : \eta \to \Theta_X$$

be an OD_X cofinal map. Let g be an OD_X surjection from ω^ω onto η. Then the map

$$\begin{aligned} \pi : \omega^\omega &\to \Theta_X \\ x &\mapsto \pi_{f\circ g((x)_0)}((x)_1) \end{aligned}$$

is an OD_X surjection, which contradicts the definition of Θ_X.

We now show that Θ_X is a strong limit in HOD_X. For each $\eta < \Theta_X$, we have to show that $|\mathscr{P}(\eta)|^{\mathrm{HOD}_X} < \Theta_X$. For this it suffices to show that there is an OD_X surjection

$$\pi : \omega^\omega \to \mathscr{P}(\eta)^{\mathrm{HOD}_X},$$

since if $|\mathscr{P}(\eta)|^{\mathrm{HOD}_X} \geqslant \Theta_X$ then there would be an OD_X surjection $\rho : \mathscr{P}(\eta) \to \Theta_X$ and so $\rho \circ \pi : \omega^\omega \to \Theta_X$ would be an OD_X surjection, which contradicts the definition of Θ_X.

Let $\pi_\eta : \omega^\omega \to \eta$ be an OD_X surjection and, for $\alpha < \eta$, let $Q_{<\alpha}$ and Q_α be the usual objects defined relative to π_η. For $e \in \omega^\omega$, let

$$S_e = \{\beta < \eta \mid U_e^{(2)}(Q_{<\beta}, Q_\beta) \neq \varnothing\}.$$

The key point is that since π_η is OD_X the game for the Uniform Coding Lemma for $Z = \bigcup\{Q_\alpha \times \omega^\omega \mid \alpha \in S\}$ is determined for each $S \in \mathscr{P}(\eta)^{\mathrm{HOD}_X}$. (See Remark 3.5.) Thus, every $S \in \mathscr{P}(\eta)^{\mathrm{HOD}_X}$ has the form S_e for some $e \in \omega^\omega$ and hence

$$\begin{aligned} \pi : \omega^\omega &\to \mathscr{P}(\eta)^{\mathrm{HOD}_X} \\ e &\mapsto S_e \end{aligned}$$

is a surjection. Moreover, π is OD_X (since π_η is OD_X), which completes the proof. □

The above theorem has the following corollary. The first part also follows from early work of Friedman and Solovay. The second part is a simple application of the Coding Lemma and Solovay's Lemma 2.23.

Theorem 3.10. *Assume* $\mathrm{ZF} + \mathrm{AD} + V = L(\mathbb{R})$. *Then*

(1) $\mathrm{HOD}^{L(\mathbb{R})} \models \Theta$ *is strongly inaccessible and*

(2) $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$.

Proof. (1) This follows immediately from Theorem 3.9 and Lemma 3.8.

(2) Since $\mathrm{HOD}^{L(\mathbb{R})}$ is Σ_1-definable over $L(\mathbb{R})$ (with the parameter $\mathbb{R}$) and since $L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R})$ (by Lemma 2.23),

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

Thus, it suffices to show

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

The right-to-left inclusion is immediate. For the left-to-right inclusion suppose $x \in \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$. We have to show that $x \in L_\Theta(\mathbb{R})$. Since Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$, x is coded by a set of ordinals $A \subseteq \alpha$ where $\alpha < \Theta$. However, by the proof of Theorem 3.9, $\mathscr{P}(\alpha) \in L_\Theta(\mathbb{R})$, for each $\alpha < \Theta$. Thus, $x \in L_\Theta(\mathbb{R})$, which completes the proof. ☐

3 Open Question. Assume $\mathrm{ZF} + \mathrm{DC} + V = L(\mathbb{R})$.

(1) Suppose that for every $\alpha < \Theta$ there is a surjection $\pi : \omega^\omega \to \mathscr{P}(\alpha)$. Must AD hold in $L(\mathbb{R})$?

(2) Suppose Θ is inaccessible. Must AD hold in $L(\mathbb{R})$?

Theorem 3.11 (Kunen). *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Suppose* $\lambda < \Theta$ *and* μ *is an ultrafilter on* λ. *Then* μ *is* OD.

Proof. Let $\leqslant$ be a prewellordering of ω^ω of length λ. Let $\pi : \omega^\omega \to \mathscr{P}(\lambda)$ be the surjection derived from $\leqslant$ as in the above proof. For $x \in \omega^\omega$, let

$$A_x = \bigcap\{\pi(y) \mid \pi(y) \in \mu \wedge y \leqslant_T x\}.$$

Since there are only countably many such y and AD implies that all ultrafilters are countably complete (Theorem 2.8), A_x is non-empty. Let

$$f(x) = \bigcap A_x.$$

Notice that A_x and $f(x)$ depend only on the Turing degree of x. In particular, we can regard f as a function from the Turing degrees $\mathscr{D}_T$ into the ordinals. Notice also that

$$A \in \mu \text{ iff for a cone of } x, f(x) \in A$$

since if $B \in \mu$ then, for any $x \geqslant_T x_0$ we have $f(x) \in B$, where x_0 is such that $\pi(x_0) = B$. We can now "erase" reference to the prewellordering by taking the ultrapower. Let μ_T be the cone ultrafilter on the Turing degrees (see Theorem 2.9) and consider the ultrapower $V^{\mathscr{D}_T}/\mu_T$. By DC the ultrapower is well-founded. So we can let M be the transitive collapse of $V^{\mathscr{D}_T}/\mu_T$ and let

$$j : V \to M$$

be the canonical map. Letting γ be the ordinal represented by f, we have

$$B \in \mu \text{ iff } \gamma \in j(B)$$

and so μ is OD. □

4. A Woodin Cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$

Our main aim in this section is to prove the following theorem:

Theorem 4.1. *Assume* ZF + DC + AD. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

This will serve as a warm-up for the proof of the Generation Theorem in the next section. The proof that we give appeals to DC at only one point (Lemma 4.8) and as we shall see in the next section one can avoid this appeal and prove the result in ZF + AD. See Theorem 5.36.

In §4.1 we will establish the reflection phenomenon that will play the role played by boundedness in §2 and we will define for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$ that is intended to witness that $\underset{\sim}{\delta}^2_1$ is λ-strong. In §4.2 we shall introduce and motivate the notion of *strong normality* by showing that the strong normality of μ_λ ensures that $\underset{\sim}{\delta}^2_1$ is λ-strong. We will then show how reflection and uniform coding combine to secure strong normality. In §4.3 we will prove the main theorem by relativizing the construction to subsets of Θ. Throughout this section we work in $L(\mathbb{R})$ and so when we write $\underset{\sim}{\delta}^2_1$ and Θ we will always be referring to these notions as interpreted in $L(\mathbb{R})$.

4.1. Reflection

We have seen that ZF+AD implies that Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$. Our next task is to show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \underset{\sim}{\delta}^2_1 \text{ is } \lambda\text{-strong},$$

for all $\lambda < \Theta$. The proof will then relativize to subsets of Θ that are in $\mathrm{HOD}^{L(\mathbb{R})}$ and thereby establish the main theorem.

The ultrafilters that witness strength cannot come from the "sup" game of Section 2 since the ultrafilters produced by this game concentrate on ω-club sets, whereas to witness strength we will need ultrafilters according to which there are measure-one many measurable cardinals below $\underset{\sim}{\delta}^2_1$. For this reason we will have to use a variant of the "sup" game. In this variant the role of boundedness will be played by a certain reflection phenomenon.

The reflection phenomenon we have in mind does not presuppose any determinacy assumptions. For the time being work in $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. The main claim is that there is a function $F : \underset{\sim}{\delta}^2_1 \to L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ which is Δ_1-definable over $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ and for which the following *reflection phenomenon* holds:

> For all $X \in L(\mathbb{R}) \cap \mathrm{OD}^{L(\mathbb{R})}$, for all Σ_1-formulas φ, and for all $z \in \omega^\omega$, if
>
> $$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$
>
> then there exists a $\delta < \underset{\sim}{\delta}^2_1$ such that
>
> $$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

One should think of F as a sequence that contains "proxies" or "generic witnesses" for each $\mathrm{OD}^{L(\mathbb{R})}$ set X: Given *any* Σ_1-fact (with a real parameter) about *any* $\mathrm{OD}^{L(\mathbb{R})}$ set X there is a "proxy" $F(\delta)$ in our fixed sequence that witnesses the same fact.

The function F is defined (much like $\Diamond$) in terms of the least counterexample. To describe this in more detail let us first recall some basic facts from Section 2.4 concerning $L(\mathbb{R})$ and the theory T_0: There are arbitrarily large α such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. In particular,

$$L_\Theta(\mathbb{R}) \models \mathrm{T}_0.$$

Moreover, since

$$L_\Theta(\mathbb{R}) \prec_1 L(\mathbb{R}),$$

there are arbitrarily large $\alpha < \Theta$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. Similarly, there are arbitrarily large $\alpha < \underset{\sim}{\delta}^2_1$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$. However, notice that it is *not* the case $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \models \mathrm{T}_0$ (by Lemma 2.27).

Because of the greater manoeuvring room provided by levels $L_\alpha(\mathbb{R})$ that satisfy T_0 we will concentrate (for example, in reflection arguments) on such levels. For example, we can use these levels to give a first-order definition of $\mathrm{OD}^{L(\mathbb{R})}$ and the natural well-ordering $<_{\mathrm{OD}^{L(\mathbb{R})}}$ on the $\mathrm{OD}^{L(\mathbb{R})}$ sets. For the latter, given $X \in \mathrm{OD}^{L(\mathbb{R})}$, let

$$\begin{aligned} \alpha_X = \ & \text{the least } \alpha \text{ such that} \\ & \quad (1)\ \ L_\alpha(\mathbb{R}) \models \mathrm{T}_0, \\ & \quad (2)\ \ X \in L_\alpha(\mathbb{R}), \text{and} \\ & \quad (3)\ \ X \text{ is definable in } L_\alpha(\mathbb{R}) \text{ from ordinal parameters;} \end{aligned}$$

let φ_X be the least formula that defines X from ordinal parameters in $L_\alpha(\mathbb{R})$; and let $\vec{\xi}_X$ be the lexicographically least sequence of ordinal parameters used to define X in $L_\alpha(\mathbb{R})$ via φ_X. Given X and Y in $\mathrm{OD}^{L(\mathbb{R})}$, working in $L(\mathbb{R})$ set

$$\begin{aligned} X <_{\mathrm{OD}} Y \ \text{ iff } \ & \alpha_X < \alpha_Y \text{ or} \\ & \alpha_X = \alpha_Y \text{ and } \varphi_X < \varphi_Y \text{ or} \\ & \alpha_X = \alpha_Y \text{ and } \varphi_X = \varphi_Y \text{ and } \vec{\xi}_X <_{\mathrm{lex}} \vec{\xi}_Y. \end{aligned}$$

Since the $L_\alpha(\mathbb{R})$ hierarchy is Σ_1-definable in $L(\mathbb{R})$, it follows that $\mathrm{OD}^{L(\mathbb{R})}$ and $(<_{\mathrm{OD}})^{L(\mathbb{R})}$ are Σ_1-definable in $L(\mathbb{R})$. (This is in contrast to the usual definitions of these notions, which are Σ_2 since they involve existential quantification over the V_α hierarchy, which is Π_1.) Notice that if $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$, then

$$(<_{\mathrm{OD}})^{L_\alpha(\mathbb{R})} \trianglelefteq (<_{\mathrm{OD}})^{L(\mathbb{R})}.$$

Furthermore, if $L_\alpha(\mathbb{R}) \prec_1 L(\mathbb{R})$, then

$$\mathrm{OD}^{L_\alpha(\mathbb{R})} = \mathrm{OD}^{L(\mathbb{R})} \cap L_\alpha(\mathbb{R}) \ \text{ and } \ \ (<_{\mathrm{OD}})^{L_\alpha(\mathbb{R})} = (<_{\mathrm{OD}})^{L(\mathbb{R})} \restriction L_\alpha(\mathbb{R}).$$

For example,

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} = \mathrm{HOD}^{L(\mathbb{R})} \cap L_\Theta(\mathbb{R}).$$

(For this it is crucial that we use the Σ_1-definition given above since the Σ_2-definition involves quantification over the V_α hierarchy and yet in $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$

even the level $V_{\omega+2}$ does not exist.) Our goal can thus be rephrased as that of showing

$$\mathrm{HOD}^{L_\Theta(\mathbb{R})} \models \underset{\sim}{\delta}^2_1 \text{ is a strong cardinal.}$$

We are now in a position to define the reflection function F. If the reflection phenomenon fails in $L(\mathbb{R})$ with respect to $F\restriction\underset{\sim}{\delta}^2_1$ then (by Replacement) there is some level $L_\alpha(\mathbb{R})$ which satisfies T_0 over which the reflection phenomenon fails with respect to $F\restriction\underset{\sim}{\delta}^2_1$. This motivates the following definition:

4.2 Definition. Assume T_0. Suppose that $F\restriction\delta$ is defined. Let $\vartheta(\delta)$ be least such that

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \mathrm{T}_0 \text{ and there is an } X \in L_{\vartheta(\delta)}(\mathbb{R}) \cap \mathrm{OD}^{L_{\vartheta(\delta)}(\mathbb{R})} \text{ such that}$$

($\star$) there is a Σ_1-formula φ and a real z such that

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \varphi[z, X, \delta, \mathbb{R}]$$

and for all $\bar{\delta} < \delta$,

$$L_{\vartheta(\delta)}(\mathbb{R}) \not\models \varphi[z, F(\bar{\delta}), \bar{\delta}, \mathbb{R}]$$

(if such an ordinal exists) and then set $F(\delta) = X$ where X is $(<_{\mathrm{OD}})^{L_{\vartheta(\delta)}}$-least such that ($\star$) holds.

We have to establish two things: First, $F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$. Second, $F(\underset{\sim}{\delta}^2_1)$ is not defined. This implies that the reflection phenomenon holds with respect to F.

Lemma 4.3. *Assume* $\mathrm{ZF} + \mathrm{AC}_\omega(\mathbb{R})$. *Then*

(1) *if* $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$, *then* $(F)^{L_\alpha(\mathbb{R})} = F\restriction\gamma$ *for some* γ,

(2) $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})} = F\restriction\underset{\sim}{\delta}^2_1$, *and*

(3) $F(\delta)$ *is defined for all* $\delta < \underset{\sim}{\delta}^2_1$.

Proof. For (1) suppose that $(F\restriction\delta)^{L_\alpha(\mathbb{R})} = F\restriction\delta$ with the aim of showing that $(F(\delta))^{L_\alpha(\mathbb{R})} = (F(\delta))^{L(\mathbb{R})}$. The point is that

$$L_\alpha(\mathbb{R}) \models \vartheta(\delta) \text{ exists}$$

if and only if

$$(\vartheta(\delta))^{L(\mathbb{R})} < \alpha,$$

in which case

$$(\vartheta(\delta))^{L_\alpha(\mathbb{R})} = (\vartheta(\delta))^{L(\mathbb{R})} \text{ and } (F(\delta))^{L_\alpha(\mathbb{R})} = (F(\delta))^{L(\mathbb{R})},$$

by the locality of the definition of F and the assumption that $(F\restriction\delta)^{L_\alpha(\mathbb{R})} = F\restriction\delta$.

For (2) first notice that we can make sense of F as defined over levels (such as $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$) that do not satisfy T_0 by letting, for an arbitrary ordinal ξ,

$$F^{L_\xi(\mathbb{R})} = \bigcup\{F^{L_\alpha(\mathbb{R})} \mid \alpha < \xi \text{ and } L_\alpha(\mathbb{R}) \models \mathrm{T}_0\}.$$

Thus, $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})} = F\restriction\gamma$ for some γ, by (1). Assume for contradiction that (2) fails, that is, for some $\gamma < \underset{\sim}{\delta}^2_1$, $F(\gamma)$ is defined and yet $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}(\gamma)$ is not defined. Since in $L(\mathbb{R})$, $\vartheta(\gamma)$ and $F(\gamma)$ are defined, the following is a true Σ_1-statement about γ:

$$\exists\alpha > \gamma\,(L_\alpha(\mathbb{R}) \models \mathrm{T}_0 + \vartheta(\gamma) \text{ exists.})$$

Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \prec_1 L(\mathbb{R})$, this statement holds in $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ and so $F^{L_\alpha(\mathbb{R})}(\gamma)$ is defined and hence $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}(\gamma)$ is defined, which is a contradiction.

For (3) assume for contradiction that $\gamma < \underset{\sim}{\delta}^2_1$, where $\gamma = \mathrm{dom}(F)$. By (2) (and the definition of $F^{L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})}$) there is an $\alpha < \underset{\sim}{\delta}^2_1$ such that $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and $F^{L_\alpha(\mathbb{R})} = F\restriction\gamma = F$. We claim that this implies that

$$L_\alpha(\mathbb{R}) \prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L(\mathbb{R}),$$

which is a contradiction (by Theorem 2.28). Suppose

$$L(\mathbb{R}) \models \psi[z, \mathbb{R}]$$

where ψ is a Σ_1-formula and $z \in \omega^\omega$. We have to show that $L_\alpha(\mathbb{R}) \models \psi[z, \mathbb{R}]$. By Replacement there is an ordinal β such that

$$L_\beta(\mathbb{R}) \models \psi[z, \mathbb{R}].$$

Consider the Σ_1-statement $\varphi[z, X, \mathbb{R}]$ expressing "There exists ξ such that $X = L_\xi(\mathbb{R})$ and $X \models \psi[z, \mathbb{R}]$". Letting $\vartheta > \beta$ be such that $L_\vartheta(\mathbb{R}) \models \mathrm{T}_0$ we have: there exists an $X \in L_\vartheta(\mathbb{R}) \cap \mathrm{OD}^{L_\vartheta(\mathbb{R})}$ (namely, $X = L_\beta(\mathbb{R})$) such that

$$L_\vartheta(\mathbb{R}) \models \mathrm{T}_0 + \varphi[z, X, \mathbb{R}].$$

Moreover, since $\vartheta(\gamma)$ does not exist, it follows (by the definition of $\vartheta(\gamma)$) that there exists a $\bar{\delta} < \gamma$ such that

$$L_\vartheta(\mathbb{R}) \models \varphi[z, F(\bar{\delta}), \mathbb{R}].$$

Thus (unpacking $\varphi[z, X, \mathbb{R}]$) there exists a ξ such that $F(\bar{\delta}) = L_\xi(\mathbb{R})$ and $L_\xi(\mathbb{R}) \models \psi[z, \mathbb{R}]$. Since $F \subseteq L_\alpha(\mathbb{R})$, $\xi < \alpha$ and so, by upward absoluteness,

$$L_\alpha(\mathbb{R}) \models \psi[z, \mathbb{R}],$$

which completes the proof. □

It follows that $F \restriction \underset{\sim}{\delta}^2_1 : \underset{\sim}{\delta}^2_1 \to L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$ is total and Δ_1-definable over $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R})$. It remains to see that $F(\underset{\sim}{\delta}^2_1)$ is not defined.

Theorem 4.4. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$. *For all* $X \in \mathrm{OD}^{L(\mathbb{R})}$, *for all* Σ_1-*formulas* φ, *and for all* $z \in \omega^\omega$ *if*

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

then there exists a $\delta < \underset{\sim}{\delta}^2_1$ *such that*

$$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Proof. The idea of the proof is straightforward but the details are somewhat involved.

Assume for contradiction that there is an $X \in \mathrm{OD}^{L(\mathbb{R})}$, a Σ_1-formula φ, and $z \in \omega^\omega$ such that

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

and for all $\delta < \underset{\sim}{\delta}^2_1$,

$$L(\mathbb{R}) \not\models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Step 1. By Replacement, let $\vartheta_0 > \underset{\sim}{\delta}^2_1$ be least such that

(1.1) $L_{\vartheta_0}(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_{\vartheta_0}(\mathbb{R}) \cap \mathrm{OD}^{L_{\vartheta_0}(\mathbb{R})}$ and

($\star$) there is a Σ_1-formula φ and a real z such that

$$L_{\vartheta_0}(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

and for all $\delta < \underset{\sim}{\delta}^2_1$

$$L_{\vartheta_0}(\mathbb{R}) \not\models \varphi[z, F(\delta), \delta, \mathbb{R}].$$

Let X_0 be least (in the order of definability) such that (1.1) and for this choice pick φ_0 and z_0 such that ($\star$). (Thus we have let $\vartheta_0 = \vartheta(\underset{\sim}{\delta}^2_1)$, $X_0 = F(\underset{\sim}{\delta}^2_1)$, and we have picked witnesses φ_0 and z_0 to the failure of reflection with respect to $F(\underset{\sim}{\delta}^2_1)$.)

Notice that $L_{\vartheta_0}(\mathbb{R}) \models \underset{\sim}{\delta}^2_1$ exists $+ F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$. Since

$$F^{L_{\vartheta_0}(\mathbb{R})} \restriction \underset{\sim}{\delta}^2_1 = F \restriction \underset{\sim}{\delta}^2_1,$$

by Lemma 4.3, (1.1) is equivalent to the internal statement $L_{\vartheta_0}(\mathbb{R}) \models \mathrm{T}_0 +$ "reflection fails with respect to $F \restriction \underset{\sim}{\delta}^2_1$". It is this internal statement that we will reflect to get a contradiction. We have that for all $\delta < \underset{\sim}{\delta}^2_1$,

(1.2) $L_{\vartheta_0}(\mathbb{R}) \not\models \varphi_0[z_0, F(\delta), \delta, \mathbb{R}]$.

Our strategy is to reflect to get $\bar{\vartheta} < \underset{\sim}{\delta}^2_1$ such that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}].$$

By upward absoluteness, this will contradict (1.2). To implement this strategy we need the appropriate Σ_1-fact (in a real) to reflect.

STEP 2. The following is a true Σ_1-statement about φ_0 and z_0 (as witnessed by taking α to be ϑ_0 from Step 1): There is an α such that

(2.1) $L_\alpha(\mathbb{R}) \models \underset{\sim}{\delta}^2_1$ exists $+$ $F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$,

(2.2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_\alpha(\mathbb{R}) \cap \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and

($\star$) there is a Σ_1-formula φ and a real z such that

$$L_\alpha(\mathbb{R}) \models \varphi[z, X, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\alpha(\mathbb{R}) \not\models \varphi[z, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}],$$

(2.3) if $\beta < \alpha$ then it is not the case that $L_\beta(\mathbb{R}) \models \mathrm{T}_0$ and there is an $X \in L_\beta(\mathbb{R}) \cap \mathrm{OD}^{L_\beta(\mathbb{R})}$ and

($\star$) there is a Σ_1-formula φ and a real z such that

$$L_\beta(\mathbb{R}) \models \varphi[z, X, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\beta(\mathbb{R}) \not\models \varphi[z, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}],$$

and

(2.4) if $\bar{X}$ is least (in the order of definability) such that (2.2) then

$$L_\alpha(\mathbb{R}) \models \varphi_0[z_0, \bar{X}, (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, \mathbb{R}]$$

and for all $\delta < (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$

$$L_\alpha(\mathbb{R}) \not\models \varphi_0[z_0, F^{L_\alpha(\mathbb{R})}(\delta), \delta, \mathbb{R}].$$

(Notice that in (2.3) the ordinal $\underset{\sim}{\delta}^2_1$ and the function F are computed in $L_\alpha(\mathbb{R})$ while the formulas are evaluated in $L_\beta(\mathbb{R})$.) Thus (2.1) ensures (by Lemma 4.3) that $F^{L_\alpha(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})} = F\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$, (2.2) says that $L_\alpha(\mathbb{R})$ satisfies "reflection is failing with respect to $F^{L_\alpha(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}$" and, because of (2.1), this ensures that $\vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$ exists, (2.3) ensures in addition that $\alpha = \vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$, and (2.4) says that φ_0 and z_0 (as chosen in Step 1) witness the existence of $\vartheta((\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})})$.

Since $L_{\underset{\sim}{\delta}^2_1}(\mathbb{R}) \prec^{\mathbb{R}}_1 L(\mathbb{R})$ and φ_0 and z_0 can be coded by a single real, the least ordinal α witnessing the existential of the above statement must be less than $\underset{\sim}{\delta}^2_1$. Let $\bar{\vartheta}$ be this ordinal.

Step 3. We claim that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}],$$

which finishes the proof since by upward absoluteness this contradicts (1.2).

The ordinal $\bar{\vartheta}$ has the Σ_1-properties listed in (2.1)–(2.4) for α. So we have: (4.1) $L_{\bar{\vartheta}}(\mathbb{R}) \models$ "$\underset{\sim}{\delta}^2_1$ exists" + "$F(\delta)$ is defined for all $\delta < \underset{\sim}{\delta}^2_1$" and so (by Lemma 4.3) $F^{L_{\bar{\vartheta}}(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})} = F\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}$, (4.2) $L_{\bar{\vartheta}}(\mathbb{R})$ satisfies "reflection is failing with respect to $F^{L_{\bar{\vartheta}}(\mathbb{R})}\restriction(\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}$" and, because of (4.1), this ensures that $\vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$ exists, (4.3) $\bar{\vartheta} = \vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$, and (4.4) φ_0 and z_0 (as chosen in Step 1) witness the existence of $\vartheta((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})})$. Therefore, by the definition of F, (4.4) implies that

$$L_{\bar{\vartheta}}(\mathbb{R}) \models \varphi_0[z_0, F((\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}), (\underset{\sim}{\delta}^2_1)^{L_{\bar{\vartheta}}(\mathbb{R})}, \mathbb{R}],$$

which contradicts (1.2). □

We will need a slight strengthening of the above theorem. This involves the notion of the *reflection filter*, which in turn involves various universal sets.

Let U_X be a good universal $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}_1^2, \mathbb{R}\})$ set of reals. So U_X is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}_1^2, \mathbb{R}\})$ subset of $\omega^\omega \times \omega^\omega$ such that each $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}_1^2, \mathbb{R} \cup \{\mathbb{R}\}\})$ subset of ω^ω is of the form $(U_X)_t$ for some $t \in \omega^\omega$. For each $\delta < \underset{\sim}{\delta}_1^2$, let U_δ be the universal $\Sigma_1(L(\mathbb{R}), \{F(\delta), \delta, \mathbb{R}\})$ set obtained using the same definition used for U_X except with X and $\underset{\sim}{\delta}_1^2$ replaced by the reflected proxies $F(\delta)$ and δ. As before, we shall identify each of U_X and U_δ with a set of reals using our recursive bijection between $\omega^\omega \times \omega^\omega$ and ω^ω.

For each Σ_1-formula φ and for each real y, there exists a $z_{\varphi,y} \in \omega^\omega$ such that

$$z_{\varphi,y} \in U_X \text{ iff } L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}_1^2, \mathbb{R}].$$

In such a situation we say that $z_{\varphi,y}$ *certifies* the Σ_1-fact φ about y. The key property is, of course, that if $z_{\varphi,y} \in U_\delta$ then $L(\mathbb{R}) \models \varphi[y, F(\delta), \delta, \mathbb{R}]$. Notice that the real $z_{\varphi,y}$ is recursive in y (uniformly).

In what follows we will drop reference to φ and y and simply write $z \in U_X$, it being understood that the formula and parameter are encoded in z. In these terms Theorem 4.4 can be recast as stating that if $z \in U_X$ then there is an ordinal $\delta < \underset{\sim}{\delta}_1^2$ such that $z \in U_\delta$, in other words, $U_X \subseteq \bigcup_{\delta < \underset{\sim}{\delta}_1^2} U_\delta$. But notice that equality fails since different X can have radically different "reflection points".

For $z \in U_X$, let

$$S_z = \{\delta < \underset{\sim}{\delta}_1^2 \mid z \in U_\delta\}$$

and set

$$\mathscr{F}_X = \{S \subseteq \underset{\sim}{\delta}_1^2 \mid \exists z \in U_X \, (S_z \subseteq S)\}.$$

Equivalently, for a Σ_1-formula φ and a real y such that

$$L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}_1^2, \mathbb{R}]$$

let

$$S_{\varphi,y} = \{\delta < \underset{\sim}{\delta}_1^2 \mid L(\mathbb{R}) \models \varphi[y, F(\delta), \delta, \mathbb{R}]\}$$

and set

$$\mathscr{F}_X = \{S \subseteq \underset{\sim}{\delta}_1^2 \mid \text{there is a } \Sigma_1\text{-formula } \varphi \text{ and a } y \in \omega^\omega \text{ such that } L(\mathbb{R}) \models \varphi[y, X, \underset{\sim}{\delta}_1^2, \mathbb{R}] \text{ and } S_{\varphi,y} \subseteq S\}.$$

Notice that we can reflect to points δ where the proxies $F(\delta), \delta$ resemble $X, \underset{\sim}{\delta}_1^2$ as much as we like. For example, suppose

$$L(\mathbb{R}) \models \psi[y, X, \underset{\sim}{\delta}_1^2, \mathbb{R}]$$

and consider the following Σ_1-statement:

There is an α such that

$L_\alpha(\mathbb{R}) \models \mathrm{T}_0$,

$\delta = (\underset{\sim}{\delta}^2_1)^{L_\alpha(\mathbb{R})}, F{\restriction}\delta = (F)^{L_\alpha(\mathbb{R})}$, and $F(\delta) \in \mathrm{OD}^{L_\alpha(\mathbb{R})}$, and

$L_\alpha(\mathbb{R}) \models \psi[y, F(\delta), \delta, \mathbb{R}]$.

If we replace δ by $\underset{\sim}{\delta}^2_1$ and $F(\delta)$ by X then this statement is true. It follows that the statement holds for $\mathscr{F}_X$-almost all δ. The second clause ensures that each such δ is a "local $\underset{\sim}{\delta}^2_1$" and that the "local computation of F up to δ" coincides with F. By altering ψ and y we can increase the degree to which the proxies $F(\delta), \delta$ resemble $X, \underset{\sim}{\delta}^2_1$.

Lemma 4.5. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$. *Then* $L(\mathbb{R}) \models \mathscr{F}_X$ *is a countably complete filter.*

Proof. Upward closure and the non-triviality condition are immediate. It remains to prove countable completeness. Suppose $\{S_n \mid n < \omega\} \subseteq \mathscr{F}_X$. For $n < \omega$, let $z_n \in U_X$ be such that $S_{z_n} \subseteq S_n$. Let $z \in \omega^\omega$ be such that $(z)_n = z_n$ for all $n < \omega$. The following is a true Σ_1-statement about z, X, $\underset{\sim}{\delta}^2_1$, and $\mathbb{R}$:

There is an α such that

(1) $\underset{\sim}{\delta}^2_1 < \alpha$,

(2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$,

(3) $X \in \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and

(4) for all $n < \omega, (z)_n \in (U_X)^{L_\alpha(\mathbb{R})}$.

Let $z^* \in U_X$ certify this statement. It follows that for each $\delta < \underset{\sim}{\delta}^2_1$ such that $z^* \in U_\delta$ the following holds:

There is an α such that

(1) $\delta < \alpha$,

(2) $L_\alpha(\mathbb{R}) \models \mathrm{T}_0$,

(3) $F(\delta) \in \mathrm{OD}^{L_\alpha(\mathbb{R})}$ and

(4) for all $n < \omega, (z)_n \in (U_\delta)^{L_\alpha(\mathbb{R})}$.

But then, by upward absoluteness, $\delta \in \bigcap\{S_{z_n} \mid n < \omega\}$ and so $S_{z^*} \subseteq \bigcap\{S_{z_n} \mid n < \omega\} \subseteq \bigcap\{S_n \mid n < \omega\}$. $\square$

We shall call $\mathscr{F}_X$ the *reflection filter* since, by definition, there are $\mathscr{F}_X$-many reflecting points in the Reflection Theorem.

We wish now to extend the Reflection Theorem by allowing various parameters $S \subseteq \underset{\sim}{\delta}^2_1$ and their "reflections" $S \cap \delta$. For this we bring in AD.

Theorem 4.6 (Reflection Theorem). *Assume* ZF + AD. *Suppose* $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ *and* $S \subseteq \underset{\sim}{\delta}^2_1$ *are in* $L(\mathbb{R})$*. For all* $X \in \mathrm{OD}^{L(\mathbb{R})}$*, for all* Σ_1*-formulas* φ*, and for all* $z \in \omega^\omega$*, if*

$$L(\mathbb{R}) \models \varphi[z, X, f, S, \underset{\sim}{\delta}^2_1, \mathbb{R}]$$

then for $\mathscr{F}_X$*-many* $\delta < \underset{\sim}{\delta}^2_1$*,*

$$L(\mathbb{R}) \models \varphi[z, F(\delta), f{\restriction}\delta, S \cap \delta, \delta, \mathbb{R}],$$

where here f *and* S *occur as predicates.*

Proof. First we show that the theorem holds for $S \subseteq \underset{\sim}{\delta}^2_1$. For each $\delta < \underset{\sim}{\delta}^2_1$, let

$$Q_\delta = U_\delta \smallsetminus \bigcup\{U_\gamma \mid \gamma < \delta\}.$$

The sequence

$$\langle Q_\delta \mid \delta < \underset{\sim}{\delta}^2_1\rangle$$

gives rise to a prewellordering of length $\underset{\sim}{\delta}^2_1$. By the Uniform Coding Lemma, there is an $e(S) \in \omega^\omega$ such that

$$U^{(2)}_{e(S)}(Q_{<\delta}, Q_\delta) \neq \varnothing \ \text{ iff } \ \delta \in S.$$

The key point is that for $\mathscr{F}_X$-almost all δ

$$F^{L_{\vartheta(\delta)}(\mathbb{R})} = F{\restriction}\delta.$$

To see this let $z \in U_X$ be such that if $z \in U_\delta$ then

$$L_{\vartheta(\delta)}(\mathbb{R}) \models \delta = \underset{\sim}{\delta}^2_1 \text{ and } F{\restriction}\delta \text{ is defined.}$$

Thus, if $\delta \in S_z$, then

$$\delta = (\underset{\sim}{\delta}^2_1)^{L_{\vartheta(\delta)}(\mathbb{R})} \quad \text{and} \quad F{\restriction}\delta = F^{L_{\vartheta(\delta)}(\mathbb{R})},$$

which implies

$$\langle Q_\gamma \mid \gamma < \delta\rangle = \langle Q_\gamma \mid \gamma < \delta\rangle^{L_{\vartheta(\delta)}(\mathbb{R})}.$$

It follows that for $\delta \in S_z$, $e(S)$ codes $S \cap \delta$.

This enables us to associate with each Σ_1-sentence φ involving the predicate S, a Σ_1-sentence φ^* involving instead the real $e(S)$ in such a way that

$$L(\mathbb{R}) \models \varphi[z, X, \underset{\sim}{\delta}^2_1, S, \mathbb{R}]$$

if and only if

$$L(\mathbb{R}) \models \varphi^*[z, X, \underset{\sim}{\delta}^2_1, e(S), \mathbb{R}]$$

and, for $\delta \in S_z \in \mathscr{F}_X$,

$$L(\mathbb{R}) \models \varphi[z, F(\delta), \delta, S \cap \delta, \mathbb{R}]$$

if and only if

$$L(\mathbb{R}) \models \varphi^*[z, F(\delta), \delta, e(S), \mathbb{R}].$$

In this fashion, the predicate S can be eliminated in favour of the real $e(S)$, thereby reducing the present version of the reflection theorem to the original version (Theorem 4.4).

To see that we can also include parameters of the form $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ simply note that $\mathscr{F}_X$-almost all δ are closed under the Gödel pairing function and so we can include functions $f : \underset{\sim}{\delta}^2_1 \to \underset{\sim}{\delta}^2_1$ by coding them as subsets of $\underset{\sim}{\delta}^2_1$. □

We are now in a position to define, for cofinally many $\lambda < \Theta$, an ultrafilter μ_λ on $\underset{\sim}{\delta}^2_1$. For the remainder of this section fix an ordinal $\lambda < \Theta$ and (by the results of Section 3.3) an OD-prewellordering $\leqslant_\lambda$ of ω^ω of length λ. Our interest is in applying the Reflection Theorem to

$$X = (\leqslant_\lambda, \lambda).$$

For each $S \subseteq \underset{\sim}{\delta}^2_1$, let $G^X(S)$ be the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \dots \\ \mathrm{II} & & y(0) & & y(1) & & \dots & \end{array}$$

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X$. If the rule is violated then, letting i be the least such that either $(x)_i \notin U_X$ or $(y)_i \notin U_X$, I wins if $(x)_i \in U_X$; otherwise II wins. If the rule is satisfied then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta$, (which exists by reflection since (as in Lemma 4.5) we can regard this as a Σ_1-statement about a single real) I wins iff $\delta \in S$. Thus, I is picking δ by steering into the δ^{th}-approximation U_δ. (Note that the winning condition is not Σ_1.)

Now set
$$\mu_X = \{S \subseteq \underset{\sim}{\delta}^2_1 \mid \text{I wins } G^X(S)\}.$$
We let $\mu_\lambda = \mu_X$ but shall typically write μ_X to emphasize the dependence on the prewellorder. For $z \in U_X$, Player I can win $G^X(S_z)$ by playing x such that $(x)_i \in U_X$ for all $i < \omega$ and, for some $i < \omega$, $(x)_i = z$. Thus,

$$\mathscr{F}_X \subseteq \mu_X.$$

It is easy to see that μ_X is upward closed and contains either S or $\underset{\sim}{\delta}^2_1 \setminus S$ for each $S \subseteq \underset{\sim}{\delta}^2_1$.

Lemma 4.7. *Assume* ZF+AD. *Then* $L(\mathbb{R}) \models \mu_X$ *is a* $\underset{\sim}{\delta}^2_1$*-complete ultrafilter.*

Proof. The proof is similar to the proof of Theorem 2.33 (which traces back to the proof of Theorem 2.13). Consider $\{S_\alpha \mid \alpha < \gamma\}$ where $S_\alpha \in \mu_X$ and $\gamma < \underset{\sim}{\delta}^2_1$. Let $S = \bigcap_{\alpha<\gamma} S_\alpha$ and assume for contradiction that $S \notin \mu_X$. Let σ' be a winning strategy for I in $G^X(\underset{\sim}{\delta}^2_1 \setminus S)$. Let

$$\begin{aligned} Z = \{(x,\sigma) \mid {} & \text{for some } \alpha < \gamma,\ x \in Q_\alpha \text{ and} \\ & \sigma \text{ is a winning strategy for I in } G^X(S_\alpha)\} \end{aligned}$$

where $Q_\alpha = \{x \in \omega^\omega \mid |x|_{\leqslant_U} = \alpha\}$ and $\leqslant_U$ is the prewellordering of length $\underset{\sim}{\delta}^2_1$ from Theorem 2.33. (One can also use the prewellordering from Theorem 4.6.)

By the Uniform Coding Lemma, let $e_0 \in \omega^\omega$ be such that for all $\alpha < \gamma$,

$$U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha) \subseteq Z \cap (Q_\alpha \times \omega^\omega) \text{ and } U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha) \neq \varnothing.$$

Let
$$\Sigma = \mathrm{proj}_2\big(\textstyle\bigcup_{\alpha<\gamma} U^{(2)}_{e_0}(Q_{<\alpha}, Q_\alpha)\big).$$
Notice that Σ is $\underset{\sim}{\Delta}^2_1$ since $\leqslant_U \restriction \gamma$ is $\underset{\sim}{\Delta}^2_1$. The key point is that (as in Lemma 2.27) we can choose a real that ensures that in a reflection argument we reflect to a level that correctly computes $\leqslant_U \restriction \gamma$ and hence Σ. We assume that all reals below have this feature.

Now we can "take control" of the output ordinal δ_0 with respect to σ' and all $\tau \in \Sigma$:

Base Case. We have

(1.1) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_X$ and

(1.2) $\forall y \in \omega^\omega \, \forall \sigma \in \Sigma \, ((\sigma * y)_I)_0 \in U_X$

since σ' and σ (as in (1.2)) are winning strategies for I. Since Σ is $\underset{\sim}{\Delta}^2_1$ this is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\} \cup \mathbb{R})$ fact about σ' and hence certified by a real $z_0 \in U_X$ such that $z_0 \leqslant_T \sigma'$; more precisely, $z_0 \leqslant_T \sigma'$ is such that for all δ if $z_0 \in U_\delta$ then

(1.3) $\forall y \in \omega^\omega \, ((\sigma' * y)_I)_0 \in U_\delta$ and

(1.4) $\forall y \in \omega^\omega \, \forall \sigma \in \Sigma \, ((\sigma * y)_I)_0 \in U_\delta$.

$(n+1)^{\text{ST}}$ STEP. Assume we have defined $z_0, \ldots, z_n$ in such a way that $z_n \leqslant_T \cdots \leqslant_T z_0$ and

(2.1) $\forall y \in \omega^\omega \left(\forall i \leqslant n \, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_X \right)$ and

(2.2) $\forall y \in \omega^\omega \, \forall \sigma \in \Sigma \left(\forall i \leqslant n \, (y)_i = z_i \to ((\sigma * y)_I)_{n+1} \in U_X \right)$.

Let $z_{n+1} \in U_X$ be such that $z_{n+1} \leqslant_T z_n$ and for all δ, if $z_{n+1} \in U_\delta$ then

(2.3) $\forall y \in \omega^\omega \left(\forall i \leqslant n \, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_\delta \right)$ and

(2.4) $\forall y \in \omega^\omega \, \forall \sigma \in \Sigma \left(\forall i \leqslant n \, (y)_i = z_i \to ((\sigma * y)_I)_{n+1} \in U_\delta \right)$.

Finally, let $z \in \omega^\omega$ be such that $(z)_i = z_i$ for all $i < \omega$ and let δ_0 be least such that $(z)_i \in U_{\delta_0}$ for all $i \in \omega$. Notice that by our choice of z_n no DC is required to define z. Then, for all $i < \omega$,

(3.1) $((\sigma' * z)_I)_i \in U_{\delta_0}$ by (1.3) and (2.3) and

(3.2) $((\sigma * z)_I)_i \in U_{\delta_0}$ for all $\sigma \in \Sigma$ by (1.4) and (2.4).

So

(4.1) δ_0 is the ordinal produced by $\sigma' * z$, i.e. $\delta_0 \in \underset{\sim}{\delta}^2_1 \setminus S$ and

(4.2) δ_0 is the ordinal produced by $\sigma_\alpha * z$ where $\sigma_\alpha \in \Sigma$ is a winning strategy for I in $G^X(S_\alpha)$, i.e. $\delta_0 \in S_\alpha$ for all $\alpha < \gamma$.

This is a contradiction. $\square$

4.2. Strong Normality

Assuming ZF + AD, in $L(\mathbb{R})$ we have defined, for cofinally many $\lambda < \Theta$, an $\mathrm{OD}^{L(\mathbb{R})}$ ultrafilter on $\underset{\sim}{\delta}^2_1$ and shown that these ultrafilters are $\underset{\sim}{\delta}^2_1$-complete. We now wish to take the ultrapower of $\mathrm{HOD}^{L(\mathbb{R})}$ with these ultrafilters and show that collectively they witness that for each $\lambda < \Theta$, $\underset{\sim}{\delta}^2_1$ is λ-strong in $\mathrm{HOD}^{L(\mathbb{R})}$. This will be achieved by showing that reflection and uniform coding combine to show that μ_λ is *strongly normal*.

We begin with the following basic lemma on the ultrapower construction, which we shall prove in greater generality than we need at the moment.

Lemma 4.8. *Assume* ZF+DC*. Suppose* μ *is a countably complete ultrafilter on* δ *and that* μ *is* OD*. Suppose* T *is a set. Let* $(\mathrm{HOD}_T)^\delta$ *be the class of all functions* $f : \delta \to \mathrm{HOD}_T$*. Then the transitive collapse* M *of* $(\mathrm{HOD}_T)^\delta/\mu$ *exists, the associated embedding*

$$j : \mathrm{HOD}_T \to M$$

is OD_T*, and*

$$M \subseteq \mathrm{HOD}_T.$$

Proof. For $f, g : \delta \to \mathrm{HOD}_T$, let $f \sim_\mu g$ iff $\{\alpha < \delta \mid f(\alpha) = g(\alpha)\} \in \mu$ and let $[f]_\mu$ be the set consisting of the members of the equivalence class of f which have minimal rank. The structure $(\mathrm{HOD}_T)^\delta/\mu$ is the class consisting of all such equivalence classes. Let E be the associated membership relation. So $[f]_\mu \ E \ [g]_\mu$ if and only if $\{\alpha < \delta \mid f(\alpha) \in g(\alpha)\} \in \mu$. Notice that both $(\mathrm{HOD}_T)^\delta/\mu$ and E are OD_T.

The map

$$j_\mu : \mathrm{HOD}_T \to (\mathrm{HOD}_T)^\delta/\mu$$
$$a \mapsto [c_a]_\mu,$$

where $c_a \in (\mathrm{HOD}_T)^\delta$ is the constant function with value a, is an elementary embedding, since Łoś's theorem holds, as HOD_T can be well-ordered. Notice that j_μ is OD_T.

Claim 1. $((\mathrm{HOD}_T)^\delta/\mu, E)$ is well-founded.

Proof. Suppose for contradiction that $((\mathrm{HOD}_T)^\delta/\mu, E)$ is not well-founded. Then, by DC, there is a sequence

$$\left\langle [f_n]_\mu \,\middle|\, n < \omega \right\rangle$$

such that $[f_{n+1}]_\mu \; E \; [f_n]_\mu$ for all $n < \omega$. For each $n < \omega$, let

$$A_n = \{\alpha < \delta \mid f_{n+1}(\alpha) \in f_n(\alpha)\}.$$

For all $n < \omega$, $A_n \in \mu$ and since μ is countable complete,

$$\bigcap\{A_n \mid n < \omega\} \in \mu.$$

This is a contradiction since for each α in this intersection, $f_{n+1}(\alpha) \in f_n(\alpha)$ for all $n < \omega$. □

Claim 2. $((\mathrm{HOD}_T)^\delta/\mu, E)$ is isomorphic to a transitive class $(M, \in)$.

Proof. We have established well-foundedness and extensionality is immediate. It remains to show that for each $a \in (\mathrm{HOD}_T)^\delta/\mu$,

$$\{b \in (\mathrm{HOD}_T)^\delta/\mu \mid b \; E \; a\}$$

is a set. Fix $a \in (\mathrm{HOD}_T)^\delta/\mu$ and choose $f \in (\mathrm{HOD}_T)^\delta$ such that $a = [f]_\mu$. Let α be such that $f \in V_\alpha$. Then for each $b \in (\mathrm{HOD}_T)^\delta/\mu$ such that bEa, letting $g \in (\mathrm{HOD}_T)^\delta$ be such that $b = [g]_\mu$,

$$\{\beta < \delta \mid g(\beta) \in V_\alpha\} \in \mu.$$

Thus,

$$\{b \in (\mathrm{HOD}_T)^\delta/\mu \mid b \; E \; a\} = \{[g]_\mu \mid [g]_\mu \; E \; [f]_\mu \text{ and } g \in V_\alpha\},$$

which completes the proof. □

Let

$$\pi : ((\mathrm{HOD}_T)^\delta/\mu, E) \to (M, \in)$$

be the transitive collapse map and let

$$j : \mathrm{HOD}_T \to M$$

be the composition map $\pi \circ j_\mu$. Since π and j_μ are OD_T, j and M are OD_T.

It remains to see that $M \subseteq \mathrm{HOD}_T$. For this it suffices to show that for all α, $M \cap V_{j(\alpha)} \subseteq \mathrm{HOD}_T$. We have

$$M \cap V_{j(\alpha)} = j(\mathrm{HOD}_T \cap V_\alpha).$$

Let $A \in \mathrm{HOD}_T \cap \mathscr{P}(\gamma)$ be such that

$$\mathrm{HOD}_T \cap V_\alpha \subseteq L[A]$$

for some γ. We have

$$M \cap V_{j(\alpha)} = j(\mathrm{HOD}_T \cap V_\alpha) \subseteq L[j(A)].$$

But j and A are OD_T. Thus, $j(A) \in \mathrm{HOD}_T$ and hence $L[j(A)] \subseteq \mathrm{HOD}_T$, which completes the proof. $\square$

4.9 Remark. The use of DC in this lemma is essential in that assuming mild large cardinal axioms (such as the existence of a strong cardinal) there are models of $\mathrm{ZF} + \mathrm{AC}_\omega$ in which the lemma is false. In these models the club filter on ω_1 is an ultrafilter and the ultrapower of On by the club filter is not well-founded.

The ultrafilter μ_X defined in Section 4.1 is $\mathrm{OD}^{L(\mathbb{R})}$. Thus, by Lemma 4.8 (with $T = \varnothing$), letting

$$\pi : (\mathrm{HOD}^{L(\mathbb{R})})^{\underset{\sim}{\delta}^2_1}/\mu_X \to M_X$$

be the transitive collapse map and letting

$$j_X : \mathrm{HOD}^{L(\mathbb{R})} \to M_X$$

be the induced elementary embedding we have that $M_X \subseteq \mathrm{HOD}^{L(\mathbb{R})}$ and the fragments of j_X are in $\mathrm{HOD}^{L(\mathbb{R})}$ (in other words, j_X is amenable to $\mathrm{HOD}^{L(\mathbb{R})}$). Moreover, since μ_X is $\underset{\sim}{\delta}^2_1$-complete, the critical point of j_X is $\underset{\sim}{\delta}^2_1$.

Our next aim is to show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \underset{\sim}{\delta}^2_1 \text{ is } \lambda\text{-strong}$$

and for this it remains to show that

$$j_X(\underset{\sim}{\delta}^2_1) > \lambda \quad \text{and} \quad \mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda \subseteq M_X.$$

From now on we will also assume that λ is such that $L_\lambda(\mathbb{R}) \prec L_\Theta(\mathbb{R})$ and $\underset{\sim}{\delta}^2_1 < \lambda$. There are arbitrarily large $\lambda < \Theta$ with this feature (by the proof of Lemma 2.20). Since

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$$

(by Theorem 3.10), it follows that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda = \mathrm{HOD}^{L_\lambda(\mathbb{R})}.$$

Thus, letting $A \subseteq \lambda$ be an $\mathrm{OD}^{L(\mathbb{R})}$ set coding $\mathrm{HOD}^{L_\lambda(\mathbb{R})}$, we have

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\lambda = L_\lambda[A].$$

Thus, it remains to show that $A \in M_X$. In fact, we will show that

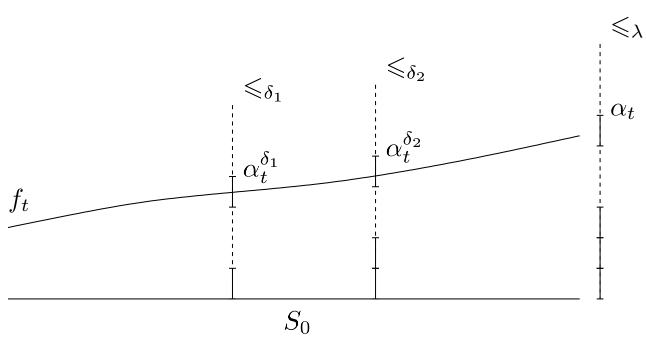
$$\mathscr{P}(\lambda) \cap \mathrm{HOD}^{L(\mathbb{R})} \subseteq M_X.$$

Let

$$S_0 = \{\delta < \underset{\sim}{\delta}^2_1 \mid F(\delta) = (\leqslant_\delta, \lambda_\delta) \text{ where } \leqslant_\delta \text{ is a prewellordering of length } \lambda_\delta \text{ and } L_{\lambda_\delta}(\mathbb{R}) \models \mathrm{T}_0\}.$$

Note that $S_0 \in \mathscr{F}_X$. For $\alpha < \lambda$, let $Q_\alpha^{\underset{\sim}{\delta}^2_1}$ be the α^{th}-component of $\leqslant_\lambda$ and, for $\delta \in S_0$ and $\alpha < \lambda_\delta$, let Q^δ_α be the α^{th}-component of $\leqslant_\delta$. Each $t \in \omega^\omega$ determines a *canonical function* f_t as follows: For $\delta \in S_0$, let α^δ_t be the unique ordinal α such that $t \in Q^\delta_\alpha$ and then set

$$\begin{aligned} f_t : S_0 &\longrightarrow \underset{\sim}{\delta}^2_1 \\ \delta &\mapsto \alpha^\delta_t. \end{aligned}$$



For $t \in \omega^\omega$, let $\alpha_t = |t|_{\leqslant_\lambda}$ be the rank of t according to $\leqslant_\lambda$, that is, $\alpha_t = |t|_{\leqslant_\lambda} = \mu\alpha\,(t \in Q_\alpha^{\underset{\sim}{\delta}^2_1})$.

Lemma 4.10. *Assume* ZF + AD. $j_X(\underset{\sim}{\delta}^2_1) > \lambda$.

Proof. Suppose $t_1, t_2 \in \omega^\omega$ and $|t_1|_{\leqslant\lambda} = |t_2|_{\leqslant\lambda}$. This is a true Σ_1-statement in $L(\mathbb{R})$ about t_1, t_2, X and $\mathbb{R}$. Thus, by reflection (Theorem 4.6), it follows that for $\mathscr{F}_X$-almost all $\delta < \underset{\sim}{\delta}^2_1$, $|t_1|_{\leqslant\delta} = |t_2|_{\leqslant\delta}$ and so the ordinal $[f_t]_{\mu_X}$ represented by f_t only depends on $|t|_{\leqslant\lambda}$. Likewise, if $|t_1|_{\leqslant\lambda} < |t_2|_{\leqslant\lambda}$ then $[f_{t_1}]_{\mu_X} < [f_{t_2}]_{\mu_X}$. Therefore, the map

$$\begin{aligned} \rho : \lambda &\to \prod \lambda_\delta/\mu_X \\ |t|_{\leqslant\lambda} &\mapsto [f_t]_{\mu_X} \end{aligned}$$

is well-defined and order-preserving and it follows that $\lambda \leqslant \prod \lambda_\delta/\mu_X < j_X(\underset{\sim}{\delta}^2_1)$. □

We now turn to showing $\mathscr{P}(\lambda) \cap \mathrm{HOD}^{L(\mathbb{R})} \subseteq M_X$. Fix $A \subseteq \lambda$ such that $A \in \mathrm{HOD}^{L(\mathbb{R})}$. By the Uniform Coding Lemma there is an index $e(A) \in \omega^\omega$ such that for all $\alpha < \lambda$,

$$U^{(2)}_{e(A)}(Q^{\underset{\sim}{\delta}^2_1}_{<\alpha}, Q^{\underset{\sim}{\delta}^2_1}_{\alpha}) \neq \varnothing \text{ iff } \alpha \in A.$$

For all $\delta \in S_0$, let

$$A^\delta = \{\alpha < \lambda_\delta \mid U^{(2)}_{e(A)}(Q^{\delta}_{<\alpha}, Q^{\delta}_{\alpha}) \neq \varnothing\}$$

be the "reflection of A". Since the statement

$$\{\alpha < \lambda \mid U^{(2)}_{e(A)}(Q^{\underset{\sim}{\delta}^2_1}_{<\alpha}, Q^{\underset{\sim}{\delta}^2_1}_{\alpha}) \neq \varnothing\} \in \mathrm{HOD}^{L(\mathbb{R})}$$

is a true Σ_1-statement about X, $\mathbb{R}$ and $e(A)$, there is a set $S \in \mathscr{F}_X$ such that for all $\delta \in S$, $A^\delta \in \mathrm{HOD}^{L(\mathbb{R})}$.

We wish to show that

$$\begin{aligned} h_A : S &\to \mathrm{HOD}^{L(\mathbb{R})} \\ \delta &\mapsto A^\delta \end{aligned}$$

represents A in the ultrapower. Notice that

$$\begin{aligned} |t|_{\leqslant\lambda} \in A \quad &\text{iff} \quad \{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) \in A^\delta\} \in \mu_X \\ &\text{iff} \quad [f_t]_{\mu_X} \in [h_A]_{\mu_X}. \end{aligned}$$

The last equivalence holds by definition. For the first equivalence note that if $|t|_{\leqslant\lambda} \in A$ then since this is a true Σ_1-statement about $e(A)$, t and X, for

μ_X-almost all δ, $|t|_{\leqslant_\delta} \in A^\delta$, that is, $\{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) \in A^\delta\} \in \mu_X$. Likewise, if $|t|_{\leqslant_\lambda} \notin A$ then since this is a true Σ_1-statement about $e(A)$, t and X, for μ_X-almost all δ, $|t|_{\leqslant_\delta} \notin A^\delta$.

So it suffices to show that the map

$$\begin{aligned} \rho : \lambda &\to \prod \lambda_\delta/\mu_X \\ |t|_{\leqslant_\lambda} &\mapsto [f_t]_{\mu_X} \end{aligned}$$

is an isomorphism since then $\pi([h_A]_{\mu_X}) = A \in M_X$, where recall that $\pi : (\mathrm{HOD}^{L(\mathbb{R})})^{\underset{\sim}{\delta}^2_1}/\mu_X \cong M_X$ is the transitive collapse map. We already know that ρ is well-defined and order-preserving (by Lemma 4.10). It remains to show that ρ is onto, that is, that every function $f \in \prod \lambda_\delta/\mu_X$ is equivalent (modulo μ_X) to a canonical function f_t. To say that this is true is to say that μ_X is *strongly normal*:

4.11 Definition (Strong normality). μ_X is *strongly normal* iff whenever $f : S_0 \to \underset{\sim}{\delta}^2_1$ is such that

$$\{\delta \in S_0 \mid f(\delta) < \lambda_\delta\} \in \mu_X$$

then there exists a $t \in \omega^\omega$ such that

$$\{\delta \in S_0 \mid f(\delta) = f_t(\delta)\} \in \mu_X.$$

Notice that normality is a special case of strong normality since if

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) < \delta\} \in \mu_X$$

then (since for $\mathscr{F}_X$-almost all δ, $\lambda_\delta > \delta$), by strong normality there is a $t \in \omega^\omega$ such that

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f_t(\delta) = f(\delta)\} \in \mu_X.$$

So if β is such that $t \in Q_\beta^{\underset{\sim}{\delta}^2_1}$ then $\beta < \underset{\sim}{\delta}^2_1$, since otherwise by reflection this would contradict the assumption that

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) < \delta\} \in \mu_X.$$

Thus,

$$\{\delta < \underset{\sim}{\delta}^2_1 \mid f(\delta) = \beta\} \in \mu_X.$$

Theorem 4.12. *Assume* ZF + AD. $L(\mathbb{R}) \models \mu_X$ *is strongly normal.*

Proof. Assume toward a contradiction that f is a counterexample to strong normality. So, for each $t \in \omega^\omega$,

$$\{\delta \in S_0 \mid f(\delta) \neq f_t(\delta)\} \in \mu_X.$$

Let

$$\eta = \min\bigl\{\beta < \lambda \mid \forall t \in Q_\beta^{\underset{\sim}{\delta}_1^2}\ \{\delta \in S_0 \mid f(\delta) < f_t(\delta)\} \in \mu_X\bigr\}$$

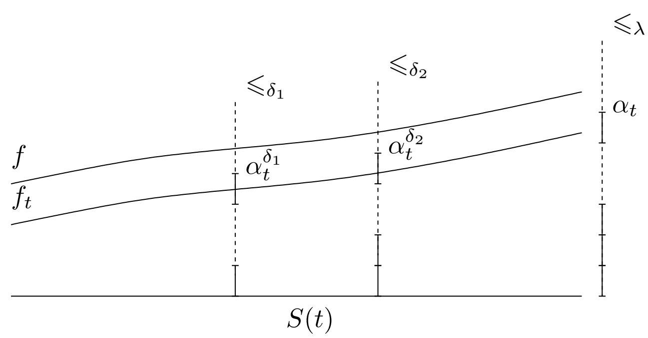
if such β exist; otherwise, let $\eta = \lambda$. Fix $y_\eta \in Q_\eta^{\underset{\sim}{\delta}_1^2}$ (unless $\eta = \lambda$, in which case we ignore this parameter) and, for $\delta \in S_0$, let $\eta_\delta = f_{y_\eta}(\delta)$ and for $\delta = \underset{\sim}{\delta}_1^2$, let $\eta_\delta = \eta$. Note that $f_{y_\eta}(\delta) > f(\delta)$ for μ_X-almost all δ. In the proof we will be working on this set and so we modify S_0 by intersecting it with this set if necessary. For convenience let

$$S(t) = \{\delta \in S_0 \mid f_t(\delta) < f(\delta)\}.$$

Notice that by the definition of η and our assumption that f is a counterexample to strong normality, we have that

$$S(t) \in \mu_X$$

for all $t \in Q_{<\eta}^{\underset{\sim}{\delta}_1^2}$.



Our aim is to compute f from a real parameter by coding relative to the various prewellorderings. Our computation will give us "$f(\underset{\sim}{\delta}_1^2)$". Then, picking a real $y_f \in Q^{\underset{\sim}{\delta}_1^2}_{\text{“}f(\underset{\sim}{\delta}_1^2)\text{”}}$ we shall have by reflection that for μ_X-almost every δ, $f(\delta) = f_{y_f}(\delta)$, which is a contradiction.

The proof involves a number of parameters which we list here. We will also give a brief description which will not make complete sense at this point but will serve as a useful reference to consult as the proof proceeds.

e_0 is the index of the universal set that selects the Z^δ_α's (represented in the diagrams as ellipses) from Z' (represented in the diagrams as chimneys).

e_1 is the index of the universal set that selects *subsets* of the Z^δ_α's (represented in the diagrams as black dots inside the ellipses).

y_η is the real in $Q^{\underset{\sim}{\delta}^2_1}_\eta$ that determines η_δ for $\delta \in S_0$.

y_f is the real in $Q^{\underset{\sim}{\delta}^2_1}_{\text{“}f(\underset{\sim}{\delta}^2_1)\text{”}}$ that determines $f(\delta)$ for $\delta \in S_3$.

We will successively shrink S_0 to S_1, S_2, and finally S_3. All four of these sets will be members of μ_X. We now proceed with the proof.

Let

$$Z' = \big\{(t, \sigma) \mid t \in Q^{\underset{\sim}{\delta}^2_1}_{<\eta} \text{ and } \sigma \text{ is a winning strategy for I in } G^X(S(t))\big\}.$$

Thus, by our assumption that f is a counterexample to strong normality and by our choice of η we have, for all $\beta < \eta$,

$$Z' \cap (Q^{\underset{\sim}{\delta}^2_1}_\beta \times \omega^\omega) \neq \varnothing,$$

since for all $t \in Q^{\underset{\sim}{\delta}^2_1}_{<\eta}$, I wins $G^X(S(t))$. By the Uniform Coding Lemma, let $e_0 \in \omega^\omega$ be such that for all $\beta < \eta$,

(1.1) $U^{(2)}_{e_0}(Q^{\underset{\sim}{\delta}^2_1}_{<\beta}, Q^{\underset{\sim}{\delta}^2_1}_\beta) \subseteq Z' \cap (Q^{\underset{\sim}{\delta}^2_1}_\beta \times \omega^\omega)$ and

(1.2) $U^{(2)}_{e_0}(Q^{\underset{\sim}{\delta}^2_1}_{<\beta}, Q^{\underset{\sim}{\delta}^2_1}_\beta) \neq \varnothing$.

By reflection, we have that for $\mathscr{F}_X$-almost all δ, for all $\beta < \eta_\delta$,

(2.1) $U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta) \subseteq Q^\delta_\beta \times \omega^\omega$ and

(2.2) $U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta) \neq \varnothing$.

Notice that in the reflected statement we have had to drop reference to Z' since we cannot reflect Z' as the games involved in its definition are not Σ_1. Let S'_1 be the set of such δ and let $S_1 = S'_1 \cap S_0$. Notice that S_1 is $\Sigma_1(L(\mathbb{R}), \{e_0, y_\eta, f, X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$.

For $\delta \in S_1 \cup \{\underset{\sim}{\delta}^2_1\}$ and $\beta < \eta_\delta$ let

$$Z^\delta_\beta = U^{(2)}_{e_0}(Q^\delta_{<\beta}, Q^\delta_\beta)$$

and for $\delta \in S_1 \cup \{\underset{\sim}{\delta}^2_1\}$ let

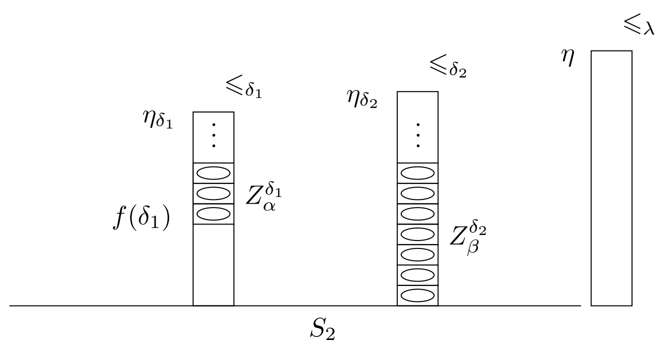
$$Z^\delta = \textstyle\bigcup_{\beta<\eta_\delta} Z^\delta_\beta.$$

Claim A (Disjointness Property). There is an $S_2 \subseteq S_1$, $S_2 \in \mu_X$ such that for $\delta_1, \delta_2 \in S_2 \cup \{\underset{\sim}{\delta}^2_1\}$ with $\delta_1 < \delta_2 \leqslant \underset{\sim}{\delta}^2_1$,

$$Z^{\delta_1}_{\alpha} \cap Z^{\delta_2}_{\beta} = \varnothing$$

for all $\alpha \in [f(\delta_1), \eta_{\delta_1})$ and $\beta \in [0, \eta_{\delta_2})$.

Proof. Here is the picture:



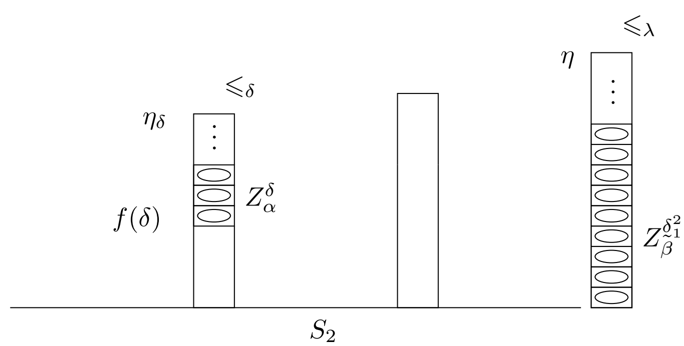
We begin by establishing a special case.

Subclaim. For μ_X-almost all δ,

$$Z^{\delta}_{\alpha} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta} = \varnothing$$

for all $\alpha \in [f(\delta), \eta_\delta)$ and $\beta \in [0, \eta)$.

Proof. The picture is similar:



Let

$$T = \left\{\delta \in S_1 \mid Z^{\delta}_{\alpha} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta} = \varnothing \text{ for all } \alpha \in [f(\delta), \eta_\delta) \text{ and } \beta \in [0, \eta)\right\}$$

and assume, toward a contradiction, that $T \notin \mu_X$. So $(\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1 \in \mu_X$. Let σ' be a winning strategy for I in $G^X((\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1)$.

Let us first motivate the main idea: Suppose z is a legal play for II against σ' (by which we mean a play for II that satisfies the Main Rule) and suppose that the ordinal associated with this play is δ_0. So $\delta_0 \in (\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1$ and (by the definition of T) there exists an $\alpha_0 \in [f(\delta_0), \eta_\delta)$ and $\beta_0 \in [0, \eta)$ such that $Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0} \neq \varnothing$. Pick $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. In virtue of the fact that $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0}$ we have

(3.1) $f_{t_0}(\delta_0) = \alpha_0 \geqslant f(\delta_0)$

and in virtue of the fact that $(t_0, \sigma_0) \in Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ we have

(3.2) σ_0 is a winning strategy for I in $G^X(S(t_0))$, where

$$S(t_0) = \{\delta \in S_0 \mid f_{t_0}(\delta) < f(\delta)\}.$$

So we get a contradiction if δ_0 happens to be in $S(t_0)$ (since then $f_{t_0}(\delta_0) < f(\delta_0)$, contradicting (3.1)). Notice that this will occur if we can arrange the play z to be such that in addition to being a legal play against σ' with associated ordinal δ_0 it is *also* a legal play against σ_0 (in the game $G^X(S(t_0))$) with associated ordinal δ_0. We can construct such a play z recursively as in the proof of completeness.

Base Case. We have

(4.1) $\forall y \in \omega^\omega \, ((\sigma' {*} y)_I)_0 \in U_X$ and

(4.2) $\forall y \in \omega^\omega \, \forall (t, \sigma) \in Z^{\underset{\sim}{\delta}^2_1} \, ((\sigma {*} y)_I)_0 \in U_X$

since σ' and σ (as in (4.2)) are winning strategies for I. Now all of this is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$-fact about σ' and e_0 (the index for $Z^{\underset{\sim}{\delta}^2_1}$) and so it is certified by a real $z_0 \in U_X$ such that $z_0 \leqslant_T \langle \sigma', e_0 \rangle$; so z_0 is such that if $z_0 \in U_\delta$ then

(4.3) $\forall y \in \omega^\omega \, ((\sigma' {*} y)_I)_0 \in U_\delta$ and

(4.4) $\forall y \in \omega^\omega \, \forall (t, \sigma) \in Z^\delta \, ((\sigma {*} y)_I)_0 \in U_\delta$.

$(n+1)^{\text{st}}$ Step. Assume we have defined $z_0, \ldots, z_n$ in such a way that $z_n \leqslant_T \cdots \leqslant_T z_0$ and

(5.1) $\forall y \in \omega^\omega \left(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma' * y)_I)_{n+1} \in U_X\right)$ and

(5.2) $\forall y \in \omega^\omega\, \forall (t,\sigma) \in Z^{\underset{\sim}{\delta}^2_1} \left(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_X\right)$.

Again, all of this is a $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$-fact about $\sigma', e_0, z_0, \ldots, z_n$ and so it is certified by a real $z_{n+1} \in U_X$ such that $z_{n+1} \leqslant_T z_n$; so z_{n+1} is such that if $z_{n+1} \in U_\delta$ then

(5.3) $\forall y \in \omega^\omega \left(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma' * y)_I)_{n+1} \in U_\delta\right)$ and

(5.4) $\forall y \in \omega^\omega\, \forall (t,\sigma) \in Z^\delta \left(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_\delta\right)$.

Finally, let $z \in \omega^\omega$ be such that $(z)_i = z_i$ for all $i < \omega$ and let δ_0 be least such that $(z)_i \in U_{\delta_0}$ for all $i \in \omega$. Notice that since we chose z_{n+1} to be recursive in z_n no DC is required to form z. Since $(z)_i \in U_X$ for all $i \in \omega$, z is a legal play for II in any of the games $G^X(S)$ relevant to the argument. Moreover, for all $i \in \omega$,

(6.1) $((\sigma' * z)_I)_i \in U_{\delta_0}$ by (4.3) and (5.3) and

(6.2) $((\sigma * z)_I)_i \in U_{\delta_0}$ for all $\sigma \in \mathrm{proj}_2(Z^{\delta_0})$ by (4.4) and (5.4)

and so

(7.1) δ_0 is the ordinal produced by $\sigma' * z$, i.e. $\delta_0 \in (\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1$ and

(7.2) δ_0 is the ordinal produced by $\sigma * z$ for any $\sigma \in \mathrm{proj}_2(Z^{\delta_0})$.

Since $\delta_0 \in (\underset{\sim}{\delta}^2_1 \smallsetminus T) \cap S_1$, by the definition of T there exists an $\alpha_0 \in [f(\delta_0), \eta_{\delta_0})$ and $\beta_0 \in [0, \eta)$ such that $Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0} \neq \varnothing$. Pick $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0} \cap Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. In virtue of the fact that $(t_0, \sigma_0) \in Z^{\delta_0}_{\alpha_0}$ we have

(8.1) $f_{t_0}(\delta_0) = \alpha_0 \geqslant f(\delta_0)$

and in virtue of the fact that $(t_0, \sigma_0) \in Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ we have

(8.2) σ_0 is a winning strategy for I in $G^X(S(t_0))$, where

$$S(t_0) = \{\delta \in S_0 \mid f_{t_0}(\delta) < f(\delta)\}.$$

Combined with (7.2) this implies $\delta_0 \in S(t_0)$, in other words, $f_{t_0}(\delta_0) < f(\delta_0)$, which contradicts (8.1). □

Thus, $T \in \mu_X$ and we have

$$\text{(9.1)}\quad \forall\delta \in T\, \forall\beta \in [0, \eta_{\underset{\sim}{\delta}^2_1})\, \forall\alpha \in [f(\delta), \eta_\delta)\, (Z^\delta_\alpha \cap Z^{\underset{\sim}{\delta}^2_1}_\beta = \varnothing).$$

This is a true Σ_1-statement in $L(\mathbb{R})$ about e_0, y_η, f, X, $\mathbb{R}$, $\underset{\sim}{\delta}^2_1$, and T. Since T is $\Sigma_1(L(\mathbb{R}), \{e_0, y_\eta, f, X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$, the above statement is $\Sigma_1(L(\mathbb{R}), \{e_0, y_\eta, f, X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$. Thus by the Reflection Theorem (Theorem 4.6) there exists $S_2 \subseteq S_1$, $S_2 \in \mu_X$ such that for all $\delta_2 \in S_2$,

$$\text{(9.2)}\quad \forall\delta_1 \in T \cap \delta_2\, \forall\beta \in [0, \eta_{\delta_2})\, \forall\alpha \in [f(\delta_1), \eta_{\delta_1})\, (Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing).$$

Notice that S_2 is $\Sigma_1(L(\mathbb{R}), \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$ in e_0, y_η and the parameters for coding. This completes the proof of Claim A. □

Claim B (Tail Computation). There exists an index $e_1 \in \omega^\omega$ such that for all $\delta \in S_2$,

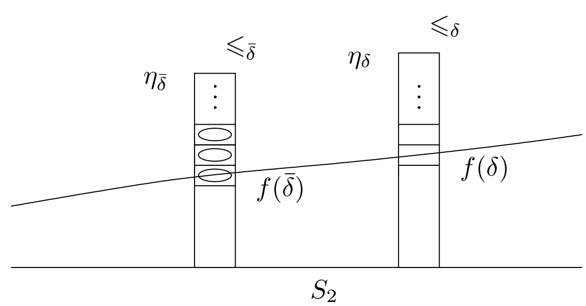
(1) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \subseteq Z^\delta_\beta$ for all $\beta < \eta_\delta$,

(2) $U^{(2)}_{e_1}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing$, and

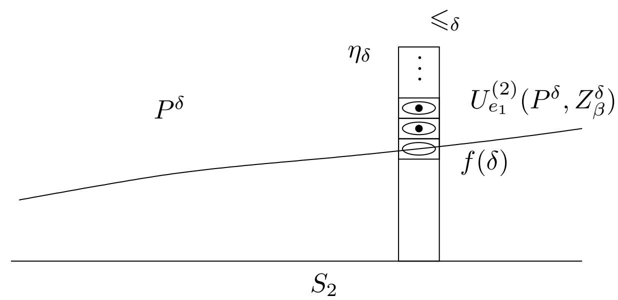
(3) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \neq \varnothing$ for β such that $f(\delta) < \beta < \eta_\delta$,

where $P^\delta = \bigcup\{Z^{\bar\delta}_\alpha \mid \bar\delta \in S_2 \cap \delta \text{ and } \alpha \in [f(\bar\delta), \eta_{\bar\delta})\}$ and S_2 is from the end of the proof of Claim A.

Proof. Here is the picture of the "tail parameter" P^δ:



Here is the picture of the statement of Claim B:



Assume toward a contradiction that there is no such e_1. We follow the proof of the Uniform Coding Lemma. To begin with, notice that it suffices to find $e_1 \in \omega^\omega$ satisfying (2) and

(3′) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta \neq \varnothing$ for β such that $f(\delta) < \beta < \eta_\delta$

since given the parameter Z^δ_β we can easily ensure (1).

Consider the set of reals such that (2) of the (revised) claim holds, that is,

$$G = \big\{e \in \omega^\omega \mid \forall \delta \in S_2 \,\big(U^{(2)}_e(P^\delta, Z^\delta_{f(\delta)}) = \varnothing\big)\big\}.$$

So, for each $e \in G$, (3′) in the claim fails for some $\delta \in S_2$ and $\beta \in (f(\delta), \eta_\delta)$. For each $e \in G$, let

$$\begin{aligned} \alpha_e = \ & \text{lexicographically least pair } (\delta, \beta) \text{ such that} \\ & \quad (1)\ \ \delta \in S_2, \\ & \quad (2)\ \ f(\delta) < \beta < \eta_\delta, \text{ and} \\ & \quad (3)\ \ U^{(2)}_e(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta = \varnothing. \end{aligned}$$

Now play the game

$$\begin{array}{llllllll} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where II wins iff $(x \in G \to (y \in G \wedge \alpha_y >_{\mathrm{lex}} \alpha_x))$

Claim 1. Player I does not have a winning strategy.

Proof. Suppose toward a contradiction that σ is a winning strategy for I. As in the proof of the Uniform Coding Lemma, we aim to “bound” all of I’s plays and then use this bound to construct a play e^* for II which defeats σ. We will make key use of the Disjointness Property.

Choose $e_\sigma \in \omega^\omega$ such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e_\sigma}(P, P') = \bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(P, P').$$

In particular, for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) = \bigcup_{y \in \omega^\omega} U^{(2)}_{(\sigma * y)_I}(P^\delta, Z^\delta_\beta).$$

Note two things: First, since σ is a winning strategy for I, $(\sigma * y)_I \in G$ for all $y \in \omega^\omega$; so $e_\sigma \in G$. Second, for all $y \in \omega^\omega$, $\alpha_{(\sigma * y)_I} \leqslant_{\text{lex}} \alpha_{e_\sigma}$. So e_σ is "at least as good" as any $(\sigma * y)_I$. We have to do "better".

Pick $x_0 \in Z^{\delta_0}_{\beta_0}$ where $(\delta_0, \beta_0) = \alpha_{e_\sigma}$. Choose e^* such that for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{e^*}(P, P') = \begin{cases} U^{(2)}_{e_\sigma}(P, P') & \text{if } x_0 \notin P' \\ U^{(2)}_{e_\sigma}(P, P') \cup \{x_0\} & \text{if } x_0 \in P'. \end{cases}$$

In particular, for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) & \text{if } x_0 \notin Z^\delta_\beta \\ U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta) \cup \{x_0\} & \text{if } x_0 \in Z^\delta_\beta. \end{cases}$$

Since we chose $x_0 \in Z^{\delta_0}_{\beta_0}$, by the Disjointness Property (and the fact that for fixed δ, $Z^\delta_\alpha \cap Z^\delta_\beta = \varnothing$ for $\alpha < \beta < \eta_\delta$) we have

(10.1) $x_0 \notin Z^\delta_\beta$ for $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$,

(10.2) $x_0 \notin Z^{\delta_0}_\beta$ for $\beta \in [0, \eta_{\delta_0}) \smallsetminus \{\beta_0\}$, and

(10.3) $x_0 \notin Z^\delta_\beta$ for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ and $\beta \in [0, \eta_\delta)$.

Thus, by the definition of e^*, we have, by (10.1–3),

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_{f(\delta)}) = U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_{f(\delta)})$$

for all $\delta \in S_2$. Since $e_\sigma \in G$, this means $e^* \in G$. So α_{e^*} exists. Similarly, by the definition of e^*, we have, by (10.1) and (10.2),

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = U^{(2)}_{e_\sigma}(P^\delta, Z^\delta_\beta)$$

for all $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$ and for $\delta = \delta_0$ and $\beta \in [f(\delta_0), \beta_0)$. So e^* is "at least as good" as e_σ. But since $x_0 \in Z^{\delta_0}_{\beta_0}$, we have that $x_0 \in U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0})$, by the definition of e^*; that is, e^* is "better" than e_σ. In other words, $\alpha_{e^*} >_{\text{lex}} \alpha_{e_\sigma} \geqslant_{\text{lex}} \alpha_{(\sigma * y)_I}$ for all $y \in \omega^\omega$ and so, by playing e^*, II defeats σ. □

Claim 2. Player II does not have a winning strategy.

Proof. Suppose toward a contradiction that τ is a winning strategy for II. We shall find an e^* such that $e^* \in G$ (Subclaim 1) and α_{e^*} does not exist (Subclaim 2), which is a contradiction.

Choose $h_0 : \omega^\omega \times (\omega^\omega \times \omega^\omega) \to \omega^\omega$ such that h_0 is Σ^1_1 and for all $(e, x) \in \omega^\omega \times (\omega^\omega \times \omega^\omega)$ and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_0(e,x)}(P, P') = \begin{cases} U^{(2)}_e(P, P') & \text{if } x \notin P \cup P' \\ \varnothing & \text{if } x \in P \cup P'. \end{cases}$$

In particular, for $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{h_0(e,x)}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_e(P^\delta, Z^\delta_\beta) & \text{if } x \notin P^\delta \cup Z^\delta_\beta \\ \varnothing & \text{if } x \in P^\delta \cup Z^\delta_\beta. \end{cases}$$

Choose $h_1 : \omega^\omega \to \omega^\omega$ such that h_1 is Σ^1_1 and for all $P, P' \subseteq \omega^\omega$,

$$U^{(2)}_{h_1(e)}(P, P') = \textstyle\bigcup_{x \in P'} U^{(2)}_{(h_0(e,x)*\tau)_{II}}(P, P').$$

In particular, for $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{h_1(e)}(P^\delta, Z^\delta_\beta) = \textstyle\bigcup_{x \in Z^\delta_\beta} U^{(2)}_{(h_0(e,x)*\tau)_{II}}(P^\delta, Z^\delta_\beta).$$

By the recursion theorem, there is an $e^* \in \omega^\omega$ such that for all $\delta \in S_2$ and $\beta < \eta_\delta$,

$$U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) = U^{(2)}_{h_1(e^*)}(P^\delta, Z^\delta_\beta).$$

Subclaim 1. $e^* \in G$.

Proof. Suppose for contradiction that $e^* \notin G$. Let $\delta_0 \in S_2$ be least such that

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing.$$

Now

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) = U^{(2)}_{h_1(e^*)}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \\ = \bigcup_{x \in Z^{\delta_0}_{f(\delta_0)}} U^{(2)}_{(h_0(e^*,x)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}).$$

So choose $x_0 \in Z^{\delta_0}_{f(\delta_0)}$ such that

$$U^{(2)}_{(h_0(e^*,x_0)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing.$$

If we can show $h_0(e^*, x_0) \in G$ then we are done since this implies that $(h_0(e^*, x_0)*\tau)_{II} \in G$ (as τ is a winning strategy for II), which contradicts the previous statement.

Subsubclaim. $h_0(e^*, x_0) \in G$, that is, for all $\delta \in S_2$,

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing.$$

Proof. By the definition of h_0, for all $\delta \in S_2$,

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \begin{cases} U^{(2)}_{e^*}(P^{\delta}, Z^{\delta}_{f(\delta)}) & \text{if } x_0 \notin P^{\delta} \cup Z^{\delta}_{f(\delta)} \\ \varnothing & \text{if } x_0 \in P^{\delta} \cup Z^{\delta}_{f(\delta)}. \end{cases}$$

Since $x_0 \in Z^{\delta_0}_{f(\delta_0)}$, by the Disjointness Property, this definition yields the following: For $\delta \in S_2 \cap [0, \delta_0)$ we have $x_0 \notin P^{\delta} \cup Z^{\delta}_{f(\delta)}$ and so,

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = U^{(2)}_{e^*}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing,$$

where the latter holds since we chose δ_0 to be least such that

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{f(\delta_0)}) \neq \varnothing;$$

for $\delta = \delta_0$ we have $x_0 \in Z^{\delta}_{f(\delta)}$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing;$$

and for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ we have $x_0 \in P^{\delta}$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta}, Z^{\delta}_{f(\delta)}) = \varnothing.$$

Thus, $h_0(e^*, x_0) \in G$. ∎

This completes the proof of Subclaim 1. $\square$

Subclaim 2. α_{e^*} does not exist.

Proof. Suppose for contradiction that α_{e^*} exists. Recall that

$$\begin{aligned}\alpha_{e^*} = \ & \text{lexicographically least pair } (\delta, \beta) \text{ such that} \\ & (1) \ \ \delta \in S_2, \\ & (2) \ \ f(\delta) < \beta < \eta_\delta, \text{ and} \\ & (3) \ \ U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) \cap Z^\delta_\beta = \varnothing.\end{aligned}$$

Let $(\delta_0, \beta_0) = \alpha_{e^*}$. We shall show $U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing$, which is a contradiction. By the definition of h_1,

$$\begin{aligned}U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) &= U^{(2)}_{h_1(e^*)}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \\ &= \textstyle\bigcup_{x \in Z^{\delta_0}_{\beta_0}} U^{(2)}_{(h_0(e^*,x)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}).\end{aligned}$$

Fix $x_0 \in Z^{\delta_0}_{\beta_0}$. Since $e^* \in G$, $h_0(e^*, x_0) \in G$, by the Disjointness Property. (This is because for $\delta \in S_2 \cap [0, \delta_0)$ we have $x_0 \notin P^\delta \cup Z^\delta_{f(\delta)}$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^\delta, Z^\delta_{f(\delta)}) = U^{(2)}_{e^*}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing,$$

where the latter holds since $e^* \in G$; for $\delta = \delta_0$ we have $x_0 \notin P^\delta \cup Z^\delta_{f(\delta)}$ and since $e^* \in G$ this implies

$$U^{(2)}_{h_0(e^*,x_0)}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing;$$

and for $\delta \in S_2 \cap (\delta_0, \underset{\sim}{\delta}^2_1)$ we have $x_0 \in P^\delta$ and so

$$U^{(2)}_{h_0(e^*,x_0)}(P^\delta, Z^\delta_{f(\delta)}) = \varnothing.)$$

So $\alpha_{h_0(e^*,x_0)}$ exists.

Subsubclaim. $\alpha_{h_0(e^*,x_0)} = \alpha_{e^*}$.

Proof. By the definition of h_0,

$$U^{(2)}_{h_0(e^*,x_0)}(P^\delta, Z^\delta_\beta) = \begin{cases} U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta) & \text{if } x_0 \notin P^\delta \cup Z^\delta_\beta \\ \varnothing & \text{if } x_0 \in P^\delta \cup Z^\delta_\beta \end{cases}$$

for $\delta \in S_2$ and $\beta < \eta_\delta$. So

$$U^{(2)}_{h_0(e^*,x_0)}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} = \varnothing,$$

since $x_0 \in P^{\delta_0} \cup Z^{\delta_0}_{\beta_0}$. And, when either $\delta = \delta_0$ and $\beta \in (f(\delta_0), \beta_0)$ or $\delta \in S_2 \cap [0, \delta_0)$ and $\beta \in [f(\delta), \eta_\delta)$, we have, by the Disjointness Property, $x_0 \notin P^\delta \cup Z^\delta_\beta$, hence

$$U^{(2)}_{h_0(e^*,x_0)}(P^\delta, Z^\delta_\beta) = U^{(2)}_{e^*}(P^\delta, Z^\delta_\beta),$$

Thus, $\alpha_{h_0(e^*,x_0)} = \alpha_{e^*}$. □

Since τ is winning for II,

$$(h_0(e^*, x_0)*\tau)_{II} \in G$$

and

$$\alpha_{(h_0(e^*,x_0)*\tau)_{II}} >_{\text{lex}} \alpha_{h_0(e^*,x_0)} = \alpha_{e^*}.$$

So

$$U^{(2)}_{(h_0(e^*,x_0)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing.$$

Since

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) = \bigcup_{x \in Z^{\delta_0}_{\beta_0}} U^{(2)}_{(h_0(e^*,x)*\tau)_{II}}(P^{\delta_0}, Z^{\delta_0}_{\beta_0})$$

we have

$$U^{(2)}_{e^*}(P^{\delta_0}, Z^{\delta_0}_{\beta_0}) \cap Z^{\delta_0}_{\beta_0} \neq \varnothing,$$

which is a contradiction. □

This completes the proof of Claim 2. □

We have a contradiction and therefore there is an e_1 is as desired. □

Notice that $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha)$, for variable α, allows us to pick out $f(\delta)$. Now we can consider the ordinal "$f(\underset{\sim}{\delta}^2_1)$" picked out in this fashion.

Claim C. There exists a $\beta_0 < \eta$ such that

(1) $U^{(2)}_{e_1}(P^{\underset{\sim}{\delta}^2_1}, Z^{\underset{\sim}{\delta}^2_1}_{\beta_0}) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^{\underset{\sim}{\delta}^2_1}, Z^{\underset{\sim}{\delta}^2_1}_{\beta}) \neq \varnothing$ for all $\beta \in (\beta_0, \eta)$, where

$$P^{\underset{\sim}{\delta}^2_1} = \bigcup\{Z^\delta_\alpha \mid \delta \in S_2 \text{ and } \alpha \in [f(\delta), \eta_\delta)\}.$$

Proof. Suppose for contradiction that the claim is false. The statement that the claim fails is a true Σ_1-statement about e_0, e_1, y_η, X, $\mathbb{R}$, f and S_2. But then by the Reflection Theorem (Theorem 4.6) this fact reflects to $\mathscr{F}_X$-almost all δ, which contradicts Claim B. □

Pick $y_f \in Q^{\underset{\sim}{\delta}^2_1}_{\beta_0}$. Now the statement that $y_f \in Q^{\underset{\sim}{\delta}^2_1}_{\beta_0}$ where β_0 is such that (1) and (2) of Claim C hold is a true Σ_1-statement about e_0, e_1, y_η, y_f, f, X, $\mathbb{R}$, and $\underset{\sim}{\delta}^2_1$. Thus, by Theorem 4.6, for $\mathscr{F}_X$-almost every $\delta < \underset{\sim}{\delta}^2_1$ this statement reflects. Let $S_3 \subseteq S_2$ be in μ_X and such that the above statement reflects to each point in S_3. Now by Claim B, for $\delta \in S_3$, the least β_0 such that $y_f \in Q^{\delta}_{\beta_0}$ is $f(\delta)$. Thus,

$$\{\delta \in S_0 \mid f_{y_f}(\delta) = f(\delta)\} \in \mu_X$$

and hence μ_X is strongly normal. □

This completes the proof of the following:

Theorem 4.13. *Assume* ZF + DC + AD. *Then, for each* $\lambda < \Theta^{L(\mathbb{R})}$,

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})} \text{ is } \lambda\text{-strong.}$$

4.14 Remark. For simplicity we proved Lemma 4.7 and Claim A of Theorem 4.12 using a proof by contradiction. This involves an appeal to determinacy. However, one can prove each result more directly, without appealing to determinacy.

Call a real y *suitable* if $(y)_i \in U_X$ for all $i < \omega$. Call a strategy σ a *proto-winning strategy* if σ is a winning strategy for I in $G^X(\underset{\sim}{\delta}^2_1)$. Thus if y is suitable and σ is a proto-winning strategy then

$$\{((\sigma * y)_I)_i, (y)_i \mid i < \omega\} \subseteq U_X$$

and so we can let

$$\delta_{(\sigma,y)} = \text{the least } \delta \text{ such that } \{((\sigma * y)_I)_i \mid i < \omega\} \cup \{(y)_i \mid i < \omega\} \subseteq U_\delta.$$

Let κ be least such that $X \in L_\kappa(\mathbb{R})$ and $L_\kappa(\mathbb{R}) \prec_1 L(\mathbb{R})$. This is the "least stable over X". It is easy to see that

$$\mathscr{P}(\mathbb{R}) \cap L_\kappa(\mathbb{R}) = \Delta_1(L(\mathbb{R}), \mathbb{R} \cup \{X, \underset{\sim}{\delta}^2_1, \mathbb{R}\})$$

and so if $\Sigma \in \mathscr{P}(\mathbb{R}) \cap L_\kappa(\mathbb{R})$ then for $\mathscr{F}_X$-almost all δ there is a reflected version Σ_δ of Σ. We can now state the relevant result:

Suppose $\Sigma \in \mathscr{P}(\mathbb{R}) \cap L_\kappa(\mathbb{R})$ is a set of proto-winning strategies for I. Then there is a proto-winning strategy σ such that for all suitable reals y, for all $\tau \in \Sigma_{\delta_{(\sigma,y)}} \cap \Sigma$, there is a suitable real y_τ such that

$$\delta_{(\sigma,y)} = \delta_{(\tau,y_\tau)}.$$

The proof of this is a variant of the above proofs and it provides a more direct proof of completeness and strong normality.

4.3. A Woodin Cardinal

We now wish to show that $\Theta^{L(\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(\mathbb{R})}$. In general, in inner model theory there is a long march up from strong cardinals to Woodin cardinals. However, in our present context, where we have the power of AD and are working with the special inner model $\mathrm{HOD}^{L(\mathbb{R})}$, this next step comes almost for free.

Theorem 4.15. *Assume* $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}$. *Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

Proof. For notational convenience let $\Theta = \Theta^{L(\mathbb{R})}$. To show that

$$\mathrm{HOD}^{L(\mathbb{R})} \models \mathrm{ZFC} + \Theta \text{ is a Woodin cardinal}$$

it suffices to show that for each $T \in \mathscr{P}(\Theta) \cap \mathrm{OD}^{L(\mathbb{R})}$, there is an ordinal δ_T such that

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC} + \delta_T \text{ is } \lambda\text{-}T\text{-strong},$$

for each $\lambda < \Theta$. Since Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$, $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies that $V_\Theta^{\mathrm{HOD}^{L(\mathbb{R})}}$ is a model of second-order ZFC. Thus, since $T \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ and $V_\Theta^{\mathrm{HOD}^{L(\mathbb{R})}} = \mathrm{HOD}^{L_\Theta(\mathbb{R})}$,

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC}.$$

It remains to establish strength. Since this is almost exactly as before we will just note the basic changes.

The model $L_\Theta(\mathbb{R})[T]$ comes with a natural Σ_1-stratification, namely,

$$\big\langle L_\alpha(\mathbb{R})[T\cap\alpha] \,\big|\, \alpha<\Theta\big\rangle.$$

Since Θ is regular in $L(\mathbb{R})$ and $L_\Theta(\mathbb{R})\models \mathrm{T}_0$, the set

$$\big\{\alpha<\Theta \mid L_\alpha(\mathbb{R})[T\cap\alpha]\prec L_\Theta(\mathbb{R})[T]\big\}$$

contains a club in Θ. To see this is note that for each $n<\omega$,

$$C_n=\big\{\alpha<\Theta \mid L_\alpha(\mathbb{R})[T\cap\alpha]\prec_n L_\Theta(\mathbb{R})[T]\big\}$$

is club (by Replacement) and, since Θ is regular, $\bigcap\{C_n \mid n<\omega\}$ is club. Thus, there are arbitrarily large $\alpha<\Theta$ such that

$$L_\alpha(\mathbb{R})[T\cap\alpha]\models \mathrm{T}_0.$$

For this reason OD_T, $<_{\mathrm{OD}_T}$ and HOD_T are Σ_1-definable in $L_\Theta(\mathbb{R})[T]$ exactly as before. (Here, as usual, we are working in the language of set theory supplemented with a predicate for T, which is assumed to be allowed in all of our definability calculations.)

Let

$$\delta_T = \text{ the least } \lambda \text{ such that } L_\lambda(\mathbb{R})[T\cap\lambda]\prec_1 L_\Theta(\mathbb{R})[T].$$

As will be evident, the relevant facts concerning $\underset{\sim}{\delta}^2_1$ carry over to the present context. For example, δ_T is the least ordinal λ such that

$$L_\lambda(\mathbb{R})[T\cap\lambda]\prec_1^{\mathbb{R}\cup\{\mathbb{R}\}} L_\Theta(\mathbb{R})[T].$$

The function $F_T:\delta_T\to L_{\delta_T}(\mathbb{R})[T\cap\delta_T]$ is defined as before as follows: Work in T_0. Suppose that $F_T{\restriction}\delta$ is defined. Let $\vartheta(\delta)$ be least such that

> $L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]\models \mathrm{T}_0$ and there is an $X\in L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]\cap \mathrm{OD}_T^{L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]}$ and
>
> ($\star$) there is a Σ_1-formula φ and a real z such that
>
> $$L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]\models\varphi[z,X,\delta_T,T\cap\vartheta(\delta),\mathbb{R}]$$
>
> and for all $\bar\delta<\delta$,
>
> $$L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]\not\models\varphi[z,F(\bar\delta),\bar\delta,T\cap\vartheta(\delta),\mathbb{R}]$$

(if such an ordinal exists) and then let $F_T(\delta)$ be the $(<_{\mathrm{OD}_T})^{L_{\vartheta(\delta)}(\mathbb{R})[T\cap\vartheta(\delta)]}$-least X such that $(\star)$.

The proof of the Reflection Theorem carries over exactly as before to establish the following: For all $X \in \mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$, for all Σ_1-formulas φ, and for all $z \in \omega^\omega$ if

$$L_\Theta(\mathbb{R})[T] \models \varphi[z, X, \delta_T, T, \mathbb{R}]$$

then there exists a $\delta < \delta_T$ such that

$$L_\Theta(\mathbb{R})[T] \models \varphi[z, F_T(\delta), \delta, T \cap \delta, \mathbb{R}].$$

Let U_X^T be a universal $\Sigma_1(L_\Theta(\mathbb{R})[T], \{X, \delta_T, T, \mathbb{R}\})$ set of reals and, for $\delta < \delta_T$, let U_δ^T be the universal $\Sigma_1(L_\Theta(\mathbb{R})[T], \{F_T(\delta), \delta, T\cap\delta, \mathbb{R}\})$ set obtained by using the same definition. For $z \in U_X^T$, let $S_z^T = \{\delta < \delta_T \mid z \in U_\delta^T\}$ and set

$$\mathscr{F}_X^T = \{S \subseteq \delta_T \mid \exists z \in U_X^T\,(S_z^T \subseteq S)\}.$$

As before, $\mathscr{F}_X^T$ is a countably complete filter and in the Reflection Theorem we can reflect to $\mathscr{F}_X^T$-many points $\delta < \delta_T$ and allow parameters $A \subseteq \delta_T$ and $f : \delta_T \to \delta_T$.

Fix an ordinal $\lambda < \Theta$. By the results of Section 3.3, there is an $\mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$-prewellordering $\leqslant_\lambda$ of ω^ω of length λ. Our interest is in applying the Reflection Theorem to

$$X = (\leqslant_\lambda, \lambda).$$

Working in $L_\Theta(\mathbb{R})[T]$, for each $S \subseteq \delta_T$, let $G_T^X(S)$ be the game

I	$x(0)$		$x(1)$		$x(2)$		$\ldots$
II		$y(0)$		$y(1)$		$\ldots$	

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X^T$. If the rule is violated then, letting i be the least such that either $(x)_i \notin U_X^T$ or $(y)_i \notin U_X^T$ I wins if $(x)_i \in U_X^T$; otherwise II wins. If the rule is satisfied then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta^T$, I wins iff $\delta \in S$.

Now set

$$\mu_X^T = \{S \subseteq \delta_T \mid \text{I wins } G_T^X(S)\}.$$

Notice that $\mu_X^T \in \mathrm{OD}_T^{L_\Theta(\mathbb{R})[T]}$. As before $\mathscr{F}_X^T \subseteq \mu_X^T$ and μ_X^T is a δ_T-complete ultrafilter.

Let

$$S_0 = \{\delta < \delta_T \mid F_T(\delta) = (\leqslant_\delta, \lambda_\delta) \text{ where } \leqslant_\delta \text{ is a prewellordering of length } \lambda_\delta \text{ and } L_{\lambda_\delta}(\mathbb{R})[T \cap \lambda_\delta] \models \mathrm{T}_0\}.$$

By reflection, $S_0 \in \mathscr{F}_X^T$.

As before we say that μ_X^T is *strongly normal* iff whenever $f : S_0 \to \delta_T$ is such that

$$\{\delta \in S_0 \mid f(\delta) < \lambda_\delta\} \in \mu_X^T$$

then there exists a $t \in \omega^\omega$ such that

$$\{\delta \in S_0 \mid f(\delta) = f_t(\delta)\} \in \mu_X^T.$$

The proof that μ_X^T is strongly normal is exactly as before. As in the proof of Lemma 4.8 we can use μ_X^T to take the ultrapower of $\mathrm{HOD}^{L_\Theta(\mathbb{R})[T]}$. In $L_\Theta(\mathbb{R})[T]$ form

$$\left(\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]}\right)^{\delta_T} / \mu_X^T.$$

As before we get an elementary embedding

$$j_\lambda : \mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \to M,$$

where M is the transitive collapse of the ultrapower. By completeness, this embedding has critical point δ_T and as in Lemma 4.10 the canonical functions witness that $j_\lambda(\delta_T) > \lambda$. Assuming further that λ is such that

$$L_\lambda(\mathbb{R})[T \cap \lambda] \prec_1 L_\Theta(\mathbb{R})[T]$$

we have that

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \subseteq M_\lambda.$$

As before, strong normality implies that

$$\begin{aligned} \rho : \lambda &\to \textstyle\prod \lambda_\delta / \mu_X^T \\ |t|_{\leqslant_\lambda} &\mapsto [f_t]_{\mu_X^T} \end{aligned}$$

is an isomorphism. It remains to establish T-strength, that is,

$$|t|_{\leqslant_\lambda} \in T \cap \lambda \ \text{ iff } \ \{\delta < \delta_T \mid f_t(\delta) \in T \cap \lambda_\delta\} \in \mu_X^T.$$

The point is that both

$$|t|_{\leqslant_\lambda} \in T \cap \lambda$$

and

$$|t|_{\leqslant\lambda} \notin T \cap \lambda$$

are $\Sigma_1(L_\Theta(\mathbb{R})[T], \{X, \delta_T, T, \mathbb{R}\})$ and so the result follows by the Reflection Theorem (Theorem 4.6) and the fact that $\mathscr{F}_X^T \subseteq \mu_X^T$.

Thus,

$$\mathrm{HOD}_T^{L_\Theta(\mathbb{R})[T]} \models \mathrm{ZFC} + \delta_T \text{ is } \lambda\text{-}T\text{-strong},$$

which completes the proof. □

In the above proof DC was only used in one place—to show that the ultrapowers were well-founded (Lemma 4.8). This was necessary since although the ultrapowers were ultrapowers of HOD and HOD satisfies AC, the ultrapowers were "external" (in that the associated ultrafilters were not in HOD) and so we had to assume DC in V to establish well-foundedness. However, this use of DC can be eliminated by using the extender formulation of being a Woodin cardinal. In this way one obtains strength through a network of "internal" ultrapowers (that is, via ultrafilters that live in HOD) and this enables one to bypass the need to assume DC in V. We will take this route in the next section.

5. Woodin Cardinals in General Settings

Our aim in this section is to abstract the essential ingredients from the previous construction and prove two abstract theorems on Woodin cardinals in general settings, one that requires DC and one that does not.

The first abstract theorem will be the subject of Section 5.1:

Theorem 5.1. *Assume* ZF + DC + AD*. Suppose X and Y are sets. Let*

$$\Theta_{X,Y} = \sup\{\alpha \mid \textit{there is an } \mathrm{OD}_{X,Y} \textit{ surjection } \pi : \omega^\omega \to \alpha\}.$$

Then

$$\mathrm{HOD}_X \models \mathrm{ZFC} + \Theta_{X,Y} \textit{ is a Woodin cardinal.}$$

There is a variant of this theorem (which we will prove in Section 5.4) where one can drop DC and assume less determinacy, the result being that Θ_X is a Woodin cardinal in HOD_X. The importance of the version involving $\Theta_{X,Y}$ is that it enables one to show that in certain settings HOD_X can have many Woodin cardinals. To describe one such key application we introduce the following notion due to Solovay. Assume ZF + $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = L(\mathscr{P}(\mathbb{R}))$ and work in $V = L(\mathscr{P}(\mathbb{R}))$. The sequence $\langle \Theta_\alpha \mid \alpha \leqslant \Omega \rangle$ is defined to be the shortest sequence such that Θ_0 is the supremum of all ordinals γ for which there is an OD surjection of ω^ω onto γ, $\Theta_{\alpha+1}$ is the supremum of all ordinals γ for which there is an OD surjection of $\mathscr{P}(\Theta_\alpha)$ onto γ, $\Theta_\lambda = \sup_{\alpha<\lambda} \Theta_\alpha$ for nonzero limit ordinals $\lambda \leqslant \Omega$, and $\Theta_\Omega = \Theta$.

Theorem 5.2. *Assume* ZF + $\mathrm{DC}_{\mathbb{R}}$ + AD + $V = L(\mathscr{P}(\mathbb{R}))$. *Then for each* $\alpha < \Omega$,

$$\mathrm{HOD} \models \mathrm{ZFC} + \Theta_{\alpha+1} \textit{ is a Woodin cardinal.}$$

The second abstract theorem provides a template that one can use in various contexts to generate inner models containing Woodin cardinals.

Theorem 5.3 (Generation Theorem). *Assume* ZF. *Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$,

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$,

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α,

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models$ *Strategic determinacy with respect to* B.

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

The motivation for the statement of the theorem—in particular, the notion of "strategic determinacy"—comes from the attempt to run the construction of Section 4.2 using lightface determinacy alone. In doing this one must simulate enough boldface determinacy to handle the real parameters that arise in that construction. To fix ideas we begin in Section 5.2 by examining a particular lightface setting, namely, $L[S, x]$ where S is a class of ordinals. Since $(\mathrm{OD}_{S,x})^{L[S,x]} = L[S,x]$ and $L[S,x]$ satisfies AC one cannot have boldface determinacy in $L[S,x]$. However, by assuming full determinacy in the background universe, strong forms of lightface determinacy hold in $L[S,x]$, for an S-cone of x. (The notion of an S-cone will be defined in Section 5.2). We will extract stronger and stronger forms of lightface determinacy until ultimately we reach the notion of "strategic determinacy", which is sufficiently rich to simulate boldface determinacy and drive the construction. With this motivation in place we will return to the general setting in Section 5.3 and prove the Generation Theorem. Finally, in Section 5.4 we will use the Generation Theorem as a template reprove the theorem of the previous section in $\mathrm{ZF}+\mathrm{AD}$ and to deduce a number of special cases, two of which are worth mentioning here:

Theorem 5.4. *Assume* $\mathrm{ZF}+\mathrm{AD}$. *Then for an S-cone of x,*

$$\mathrm{HOD}_S^{L[S,x]} \models \mathrm{ZFC} + \omega_2^{L[S,x]} \textit{ is a Woodin cardinal.}$$

Theorem 5.5. *Assume* $\mathrm{ZF}+\mathrm{AD}$. *Suppose Y is a set and $a \in H(\omega_1)$. Then for a Y-cone of x,*

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \mathrm{ZFC} + \omega_2^{\mathrm{HOD}_{Y,a,x}} \textit{ is a Woodin cardinal,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$.

(The notion of a Y-cone will be defined in Section 5.4). In Section 6 these two results will be used as the basis of a calibration of the consistency strength of lightface and boldface definable determinacy in terms of the large cardinal hierarchy. The second result will also be used to reprove and generalize Kechris' classical result that $\mathrm{ZF}+\mathrm{AD}$ implies that DC holds in $L(\mathbb{R})$. For this reason it is important to note that the theorem does not presuppose DC.

5.1. First Abstraction

Theorem 5.6. *Assume* $\mathrm{ZF}+\mathrm{DC}+\mathrm{AD}$. *Suppose X and Y are sets. Then*

$$\mathrm{HOD}_X \models \mathrm{ZFC} + \Theta_{X,Y} \textit{ is a Woodin cardinal.}$$

Proof. By Theorem 3.9,

$$\mathrm{HOD}_{X,Y} \models \Theta_{X,Y} \text{ is strongly inaccessible}$$

and so

$$\mathrm{HOD}_X \models \Theta_{X,Y} \text{ is strongly inaccessible.}$$

A direct approach to showing that in addition

$$\mathrm{HOD}_X \models \Theta_{X,Y} \text{ is a Woodin cardinal}$$

would be to follow Section 4.3 by showing that for each $T \in \mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_X$ there is an ordinal δ_T such that

$$\mathrm{HOD}_X \cap V_{\Theta_{X,Y}} \models \delta_T \text{ is } \lambda\text{-}T\text{-strong}$$

for each $\lambda < \Theta_{X,Y}$. However, such an approach requires that for each $\lambda < \Theta_{X,Y}$, there is a prewellordering of ω^ω of length λ which is OD in $L_{\Theta_{X,Y}}(\mathbb{R})[T]$ and in our present, more general setting we have no guarantee that this is true. So our strategy is to work with a larger model (where such prewellorderings exist), get the ultrafilters we need, and then pull them back down to $L_{\Theta_{X,Y}}(\mathbb{R})[T]$ by Kunen's theorem (Theorem 3.11).

We will actually first show that

$$\mathrm{HOD}_{X,Y} \models \Theta_{X,Y} \text{ is a Woodin cardinal.}$$

Let T be an element of $\mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_{X,Y}$ and let (by Lemma 3.7)

$$A = \langle A_\alpha \mid \alpha < \Theta_{X,Y} \rangle$$

be an $\mathrm{OD}_{X,Y}$ sequence such that each A_α is a prewellordering of ω^ω of length α. We will work with the structure

$$L_{\Theta_{X,Y}}(\mathbb{R})[T, A]$$

and the natural hierarchy of structures that it provides.

To begin with we note some basic facts. First, notice that

$$\Theta_{X,Y} = (\Theta_{T,A})^{L(\mathbb{R})[T,A]} = \Theta^{L(\mathbb{R})[T][A]}.$$

(For the first equivalence we have

$$(\Theta_{T,A})^{L(\mathbb{R})[T,A]} \geqslant \Theta_{X,Y}$$

because of A and we have

$$(\Theta_{T,A})^{L(\mathbb{R})[T,A]} \leqslant \Theta_{X,Y}$$

because $L(\mathbb{R})[T, A]$ is $\mathrm{OD}_{X,Y}$. The second equivalence holds since every element in $L(\mathbb{R})[T, A]$ is $\mathrm{OD}_{T,A,y}^{L(\mathbb{R})[T][A]}$ for some $y \in \omega^\omega$. So the "averaging over reals" argument of Lemma 3.8 applies.) It follows that our earlier arguments generalize. For example, by the proof of Theorem 3.10,

$$\Theta_{X,Y} \text{ is strongly inaccessible in } \mathrm{HOD}^{L(\mathbb{R})[T,A]}$$

and

$$\Theta_{X,Y} \text{ is regular in } L(\mathbb{R})[T, A].$$

(Note that $\Theta_{X,Y}$ need not be regular in V. For example, assuming $\mathrm{ZF} + \mathrm{DC} + \mathrm{AD}_{\mathbb{R}}$, Θ_0 has cofinality ω in V.) Moreover, the proof of Lemma 2.21 shows that

$$L_{\Theta_{X,Y}}(\mathbb{R})[T, A] \models \mathrm{T}_0$$

and the proof of Lemma 2.23 shows that

$$L_{\Theta_{X,Y}}(\mathbb{R})[T, A] \prec_1 L(\mathbb{R})[T, A].$$

This implies (in conjunction with the fact that $\Theta_{X,Y}$ is regular in $L(\mathbb{R})[T, A]$) that

$$\{\alpha < \Theta_{X,Y} \mid L_\alpha(\mathbb{R})[T{\restriction}\alpha, A{\restriction}\alpha] \prec L_{\Theta_{X,Y}}(\mathbb{R})[T, A]\}$$

is club in $\Theta_{X,Y}$ and hence that each such level satisfies T_0.

So we are in exactly the situation of Section 4.3 except that now the prewellorderings are explicitly part of the structure. The proof of Theorem 4.15 thus shows that: For each $T \in \mathscr{P}(\Theta_{X,Y}) \cap \mathrm{OD}_{X,Y}$ there is an ordinal $\delta_{T,A}$ such that

$$\mathrm{HOD}_{T,A}^{L_{\Theta_{X,Y}}(\mathbb{R})[T,A]} \models \delta_{T,A} \text{ is } \lambda\text{-}T\text{-strong}$$

for each $\lambda < \Theta_{X,Y}$, as witnessed by an ultrafilter μ_λ^T on $\delta_{T,A}$. These ultrafilters are $\mathrm{OD}_{T,A}^{L_{\Theta_{X,Y}}(\mathbb{R})[T,A]}$.

The key point is that all of these ultrafilters μ_λ^T are actually OD by Kunen's theorem (Theorem 3.11). This is where DC is used.

Now we return to the smaller model $L_{\Theta_{X,Y}}(\mathbb{R})[T]$. Since $\Theta_{X,Y}$ is strongly inaccessible in HOD_X there is a set $H \in \mathscr{P}(\Theta_{X,Y})^{L_{\Theta_{X,Y}}(\mathbb{R})[T]}$ such that

$$\mathrm{HOD}_X \cap V_{\Theta_{X,Y}} = L_{\Theta_{X,Y}}[H].$$

We may assume without loss of generality that H is folded into T. Thus

$$\mathrm{HOD}_T^{L_{\Theta_{X,Y}}(\mathbb{R})[T]} = \mathrm{HOD}_X \cap V_{\Theta_{X,Y}}$$

and this structure contains all of the ultrafilters μ_λ^T. These ultrafilters can now be used (as in the proof of Lemma 4.8) to take the ultrapower and so we have

$$\mathrm{HOD}_T^{L_{\Theta_{X,Y}}(\mathbb{R})[T]} \models \delta_{T,A} \text{ is } \lambda\text{-}T\text{-strong},$$

which completes the proof. □

5.2. Strategic Determinacy

Let us now turn to the Generation Theorem. We shall begin by motivating the notion of "strategic determinacy" by examining the special case of $L[S,x]$ where S is a class of ordinals.

For $x \in \omega^\omega$, the *S-degree of x* is $[x]_S = \{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$. The *$S$-degrees* are the sets of the form $[x]_S$ for some $x \in \omega^\omega$. Let $\mathscr{D}_S = \{[x]_S \mid x \in \omega^\omega\}$. Define $x \leqslant_S y$ to hold iff $x \in L[S,y]$ and define the notions $x \equiv_S y$, $x <_S y$, $x \geqslant_S y$, $[x]_S \leqslant_S [y]_S$ in the obvious way. A *cone of S-degrees* is a set of the form $\{[y]_S \mid y \geqslant_S x_0\}$ for some $x_0 \in \omega^\omega$. An *S-cone of reals* is a set of form $\{y \in \omega^\omega \mid y \geqslant_S x_0\}$ for some $x_0 \in \omega^\omega$. The *cone filter on $\mathscr{D}_S$* is the filter consisting of sets of S-degrees that contain a cone of S-degrees. Given a formula $\varphi(x)$ we say that *φ holds for an S-cone of x* if there is a real x_0 such that for all $y \geqslant_S x_0$, $L[S,y] \models \varphi(y)$. The proof of the Cone Theorem (Theorem 2.9) generalizes.

Theorem 5.7 (Martin). *Assume* ZF + AD. *The cone filter on $\mathscr{D}_S$ is an ultrafilter.*

Proof. For $A \subseteq \mathscr{D}_S$ consider the game

I	$x(0)$		$x(1)$		$x(2)$		...
II		$y(0)$		$y(1)$		...	

where I wins iff $[x*y]_S \in A$. If I has a winning strategy σ_0 then σ_0 witnesses that A is in the S-cone filter since for $y \geqslant_S \sigma_0$, $[y]_S = [\sigma_0 * y]_S \in A$. If II has a winning strategy τ_0 then τ_0 witnesses that $\mathscr{D}_S \smallsetminus A$ is in the S-cone filter since for $x \geqslant_S \tau_0$, $[x]_S = [x * \tau_0]_S \in \mathscr{D}_S \smallsetminus A$. □

It follows that each statement φ either holds on an S-cone or fails on an S-cone. In fact, the entire theory stabilizes. However, in order to fully articulate this fact one needs to invoke second-order assumptions (like the existence of a satisfaction relation). Without invoking second-order assumptions one has the following:

Corollary 5.8. *Assume* ZF + AD. *For each $n < \omega$, there is an x_n such that for all $x \geqslant_S x_n$,*

$$L[S,x] \models \varphi \quad \textit{iff} \quad L[S,x_n] \models \varphi,$$

for all Σ^1_n-sentences φ.

Proof. Let $\langle \varphi_i \mid i < \omega \rangle$ enumerate the Σ^1_n-sentences of the language of set theory and, for each i, let y_i be the base of an S-cone settling φ_i. Now using $\mathrm{AC}_\omega(\mathbb{R})$ (which is provable in ZF + AD) let x_n encode $\langle y_i \mid i < \omega \rangle$. □

A natural question then is: "What is the stable theory?"

Theorem 5.9. *Assume* ZF + AD. *Then for an S-cone of x,*

$$L[S,x] \models \mathrm{CH}.$$

Proof. Suppose for contradiction (by Theorem 5.7) that ¬CH holds for an S-cone of x. Let x_0 be the base of this cone.

We will arrive at a contradiction by producing an $x \geqslant_S x_0$ such that $L[S,x] \models \mathrm{CH}$. This will be done by forcing over $L[S,x_0]$ in two stages, first to get CH and then to get a real coding this generic (while preserving CH). It will be crucial that the generics actually exist.

Claim. ω_1^V is strongly inaccessible in any inner model M of AC.

Proof. We first claim that there is no ω_1^V-sequence of distinct reals: Let μ be the club filter on ω_1^V. By Solovay's theorem (Theorem 2.12, which doesn't require DC) μ is a countably complete ultrafilter on ω_1^V. Suppose $\langle a_\alpha \mid \alpha < \omega_1^V \rangle$ is a sequence of characteristic functions for distinct reals. By countable completeness there is a μ-measure one set X_n of elements of this sequence that agree on their n^{th}-coordinate. Thus, $\bigcap_{n<\omega} X_n$ has μ-measure one, which is impossible since it only has one member.

It follows that for each $\gamma < \omega_1^V$, $(2^\gamma)^M < \omega_1^V$ since otherwise (γ being countable) there would be an ω_1^V sequence of distinct reals. Since ω_1^V is clearly regular in M the result follows. □

Step 1. Let G be $L[S,x_0]$-generic for $\mathrm{Col}(\omega_1^{L[S,x_0]}, \mathbb{R}^{L[S,x_0]})$. (By the Claim this generic exists in V). So

$$L[S,x_0][G] \models \mathrm{CH} \text{ and } \mathbb{R}^{L[S,x_0][G]} = \mathbb{R}^{L[S,x_0]}.$$

The trouble is that $L[S, x_0][G]$ is not of the form $L[S, x]$ for $x \in \mathbb{R}$. (We could code G via a real by brute force but doing so might destroy CH. A more delicate approach is needed.)

STEP 2. Code G using almost disjoint forcing: First, view G as a subset of $\omega_1^{L[S,x_0]}$ by letting $A \subseteq \omega_1^{L[S,x_0]}$ be such that

$$L[S, x_0][G] = L[S, x_0, A].$$

Now let

$$\langle \sigma_\alpha \mid \alpha < \omega_1^{L[S,x_0]} \rangle \in L[S, x_0]$$

be a sequence of infinite almost disjoint subsets of ω (that is, such that if $\alpha \neq \beta$ then $\sigma_\alpha \cap \sigma_\beta$ is finite). By almost disjoint forcing, in $L[S, x_0, A]$ there is a c.c.c. forcing $\mathbb{P}_A$ of size $\omega_1^{L[S,x_0,A]}$ such that if $H \subseteq \mathbb{P}_A$ is $L[S, x_0, A]$-generic then there is a $c(A) \subseteq \omega$ such that

$$\alpha \in A \text{ iff } c(A) \cap \sigma_\alpha \text{ is infinite.}$$

(See [1, pp.267-8] for details concerning this forcing notion.) Also

$$L[S, x_0, A][H] = L[S, x_0, A][c(A)] = L[S, x_0, c(A)].$$

Finally,

$$L[S, x_0, c(A)] \models \mathrm{CH}$$

as $\mathbb{P}_A$ is c.c.c., $|\mathbb{P}_A| = \omega_1^{L[S,x_0,A]}$, and $L[S, x_0, A] \models \mathrm{CH}$, and so there are, up to equivalence, only $\omega_1^{L[S,x_0,A]}$-many names for reals. □

Corollary 5.10. *Assume* $\mathrm{ZF} + \mathrm{AD}$. *For an* S*-cone of* x*,*

$$L[S, x] \models \mathrm{GCH} \textit{ below } \omega_1^V.$$

Proof. Let x_0 be such that for all $x \geqslant_S x_0$,

$$L[S, x] \models \mathrm{CH}.$$

Fix $x \geqslant_S x_0$. We claim that $L[S, x] \models \mathrm{GCH}$ below ω_1^V: Suppose for contradiction that there is a $\lambda < \omega_1^V$ such that $L[S, x] \models 2^\lambda > \lambda^+$. Let $G \subseteq \mathrm{Col}(\omega, \lambda)$ be $L[S, x]$-generic. Thus $L[S, x][G] \models \neg\mathrm{CH}$. But $L[S, x][G] = L[S, y]$ for some real y and so $L[S, x][G] \models \mathrm{CH}$. □

A similar proof shows that $\Diamond$ holds for an S-cone of x, the point being that adding a Cohen subset of ω_1 forces $\Diamond$ and this forcing is c.c.c. and of size ω_1. See [1], Exercises 15.23 and 15.24.

5.11 Conjecture. Assume ZF + AD. For an S-cone of x,

$$L[S,x] \cap V_{\omega_1^V}$$

is an "L-like" model in that it satisfies Condensation, $\Box$, Morasses, etc.

Corollary 5.10 tells us that for an S-cone of x,

$$\Theta^{L[S,x]} = (\mathfrak{c}^+)^{L[S,x]} = \omega_2^{L[S,x]}.$$

Thus, to prove that for an S-cone of x,

$$L[S,x] \models \omega_2 \text{ is a Woodin cardinal in } \mathrm{HOD}_S,$$

we can apply our previous construction concerning Θ provided we have enough determinacy in $L[S,x]$.

Theorem 5.12 (Kechris and Solovay). *Assume* ZF + AD. *For an S-cone of x,*

$$L[S,x] \models \mathrm{OD}_S\textit{-determinacy.}$$

Proof. Play the following game

I	a, b
II	c, d

where, letting $p = \langle a,b,c,d\rangle$, I wins if $L[S,p] \not\models \mathrm{OD}_S$-determinacy and $L[S,p] \models$ "$a{*}d \in A^p$", where A^p is the least (in the canonical ordering) undetermined $\mathrm{OD}_S^{L[S,p]}$ set in $L[S,p]$. In such a game the reals are played so as to be "interleaved" in the pattern $(a(0), c(0), b(0), d(0), \dots)$. Here the two players are to be thought of as cooperating to determine the playing field $L[S,p]$ in which they will simultaneously play (via a and d) an auxiliary round of the game on the least undetermined OD_S set A^p (assuming, of course, that such a set exists, as I is trying to ensure.)

Case 1: I has a winning strategy σ_0.

We claim that for all $x \geqslant_S \sigma_0$, $L[S,x] \models \mathrm{OD}_S$-determinacy, which contradicts the assumption that σ_0 is a winning strategy for I. For consider such a real x and suppose for contradiction that $L[S,x] \not\models \mathrm{OD}_S$-determinacy. As above let $A^x \in \mathrm{OD}_S^{L[S,x]}$ be least such that A^x is not determined. We will arrive at a contradiction by deriving a winning strategy σ for I in A^x from the strategy σ_0. Run the game according to σ_0 while having Player II feed in x for c and playing some auxiliary play $d \in L[S,x]$. This ensures that the resulting model $L[S,p]$ that the two players jointly determine is just $L[S,x]$ and so $A^p = A^x$. We can now derive a winning strategy σ for I in A^x from σ_0 as follows: For $d \in L[S,x]$, let σ be the strategy such that $(\sigma * d)_I$ is the a such that $(\sigma_0 * \langle x, d\rangle)_I = \langle a, b\rangle$.

(It is crucial that we have II play $c = x$ and $d \in L[S,x]$ since otherwise we would get $a * d \in A^p$ for varying p. By having II play $c = x$ and $d \in L[S,x]$, II has "steered into the right model", namely $L[S,x]$, and we have "fixed" the set A^x. This issue will become central later on when we refine this proof.)

Case 2: II has a winning strategy τ_0.

We claim that for $x \geqslant_S \tau_0$, $L[S,x] \models \mathrm{OD}_S$-determinacy. This is as above except that now we run the game according to τ_0, having I steer into $L[S,x]$ by playing x for b and some $a \in L[S,x]$. Then, as above, we derive a winning strategy for I in $\omega^\omega \smallsetminus A^x$ and hence a winning strategy τ for II in A^x. $\square$

To drive the construction of a model containing a Woodin cardinal we need more than OD_S-determinacy since some of the games in the construction are definable in a real parameter. Unfortunately, we cannot hope to get

$$L[S,x] \models \mathrm{OD}_{S,y}\text{-determinacy}$$

for each y since $(\mathrm{OD}_{S,x})^{L[S,x]} = L[S,x]$ and $L[S,x]$ is a model of AC. Nevertheless, it is possible to have $\mathrm{OD}_{S,y}$-determinacy in $L[S,x]$ for certain specially chosen reals y. There is therefore hope of approximating a sufficient amount of boldface definable determinacy to drive the construction. To make precise the approximation we need, we introduce the notion of a "prestrategy".

Let A and B be sets of reals. A *prestrategy for I* (*respectively II*) *in* A is a continuous function f such that for all $x \in \omega^\omega$, $f(x)$ is a strategy for I (respectively II) in A. A prestrategy f in A (for either I or II) is *winning with respect to the basis* B if, in addition, for all $x \in B$, $f(x)$ is a winning strategy in A. The *strategic game with respect to the predicates* $P_1, \ldots, P_k$

and the basis B is the game $SG^B_{P_0,\dots,P_k}$

$$\begin{array}{lcccccc} \text{I} & A_0 & & \cdots & A_n & \cdots & \\ \text{II} & & f_0 & & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{P_0,\dots,P_k}$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{P_0,\dots,P_k,f_0,\dots,f_n}$ and

(2) f_n is a prestrategy for A_n that is winning with respect to B,

and II wins iff II can play all ω rounds. We say that *strategic definable determinacy holds with respect to the predicates $P_0, \dots, P_k$ and the basis B* ($\mathrm{ST}^B_{P_0,\dots,P_k}$*-determinacy*) if II wins $SG^B_{P_0,\dots,P_k}$ and we say that *strategic definable determinacy for n moves holds with respect to the predicates $P_0, \dots, P_k$ and the basis B* ($\mathrm{ST}^B_{P_0,\dots,P_k}$*-determinacy for n moves*) if II can play n rounds of $SG^B_{P_0,\dots,P_k}$. When these parameters are clear from context we shall often simply refer to SG and ST-determinacy.

In the context of $L[S,x]$ the predicate will be S and the basis B will be the S-degree of x. Thus to say that $L[S,x]$ satisfies ST^B_S-determinacy (or ST-determinacy for short) is to say that II can play all rounds of the game

$$\begin{array}{lcccccc} \text{I} & A_0 & & \cdots & A_n & \cdots & \\ \text{II} & & f_0 & & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^{L[S,x]}_S$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^{L[S,x]}_{S,f_0,\dots,f_n}$, and

(2) $f_n \in L[S,x]$ is a prestrategy for A_n that is winning with respect to $[x]_S$.

The ability to survive a single round of this game implies that $L[S,x]$ satisfies OD_S-determinacy. So this notion is indeed a generalization of OD_S-determinacy.

Before turning to the main theorems, some remarks are in order. First, notice that the games $\mathrm{ST}^B_{P_0,\dots,P_k}$ are closed for Player II, hence determined. The only issue is whether II wins.

Second, notice also that if I wins then I has a *canonical* strategy. This can be seen as follows: Player I can rank partial plays, assigning rank 0 to partial plays in which he wins; Player I can then play by reducing rank. The result is a quasi-strategy that is definable in terms of the tree of partial plays which in turn is ordinal definable. Since I is essentially playing ordinals this quasi-strategy can be converted into a strategy in a definable fashion. We take this to be I's canonical strategy.

Third, notice that each prestrategy can be coded by a real number in a canonical manner. We assume that such a coding has been fixed and, for notational convenience, we will identify a prestrategy with its code.

Fourth, it is important to note that if II is to have a hope of winning then we must allow II to play prestrategies and not strategies. To see this, work in $L[S,x]$ and consider the variant of SG^B_S where we have II play strategies τ_0, $\tau_1, \ldots$ instead of prestrategies. The set $A_0 = \{y \in \omega^\omega \mid L[S, y_{\text{even}}] = L[S,x]\}$ is $\mathrm{OD}^{L[S,x]}_S$ and hence a legitimate first move for I. But then II's response must be a winning strategy for I in A_0 since I can win a play of A_0 by playing x. However, $\mathrm{OD}^{L[S,x]}_{S,\tau_0} = L[S,x]$ and so in the next round I is allowed to play any $A_1 \in L[S,x]$. But then II cannot hope to always respond with a winning strategy since $L[S,x] \not\models \mathrm{AD}$. The upshot is that if II is to have a hope of winning a game of this form then we must allow II to be less committal.

Fifth, although one can use a base B which is slightly larger than $[x]_S$, the previous example motivates the choice of $B = [x]_S$. Let A_0 be as in the previous paragraph and let f_0 be II's response. By the above argument, it follows that for all $z \in B$, $\langle f_0, z\rangle \in [x]_S$ and so in a sense we are "one step away" from showing that one must have $B \subseteq [x]_S$.

Finally, as we shall show in the next section, for every OD basis $B \subseteq \omega^\omega$ there is an OD set $A \subseteq \omega^\omega$ such that there is no OD prestrategy which is winning for A with respect to B (Theorem 6.11). Thus, for each basis B, ST^B-determinacy does not trivially reduce to OD-determinacy.

Theorem 5.13. *Assume* ZF + AD*. Then for an S-cone of x, for each n,*

$$L[S,x] \models \mathrm{ST}^B_S\text{-}determinacy\ for\ n\ moves,$$

where $B = [x]_S$.

Theorem 5.14. *Assume* $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD}$*. Then for an S-cone of x,*

$$L[S,x] \models \mathrm{ST}^B_S\text{-}determinacy,$$

where $B = [x]_S$.

Proofs of theorems 5.13 and 5.14. Assume toward a contradiction that the statement of Theorem 5.14 is false. By Theorem 5.7, there is a real x_0 such that if $x \geqslant_T x_0$,

$$L[S,x] \models \text{I wins } SG,$$

(where here and below we drop reference to S and B since these are fixed throughout). For $x \geqslant_T x_0$, let σ^x be I's canonical winning strategy in $SG^{L[S,x]}$. Note that the strategy depends only on the model, that is, if $y \equiv_S x$ then $\sigma^y = \sigma^x$.

Our aim is to construct a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that the winning strategies (for whichever player wins) enable us to define, for an S-cone of x, prestrategies $f^x_0, f^x_1, \ldots, f^x_n, \ldots$ which constitute a non-losing play against σ^x in $SG^{L[S,x]}$.

Step 0. Consider (in V) the game G_0

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d, x_0\rangle$ and $A^p_0 = \sigma^p(\varnothing)$, I wins iff $a{*}d \in A^p_0$. Notice that by including x_0 in p we have ensured that σ^p is defined and hence that the winning condition makes sense. In this game I and II are cooperating to steer into the model $L[S,p]$ and they are simultaneously playing (via a and d) an auxiliary round of the game A^p_0, where A^p_0 is I's first move according to the canonical strategy in the strategic game $SG^{L[S,p]}$. I wins a round iff I wins the auxiliary round of this auxiliary game.

Claim 1. There is a real x_1 such that for all $x \geqslant_S x_1$ there is a prestrategy f^x_0 that is a non-losing first move for II against σ^x in $SG^{L[S,x]}$.

Proof. Case 1: I has a winning strategy σ_0 in G_0.

For $x \geqslant_T \sigma_0$, let f^x_0 be the prestrategy derived from σ_0 by extracting the response in the auxiliary game where we have II feed in y for c, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f^x_0(y)$ be such that $f^x_0(y){*}d = a{*}d$ where a is such that $(\sigma_0{*}\langle y, d\rangle)_I = \langle a, b\rangle$. Note that $f^x_0 \in L[S,x]$ as it is definable from σ_0. Let $x_1 = \langle \sigma_0, x_0\rangle$ and for $x \geqslant_S x_1$ let $A^x_0 = \sigma^x(\varnothing)$. We claim that for $x \geqslant_S x_1$, f^x_0 is a prestrategy for I in A^x_0 that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f^x_0 is a non-losing first move for II against σ^x in $SG^{L[S,x]}$. To see this fix $x \geqslant_S x_1$ and y such that $L[S,y] = L[S,x]$ and consider $d \in L[S,x]$. The value $f^x_0(y)$ of the prestrategy was defined by running G_0, having II feed in y for c:

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & y, d \end{array}$$

By our choice of y and d, we have solved the "steering problem", that is, we have $L[S,p] = L[S,x]$ and $A_0^p = A_0^x$ where $p = \langle a, b, y, d, x_0 \rangle$. Now, f_0^x is such that $f_0^x(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d \rangle)_I = \langle a, b \rangle$. Since σ_0 is winning for I, we have $f_0^x(y) * d = a * d \in A_0^p = A_0^x$.

Case 2: II has a winning strategy τ_0 in G_0.

Let f_0^x be the prestrategy derived from τ_0 by extracting the response in the auxiliary game where we have I feed in y for b, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f_0^x(y)$ be such that $a * f_0^x(y) = a * d$ where d is such that $(\langle a, y \rangle * \tau_0)_{II} = \langle c, d \rangle$. Let $x_1 = \langle \tau_0, x_0 \rangle$ and for $x \geqslant_S x_1$ let $A_0^x = \sigma^x(\varnothing)$. As before, we have that for $x \geqslant_S x_1$, f_0^x is a prestrategy for II in A_0^x that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f_0^x is a non-losing first move for II against σ^x in $SG^{L[S,x]}$.

Let x_1 be as described in whichever case holds. □

Step n+1. Assume that we have defined games $G_0, \ldots, G_n$, reals $x_0, \ldots, x_{n+1}$ such that $x_0 \leqslant_S x_1 \leqslant_S \cdots \leqslant_S x_{n+1}$, and prestrategies $f_0^x, \ldots, f_n^x$ which depend only on the degree of x and such that for all $x \geqslant_S x_{n+1}$,

$$f_0^x, \ldots, f_n^x$$

is a non-losing partial play for II against σ^x in $SG^{L[S,x]}$.

Consider (in V) the game G_{n+1}

$$\begin{array}{ll} \text{I} & a, b \\ \text{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d, x_{n+1} \rangle$ and A_{n+1}^p be I's response via σ^p to the partial play $f_0^p, \ldots, f_n^p$, I wins iff $a * d \in A_{n+1}^p$. Notice that we have included x_{n+1} in p to ensure that $\sigma^p, f_0^p, \ldots, f_n^p$ are defined and hence that the winning condition makes sense. In this game I and II are cooperating to steer into the model $L[S,p]$ and they are simultaneously playing an auxiliary round (via a and d) on A_{n+1}^p, where A_{n+1}^p is I's response via σ^p to II's non-losing partial play $f_0^p, \ldots, f_n^p$ in the strategic game $SG^{L[S,p]}$. I wins a round iff he wins the auxiliary round of this auxiliary game.

Claim 2. There is a real x_{n+2} such that for all $x \geqslant_S x_{n+2}$ there is a prestrategy f_{n+1}^x such that $f_0^x, \ldots, f_n^x, f_{n+1}^x$ is a non-losing partial play for II against σ^x in $SG^{L[S,x]}$.

Proof. Case 1: I has a winning strategy σ_{n+1} in G_{n+1}.

Let f^x_{n+1} be the prestrategy derived from σ_{n+1} by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f^x_{n+1}(y)$ be such that $f^x_{n+1}(y)*d = a*d$ where a is such that $(\sigma_{n+1}*\langle y,d\rangle)_I = \langle a,b\rangle$. Let $x_{n+2} = \langle \sigma_{n+1}, x_{n+1}\rangle$ and for $x \geqslant_S x_{n+2}$ let $A^x_{n+1} = \sigma^x(\langle f^x_0,\ldots,f^x_n\rangle)$, i.e. A^x_{n+1} is the $(n+2)^{\text{nd}}$ move of I in $SG^{L[S,x]}$ following σ^x against II's play of $f^x_0,\ldots,f^x_n$. As in Claim 1, f^x_{n+1} is a prestrategy for I in A^x_{n+1} that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f^x_{n+1} is a non-losing $(n+2)^{\text{nd}}$ move for II against σ^x in $SG^{L[S,x]}$.

Case 2: II has a winning strategy τ_{n+1} in G_{n+1}.

Let f^x_{n+1} be the prestrategy derived from τ_{n+1} by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L[S,x]}$ let $f^x_{n+1}(y)$ be such that $a*f^x_{n+1}(y) = a*d$ where d is such that $(\langle a,y\rangle*\tau_{n+1})_{II} = \langle c,d\rangle$. Let $x_{n+2} = \langle \tau_{n+1}, x_{n+1}\rangle$ and for $x \geqslant_S x_{n+2}$ let $A^x_{n+1} = \sigma^x(\langle f^x_0,\ldots,f^x_n\rangle)$, as above. As before, we have that for $x \geqslant_S x_{n+2}$, f^x_{n+1} is a prestrategy for II in A^x_{n+1} that is winning with respect to $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$, that is, f^x_{n+1} is a non-losing $(n+2)^{\text{nd}}$ move for II against σ^x in $SG^{L[S,x]}$.

Let x_{n+2} be as described in whichever case holds. □

Finally, using $\mathrm{DC}_{\mathbb{R}}$, we get a sequence of reals $x_0,\ldots,x_n,\ldots$ and prestrategies $f^x_0,\ldots,f^x_n,\ldots$ as in each of the steps. Letting $x^\infty \geqslant_S x_n$, for all n, we have that for all $x \geqslant_S x^\infty$, $f^x_0,\ldots,f^x_n,\ldots$ is a non-losing play for II against σ^x in $SG^{L[S,x]}$, which is a contradiction. This completes the proof of Theorem 5.14.

For Theorem 5.13 simply note that $\mathrm{DC}_{\mathbb{R}}$ is not needed to define the finite sequences $x_0,\ldots,x_n,x_{n+1}$ and $f^x_0,\ldots,f^x_n$ for $x \geqslant_S x_{n+1}$ (as these prestrategies are definable from $x_0,\ldots,x_n,x_{n+1}$). □

5.3. Generation Theorem

In the previous section we showed (assuming ZF + AD) that for an S-cone of x,

$$L[S,x] \models \mathrm{OD}_S\text{-determinacy},$$

and (even more) that for each n,

$$L[S,x] \models \mathrm{ST}^B_S\text{-determinacy for } n \text{ moves},$$

where $B = [x]_S$. It turns out that for a sufficiently large choice of n this degree of determinacy is sufficient to implement the previous arguments and show that

$$L[S, x] \models \omega_2 \text{ is a Woodin cardinal in } \mathrm{HOD}_S.$$

At this stage we could proceed directly to this result but instead, with this motivation behind us, we return to the more general setting. The main theorem to be proved is the Generation Theorem:

Theorem 5.15 (Generation Theorem). *Assume* ZF. *Suppose*

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

is such that

(1) $M \models \mathrm{T}_0$,

(2) Θ_M *is a regular cardinal,*

(3) $T \subseteq \Theta_M$,

(4) $A = \langle A_\alpha \mid \alpha < \Theta_M \rangle$ *is such that* A_α *is a prewellordering of the reals of length greater than or equal to* α,

(5) $B \subseteq \omega^\omega$ *is nonempty, and*

(6) $M \models \mathrm{ST}^B_{T,A,B}$*-determinacy for four moves.*

Then

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

The importance of the restriction to strategic determinacy for four moves is that in a number of applications of this theorem strategic determinacy for n moves (for each n) can be established without any appeal to DC (as for example in Theorem 5.13) in contrast to full strategic determinacy which (just as in Theorem 5.14) uses $\mathrm{DC}_\mathbb{R}$.

The external assumption that Θ_M is a regular cardinal is merely for convenience—it ensures that there are cofinally many stages in the stratification of M where T_0 holds. The dedicated reader can verify that this assumption can be dropped by working instead with the theory $\mathrm{ZF_N} + \mathrm{AC}_\omega(\mathbb{R})$ for some sufficiently large N.

The remainder of this section is devoted to a proof of the Generation Theorem.

Proof. Let us start by showing that $\mathrm{HOD}^M_{T,A,B}$ satisfies ZFC. When working with structures of the form $L_{\Theta_M}(\mathbb{R})[T,A,B]$ it is to be understood that we are working in the language of ZFC augmented with constant symbols for T, A, B, and $\mathbb{R}$. The first step is to show that $\mathrm{HOD}^M_{T,A,B}$ is first-order over M. For $\gamma < \Theta_M$, let

$$M_\gamma = L_\gamma(\mathbb{R})[T\restriction\gamma, A\restriction\gamma, B],$$

it being understood that the displayed predicates are part of the structure. Since Θ_M is regular and $M \models \mathrm{T}_0$ there are cofinally many $\gamma < \Theta_M$ such that $M_\gamma \models \mathrm{T}_0$. So a set $x \in M$ is $\mathrm{OD}^M_{T,A,B}$ if and only if there is a $\gamma < \Theta_M$ such that $M_\gamma \models \mathrm{T}_0$ and x is definable in M_γ from ordinal parameters (and the constant symbols for the parameters). It follows that $\mathrm{OD}^M_{T,A,B}$ and $\mathrm{HOD}^M_{T,A,B}$ are Σ_1-definable over M (in the expanded language).

With this first-order characterization of $\mathrm{HOD}^M_{T,A,B}$ all of the standard results carry over to our present setting. For example, since $M \models \mathrm{ZF} -$ Power Set we have that $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} -$ Power Set. (The proofs that AC holds in $\mathrm{HOD}^M_{T,A,B}$ and that for all $\alpha < \Theta_M$, $V_\alpha \cap \mathrm{HOD}^M_{T,A,B} \in \mathrm{HOD}^M_{T,A,B}$ require that $\mathrm{OD}^M_{T,A,B}$ be ordinal definable.)

Lemma 5.16. $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC}$.

Proof. We have seen that $\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} - \text{Power Set}$. Since $\mathrm{HOD}^M_{T,A,B} \models$ AC it remains to show that for all $\lambda < \Theta_M$,

$$\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in \mathrm{HOD}^M_{T,A,B}.$$

The point is that since $M \models \mathrm{OD}_{T,A,B}$-determinacy, for each $S \in \mathrm{OD}^M_{T,A,B} \cap \mathscr{P}(\lambda)$ the game for coding S relative to the prewellordering A_λ is determined: Without loss of generality, we may assume A_λ has length λ. For $\alpha < \lambda$, let $Q^\kappa_{<\alpha}$ and Q^κ_α be the usual objects defined relative to A_λ. For $e \in \omega^\omega$, let

$$S_e = \{\alpha < \lambda \mid U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing\}.$$

Since A_λ is trivially $\mathrm{OD}^M_{T,A,B}$ the game for the Uniform Coding Lemma for $Z = \bigcup\{Q^\kappa_\alpha \times \omega^\omega \mid \alpha \in S\}$ is determined for each $S \in \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}}$. Thus, every $S \in \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}}$ has the form S_e for some $e \in \omega^\omega$ and hence

$$\begin{aligned} \pi : \omega^\omega &\to \mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \\ e &\mapsto S_e \end{aligned}$$

is an $\mathrm{OD}^M_{T,A,B}$ surjection. Thus, $\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in M$ and so, by our first-order characterization, $\mathscr{P}(\lambda)^{\mathrm{HOD}^M_{T,A,B}} \in \mathrm{HOD}^M_{T,A,B}$. □

The ordinal κ that we will show to be T-strong in $\mathrm{HOD}^M_{T,A,B}$ is "the least stable in M":

5.17 Definition. Let κ be least such that

$$M_\kappa \prec_1 M.$$

As before the $\diamondsuit$-like function $F : \kappa \to M_\kappa$ is defined inductively in terms of the least counterexample: Given $F\restriction\delta$ let $\vartheta(\delta)$ be least such that

$$M_{\vartheta(\delta)} \models \mathrm{T}_0 \text{ and there is an } X \in M_{\vartheta(\delta)} \cap \mathrm{OD}^{M_{\vartheta(\delta)}}_{T\restriction\vartheta(\delta),A\restriction\vartheta(\delta),B} \text{ and}$$

($\star$) there is a Σ_1-formula φ and a $t \in \omega^\omega$ such that

$$M_{\vartheta(\delta)} \models \varphi[t, X, \delta, \mathbb{R}]$$

and for all $\bar{\delta} < \delta$

$$M_{\vartheta(\delta)} \not\models \varphi[t, F(\bar{\delta}), \bar{\delta}, \mathbb{R}]$$

(if such an ordinal exists) and then let $F(\delta)$ be the $<^{M_{\vartheta(\delta)}}_{\mathrm{OD}_{T\restriction\vartheta(\delta),A\restriction\vartheta(\delta),B}}$-least X such that ($\star$) holds.

Theorem 5.18. *For all $X \in \mathrm{OD}^M_{T,A,B}$, for all Σ_1-formulas φ, and for all $t \in \omega^\omega$, if*

$$M \models \varphi[t, X, \kappa, \mathbb{R}]$$

then there exists a $\delta < \kappa$ such that

$$M \models \varphi[t, F(\delta), \delta, \mathbb{R}].$$

Proof. Same as the proof of Theorem 4.6. □

Our interest is in applying Theorem 5.18 to

$$X = (\leqslant_\lambda, \lambda)$$

where $\leqslant_\lambda = A_\lambda$ is the prewellordering of length λ, for $\lambda < \Theta_M$. Clearly X is $\mathrm{OD}^M_{T,A,B}$.

Let U_X be a universal $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$ set of reals and, for $\delta < \kappa$, let U_δ be the reflected version (using the same definition used for U except with $F(\delta)$ and δ in place of X and κ). For $z \in U_X$, let $S_z = \{\delta < \kappa \mid z \in U_\delta\}$ and set

$$\mathscr{F}_X = \{S \subseteq \kappa \mid \exists z \in U_X\, (S_z \subseteq S)\}.$$

As before, $\mathscr{F}_X$ is a countably complete filter and in Theorem 5.18 we can reflect to $\mathscr{F}_X$-many points $\delta < \kappa$. Let

$$S_0 = \{\delta < \kappa \mid F(\delta) = (A_{\lambda_\delta}, \lambda_\delta) \text{ for some } \lambda_\delta > \delta\}.$$

Notice that $S_0 \in \mathscr{F}_X$. For notational conformity let $\leqslant_\delta$ be A_{λ_δ}. For $\alpha < \lambda$, let Q^κ_α be the α^{th}-component of $\leqslant_\lambda$ and, for $\delta \in S_0$ and $\alpha < \lambda_\delta$, let Q^δ_α be the α^{th}-component of $\leqslant_\delta$ (where without loss of generality we may assume that each A_α has length exactly α).

In our previous settings we went on to do two things. First, using the Uniform Coding Lemma we showed that one can allow parameters of the form $A \subseteq \kappa$ and $f : \kappa \to \kappa$ in the Reflection Theorem. Second, for $S \subseteq \kappa$, we defined the games $G^X(S)$ that gave rise to the ultrafilter extending the reflection filter, an ultrafilter that was either explicitly OD in the background universe (as in Section 4.3) or shown to be OD by appeal to Kunen's theorem (as in Section 5.1). In our present setting (where we have a limited amount of determinacy at our disposal) we will have to manage our resources more carefully. The following notion will play a central role.

5.19 Definition. A set $x \in M$ is *n-good* if and only if II can play n rounds of $(SG^B_{T,A,B,x})^M$. For $y \in M$, a set $x \in M$ is *n-y-good* if and only if (x, y) is n-good.

Notice that if M satisfies $\mathrm{ST}^B_{T,A,B,y}$-determinacy for $n + 1$ moves then II's first move f_0 is n-y-good. Notice also that if x is 1-y-good then every $\mathrm{OD}^M_{T,A,B,x,y}$ set of reals is determined. For example, if $S \subseteq \kappa$ is 1-good then the game for coding S relative to A_κ using the Uniform Coding Lemma is determined. Thus we have the following version of the Reflection Theorem.

Theorem 5.20. *Suppose $f : \kappa \to \kappa$, $G \subseteq \kappa$, $S \subseteq \kappa$ and (f, G, S) is 1-good. For all $X \in M \cap \mathrm{OD}^M_{T,A,B}$, for all Σ_1-formulas φ, and for all $t \in \omega^\omega$, if*

$$M \models \varphi[t, X, \kappa, \mathbb{R}, f, G, S]$$

then for $\mathscr{F}_X$-many $\delta < \kappa$,

$$M \models \varphi[t, F(\delta), \delta, \mathbb{R}, f{\restriction}\delta, G \cap \delta, S \cap \delta].$$

For each $S \subseteq \kappa$, let $G^X(S)$ be the game

$$\begin{array}{lccccccc} \text{I} & x(0) & & x(1) & & x(2) & & \dots \\ \text{II} & & y(0) & & y(1) & & \dots & \end{array}$$

with the following winning conditions: Main Rule: For all $i < \omega$, $(x)_i, (y)_i \in U_X$. If the rule is violated, then, letting i be the least such that either $(x)_i \notin U_X$ or $(y)_i \notin U_X$, I wins if $(x)_i \in U_X$; otherwise II wins. If the rule is satisfied, then, letting δ be least such that for all $i < \omega$, $(x)_i, (y)_i \in U_\delta$, I wins iff $\delta \in S$.

As before, if $S \in \mathscr{F}_X$ then I wins $G^X(S)$ by playing any x such that for all $i < \omega$, $(x)_i \in U_X$ and for some $i < \omega$, $(x)_i = z$, where z is such that $S_z \subseteq S$. But we cannot set

$$\mu_X = \{S \subseteq \kappa \mid \text{I wins } G^X(S)\}$$

since we have no guarantee that $G^X(S)$ is determined for an arbitrary $S \subseteq \kappa$.

However, if S is 1-good then $G^X(S)$ is determined. In particular, $G^X(S)$ is determined for each $S \in \mathscr{P}(\kappa) \cap \text{HOD}^M_{T,A,B}$. Thus, setting

$$\mu = \{S \in \mathscr{P}(\kappa) \cap \text{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S)\}$$

we have directly shown that κ is measurable in $\text{HOD}^M_{T,A,B}$.

It is useful at this point to stand back and contrast the present approach with the two earlier approaches. In both of the earlier approaches (namely, that of Section 4.3 and that of Section 5.1) the ultrafilters were ultrafilters in V and seen to be complete and normal in V and the ultrafilters were OD^V, the only difference being that in the first case the ultrafilters were *directly* seen to be OD^V, while in the second case they were *indirectly* seen to be OD^V by appeal to Kunen's theorem (Theorem 3.11). Now, in our present setting, there is no hope of getting such ultrafilters in V since we do not have enough determinacy. Instead we will get ultrafilters in $\text{HOD}^M_{T,A,B}$. However, the construction will still be "external" in some sense since we will be defining the ultrafilters in V.

We also have to take care to ensure that the ultrafilters "fit together" in such a way that they witness that κ is T-strong. In short, we will define a (κ, λ)-pre-extender $E_X \in \text{HOD}^M_{T,A,B}$, a notion we now introduce.

For $n \in \omega$ and $z \in [\text{On}]^n$, we write $z = \{z_1, \dots, z_n\}$, where $z_1 < \dots < z_n$. Suppose $b \in [\text{On}]^n$ and $a \subseteq b$ is such that $a = \{b_{i_1}, \dots b_{i_k}\}$, where $i_1 < \dots < i_k$. For $z \in [\text{On}]^n$, set

$$z_{a,b} = \{z_{i_1}, \dots, z_{i_k}\}.$$

Thus the elements of $z_{a,b}$ sit in z in the same manner in which the elements of a sit in b. For $\alpha \in \mathrm{On}$ and $X \subseteq [\alpha]^k$, let

$$X^{a,b} = \{z \in [\alpha]^n \mid z_{a,b} \in X\}.$$

For $\alpha \in \mathrm{On}$ and $f : [\alpha]^k \to V$, let $f^{a,b} : [\alpha]^n \to V$ be such that

$$f^{a,b}(z) = f(z_{a,b}).$$

Thus we use 'a, b' as a subscript to indicate that $z_{a,b}$ is the "drop of z from b to a" and we use 'a, b' as a superscript to indicate that $X^{a,b}$ is the "lift of X from a to b".

5.21 Definition. Let κ be an uncountable cardinal and let $\lambda > \kappa$ be an ordinal. The sequence

$$E = \langle E_a \mid a \in [\lambda]^{<\omega}\rangle$$

is a (κ, λ)-*extender* provided:

(1) For each $a \in [\lambda]^{<\omega}$,

$$E_a \text{ is a } \kappa\text{-complete ultrafilter on } [\kappa]^{|a|}$$

that is principal if and only if $a \subseteq \kappa$.

(2) (Coherence) If $a \subseteq b \in [\lambda]^{<\omega}$ and $X \in E_a$, then $X^{a,b} \in E_b$.

(3) (Countable Completeness) If $X_i \in E_{a_i}$ where $a_i \in [\lambda]^{<\omega}$ for each $i < \omega$, then there is an order preserving map

$$h : \bigcup_{i<\omega} a_i \to \kappa$$

such that $h``a_i \in X_i$ for all $i < \omega$.

(4) (Normality) If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$ is such that

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a$$

for some $i \leqslant |a|$, then there is a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|a\cup\{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$$

where k is such that β is the k^{th} element of $a \cup \{\beta\}$.

If conditions (1) and (2) alone are satisfied then we say that E is a (κ, λ)-*pre-extender*.

We need to ensure that the ultrafilter E_a on $[\kappa]^{|a|}$ depends on a in such a way that guarantees coherence and the other properties. The most natural way to define an ultrafilter E_a on $[\kappa]^{|a|}$ that depends on a is as follows:

(1) For $\mathscr{F}_X$-almost all δ define a "reflected version" $a^\delta \in [\lambda_\delta]^{<\omega}$ of the "generator" a.

(2) For $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$, let

$$S(a, Y) = \{\delta < \kappa \mid a^\delta \in Y\}$$

and set

$$E_a = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a, Y))\}.$$

In other words, we regard Y as "E_a-large" if and only if it contains the "reflected generators" on a set which is large from the point of view of the game.

The trouble is that we have not guaranteed that $S(a, Y)$ is determined. This set will be determined if it is 1-good but we have not ensured this. So we need to "reflect" a in such a way that $S(a, Y)$ is 1-good. Now the most natural way to reflect $a \in [\lambda]^k$ is as follows: Choose

$$(y_1, \ldots, y_k) \in Q^\kappa_{a_1} \times \cdots \times Q^\kappa_{a_k}$$

and, for $\delta \in S_0$, let $a^\delta = \{a^\delta_1, a^\delta_2, \ldots, a^\delta_k\}$ be such that

$$(y_1, \ldots, y_k) \in Q^\delta_{a^\delta_1} \times \cdots \times Q^\delta_{a^\delta_k}.$$

There is both a minor difficulty and a major difficulty with this approach. The minor difficulty is that we have to ensure that there is no essential dependence on our particular choice of $(y_1, \ldots, y_k)$. The major difficulty is that unless $(y_1, \ldots, y_k)$ is 1-good we still have no guarantee that $S(a, Y)$ is 1-good. The trouble is that there is in general no way of choosing such 1-good reals. However, assuming that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves, there *is* a way of generating 1-good prestrategies which (for all $x \in B$) hand us the reals we want. We will prove something slightly more general.

Lemma 5.22. *Assume z is $(n+1)$-good. Then for each $a \in [\lambda]^{<\omega}$ there is a function $f_a : \omega^\omega \to (\omega^\omega)^k$ such that*

(1) *f_a is n-z-good and*

(2) *for all $x \in B$,*
$$f_a(x) \in Q^\kappa_{a_1} \times \cdots \times Q^\kappa_{a_k},$$
where $k = |a|$.

Proof. The set
$$A_0 = \{x \in \omega^\omega \mid (x_{\text{even}})_i \in Q^\kappa_{a_{i+1}} \text{ for all } i < k\}$$
is $\mathrm{OD}^M_{T,A,B}$ and clearly I wins A_0. Let A_0 be I's first move in $(SG^B_{T,A,B,z})^M$ and let f_0 be II's response. Notice that f_0 is n-z-good. We have
$$\forall x \in B\, \forall y \in \omega^\omega \left(f_0(x){*}y \in A_0\right)$$
and hence
$$\forall x \in B\, \forall y \in \omega^\omega \left(((f_0(x){*}y)_{\text{even}})_i \in Q^\kappa_{a_{i+1}} \text{ for all } i < k\right).$$
Thus the function
$$\begin{aligned} f_a : \omega^\omega &\to (\omega^\omega)^k \\ x &\mapsto \begin{cases} \left(((f_0(x){*}0)_{\text{even}})_0, \ldots, ((f_0(x){*}0)_{\text{even}})_{k-1}\right) & \text{if } x \in B \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$
is n-z-good (since it is definable from the n-z-good object f_0) and has the desired property. □

5.23 Definition. Assume M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. For $a \in [\lambda]^{<\omega}$, we call a 1-good function $f_a : \omega^\omega \to (\omega^\omega)^{|a|}$ given by Lemma 5.22, a 1-*good code* for a.

The importance of a 1-good code f_a is twofold. First, any game defined in terms of f_a is determined. Second, for $\mathscr{F}_X$-almost all δ a 1-good code f_a selects a reflected version a^δ of a in a manner that is independent of $x \in B$; moreover, we can demand that a^δ inherits any $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$-property that a has. To see this, consider a statement such as the following: For all $x, x' \in B$, if $\alpha_1, \ldots, \alpha_k$ are such that

$$f_a(x) \in Q^\kappa_{\alpha_1} \times \cdots \times Q^\kappa_{\alpha_k}$$

then

$$f_a(x') \in Q^\kappa_{\alpha_1} \times \cdots \times Q^\kappa_{\alpha_k}$$

and

$$\alpha_1 < \cdots < \alpha_k.$$

This is a true $\Sigma_1(M, \{X, \mathbb{R}, \kappa\})$ statement. Thus, for $\mathscr{F}_X$-almost all δ, the statement reflects.

5.24 Definition. Suppose $a \in [\lambda]^{<\omega}$ and f_a is a 1-good code for a. Let

$$S_0(f_a) = \big\{\delta < \kappa \mid \forall x, x' \in B \, \forall \alpha_1, \ldots, \alpha_k \, (f_a(x) \in Q^\delta_{\alpha_1} \times \cdots \times Q^\delta_{\alpha_k} \\ \to f_a(x') \in Q^\delta_{\alpha_1} \times \cdots \times Q^\delta_{\alpha_k} \wedge \alpha_1 < \cdots < \alpha_k)\big\}.$$

Notice that $S_0(f_a) \in \mathscr{F}_X$ and $S_0(f_a)$ is $\mathrm{OD}^M_{T,A,B,f_a}$.

5.25 Definition. Suppose $a \in [\lambda]^{<\omega}$ and f_a is a 1-good code for a. For $\delta \in S_0(f_a)$ and some (any) $x \in B$, let

$$a^\delta_{f_a} = \{|(f_a(x))_1|_{\leqslant\delta}, \ldots, |(f_a(x))_{|a|}|_{\leqslant\delta}\}$$

be the *reflected generator* of a.

5.26 Definition. For $a \in [\lambda]^{<\omega}$, f_a a 1-good code for a, and $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$, let

$$S(a, f_a, Y) = \{\delta \in S_0(f_a) \mid a^\delta_{f_a} \in Y\}.$$

Since f_a is 1-good and $S(a, f_a, Y)$ is $\mathrm{OD}^M_{T,A,B,f_a}$ it follows that $S(a, f_a, Y)$ is 1-good and hence $G^X(S(a, f_a, Y))$ is determined.

For $a \in [\lambda]^{<\omega}$ and f_a a 1-good code for a, let

$$E_a(f_a) = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a, f_a, Y))\}$$

and let

$$E_X(f_a) : [\lambda]^{<\omega} \to \mathrm{HOD}^M_{T,A,B} \\ a \mapsto E_a(f_a).$$

The only trouble with this definition is that there is no guarantee that $E_X(f_a)$ is in $\mathrm{HOD}^M_{T,A,B}$ because there is no guarantee that $E_a(f_a)$ is in $\mathrm{HOD}^M_{T,A,B}$. We have to "erase" the dependence on the choice of f_a in the definition of E_a.

Lemma 5.27. *Suppose $a \in [\lambda]^{<\omega}$ and f_a and $\hat{f}_a$ are 1-good codes for a. Suppose $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$. Then*

(1) *I wins $G^X(\{\delta \in S_0(f_a) \cap S_0(\hat{f}_a) \mid a^\delta_{f_a} = a^\delta_{\hat{f}_a}\})$.*

(2) *I wins $G^X(S(a, f_a, Y))$ iff I wins $G^X(S(a, \hat{f}_a, Y))$, and*

(3) *$E_a(f_a) = E_a(\hat{f}_a)$.*

Proof. (1) The statement

$$\forall x \in B\, \forall i \leqslant |a| \left((f_a(x))_i =_\lambda (\hat{f}_a(x))_i\right)$$

is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}\})$-statement about f_a and $\hat{f}_a$. So, by reflection, the set $\{\delta \in S_0(f_a) \cap S_0(\hat{f}_a) \mid a^\delta_{f_a} = a^\delta_{\hat{f}_a}\}$ is in $\mathscr{F}_X$ and hence in μ_X.

(2) Assume I wins $G^X(S(a, f_a, Y))$. We have that I wins the game in (1). Let $G^X(S_0(f_a, \hat{f}_a))$ abbreviate this game. So I wins $G^X(S(a, f_a, Y) \cap S_0(f_a, \hat{f}_a))$. But

$$S(a, f_a, Y) \cap S_0(f_a, \hat{f}_a) \subseteq S(a, \hat{f}_a, Y).$$

So I wins $G^X(S(a, \hat{f}_a, Y))$. Likewise if I wins $G^X(S(a, \hat{f}_a, Y))$ then I wins $G^X(S(a, f_a, Y))$.

(3) This follows immediately from (2). □

Thus, we may wash out reference to f_a by setting

$$\begin{aligned} E_a &= \bigcap\{E_a(f_a) \mid f_a \text{ is a 1-good code of } a\} \\ &= E_a(f_a) \text{ for some (any) 1-good code } f_a \text{ of } a. \end{aligned}$$

Let

$$\begin{aligned} E_X : [\lambda]^{<\omega} &\to \mathrm{HOD}^M_{T,A,B} \\ a &\mapsto E_a \end{aligned}$$

Note that $E_a \in \mathrm{OD}^M_{T,A,B}$ and $E_a \subseteq \mathrm{HOD}^M_{T,A,B}$. Thus, $E_a \in \mathrm{HOD}^M_{T,A,B}$ and $E_X \in \mathrm{HOD}^M_{T,A,B}$.

Our definition of the extender E_X presupposes that for each $a \in [\lambda]^{<\omega}$ there is a 1-good code f_a of a and the existence of such codes is guaranteed by the assumption that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. Thus we have proved the following:

Lemma 5.28. *Assume that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. Then E_X is well-defined and $E_X \in \mathrm{HOD}^M_{T,A,B}$.*

It is important to stress that although the extender E_X is in $\mathrm{HOD}^M_{T,A,B}$ it is defined in M. For example, the certification that a certain set Y is in E_a depends on the existence of a winning strategy for a game in M. In general both the strategy and the game will not be in $\mathrm{HOD}^M_{T,A,B}$. So in establishing properties of E_X that hold in $\mathrm{HOD}^M_{T,A,B}$ we nevertheless have to consult the parent universe M.

Lemma 5.29. *Assume that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. Then*

$$\mathrm{HOD}^M_{T,A,B} \models E_X \text{ is a pre-extender,}$$

that is, $\mathrm{HOD}^M_{T,A,B}$ satisfies

(1) *for each $a \in [\lambda]^{<\omega}$,*

  (a) *E_a is a κ-complete ultrafilter on $[\kappa]^{|a|}$ and*

  (b) *E_a is principal iff $a \subseteq \kappa$, and*

(2) *if $a \subseteq b \in [\lambda]^{<\omega}$ and $Y \in E_a$ then $Y^{a,b} \in E_b$.*

Proof. (1)(a) It is easy to see that E_a is an ultrafilter in $\mathrm{HOD}^M_{T,A,B}$. It remains to see that E_a is κ-complete in $\mathrm{HOD}^M_{T,A,B}$. The proof is similar to that of Lemma 4.7. Let f_a be a 1-good code of a such that $E_a = E_a(f_a)$ and recall that

$$E_a(f_a) = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid \text{I wins } G^X(S(a, f_a, Y))\}.$$

Consider $\{Y_\alpha \mid \alpha < \gamma\} \in \mathrm{HOD}^M_{T,A,B}$ such that $\gamma < \kappa$ and for each $\alpha < \gamma$, $Y_\alpha \in E_\alpha(f_a)$. We have to show that

$$Y = \bigcap\{Y_\alpha \mid \alpha < \gamma\} \in E_a(f_a).$$

The key point is that

$$S(a, f_a, Y) = \bigcap\{S(a, f_a, Y_\alpha) \mid \alpha < \gamma\}$$

and so we are in almost exactly the situation as Lemma 4.7, only now we have to carry along the parameter f_a.

Since $Y \in \mathrm{HOD}^M_{T,A,B}$, $S(a, f_a, Y) \in \mathrm{OD}^M_{T,A,B,f_a}$. Since f_a is 1-good it follows that $G^X(S(a, f_a, Y))$ is determined. Assume for contradiction that I does not win $G^X(S(a, f_a, Y))$ and let σ' be a winning strategy for I in $G^X(\kappa \smallsetminus S(a, f_a, Y))$. We will derive a contradiction by finding a play that is legal against σ' and against winning strategies for I in each game $G^X(S(a, f_a, Y_\alpha))$, for $\alpha < \gamma$.

As in the case of Lemma 4.7, for the purposes of coding the winning strategies (in the games $G^X(S(a, f_a, Y_\alpha))$ for $\alpha < \gamma$) we need a prewellordering of length γ which is such that in a reflection argument we can ensure that it reflects to itself. For this purpose, for $\delta < \kappa$, let

$$Q_\delta = U_\delta \smallsetminus \bigcup\{U_\xi \mid \xi < \delta\}.$$

The sequence

$$\langle Q_\xi \mid \xi < \kappa\rangle$$

gives rise to an $\mathrm{OD}^M_{T,A,B}$ prewellordering with the feature that for $\mathscr{F}_X$-almost all δ,

$$\langle Q_\xi \mid \xi < \delta\rangle = \langle Q_\xi \mid \xi < \delta\rangle^{M_{\vartheta(\delta)}}$$

and, by choosing a real, we can ensure that we always reflect to some such point $\delta > \gamma$.

Now set

$$Z = \{(x, \sigma) \mid \text{for some } \alpha < \gamma,\ x \in Q_\alpha \text{ and } \sigma \text{ is a winning strategy for I in } G^X(S(a, f_a, Y_\alpha))\}.$$

This set is $\mathrm{OD}^M_{T,A,B,f_a}$, hence determined (as f_a is 1-good). So the game in the Uniform Coding Lemma is determined. The rest of the proof is exactly as before.

(b) By κ-completeness, E_a is principal if and only if there exists $b \in [\kappa]^{|a|}$ such that

$$E_a = \{Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B} \mid b \in Y\}.$$

Suppose that $a \in [\kappa]^{|a|}$. We claim that $b = a$ witnesses that E_a is principal. Let f_a be a 1-good code of a. For $\mathscr{F}_X$-almost all δ, $a^\delta_{f_a} = a$. So, for $Y \in \mathscr{P}([\kappa]^{|a|}) \cap \mathrm{HOD}^M_{T,A,B}$,

$$\begin{aligned} Y \in E_a &\leftrightarrow \text{I wins } G^X(S(a, f_a, Y)) \\ &\leftrightarrow \text{I wins } G^X(\{\delta \in S_0(f_a) \mid a^\delta_{f_a} = a \in Y\}) \\ &\leftrightarrow a \in Y. \end{aligned}$$

Suppose that $a \notin [\kappa]^{|a|}$. We claim that no $\beta \in [\kappa]^{|a|}$ witnesses that E_a is principal. Consider $b \in [\kappa]^{|a|}$ and let f_b be a 1-good code for b and let f_a be a 1-good code for a. For $\mathscr{F}_X$-almost all δ, $a^\delta_{f_a} \neq b^\delta_{f_b} = b$. Let S be the set of such δ and let $Y = \{a^\delta_{f_a} \mid \delta \in S\}$. Then $Y \in E_a$ and $b \notin Y$. Hence E_a is not principal.

(2) Suppose $a \subseteq b \in [\lambda]^{<\omega}$ and $Y \in E_a$. So I wins $G^X(S(a, f_a, Y))$ for some (any) 1-good code f_a of a. We must show that I wins $G^X(S(b, f_b, Y^{a,b}))$ for some (any) 1-good code f_b of b. Let f_b be a 1-good code of b and consider the statement describing the manner in which a sits inside b. This is a $\Sigma_1(M, \{X, \mathbb{R}, \kappa\})$-statement about f_a and f_b. So, by reflection, there exists an $S_0(f_a, f_b) \in \mathscr{F}_X$ such that for all $\delta \in S_0(f_a, f_b)$,

$$\langle a^\delta_{f_a}, b^\delta_{f_b}, \in\rangle \cong \langle a, b, \in\rangle.$$

We claim that $S(a, f_a, Y) \cap S_0(f_a, f_b) \subseteq S(b, f_b, Y^{a,b})$. Let δ be an ordinal in $S(a, f_a, Y) \cap S_0(f_a, f_b)$. We have $a^\delta_{f_a} \in Y$ and $\langle a^\delta_{f_a}, b^\delta_{f_b}, \in\rangle \cong \langle a, b, \in\rangle$. Since, by definition,

$$Y^{a,b} = \{z \in [\kappa]^{|b|} \mid z_{a,b} \in Y\},$$

this means that $b^\delta_{f_b} \in Y^{a,b}$ (as $(b^\delta_{f_b})_{a,b} = a^\delta_{f_a}$), that is, $\delta \in S(b, f_b, Y^{a,b})$. Finally, since I wins $G^X\big(S(a, f_a, Y) \cap S_0(f_a, f_b)\big)$, I wins $G^X(S(b, f_b, Y^{a,b}))$. □

Lemma 5.30. *Assume that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. Then*

$$\mathrm{HOD}^M_{T,A,B} \models E_X \textit{ is countably complete.}$$

Proof. Let $\{a_i \mid i < \omega\} \in \mathrm{HOD}^M_{T,A,B}$ and suppose that for each $i < \omega$, $X_i \in E_{a_i}$, that is, I wins $G^X(S(a_i, f_{a_i}, X_i))$ for some (any) 1-good code f_{a_i} of a_i. Let $S = \bigcap_{i<\omega} S(a_i, f_{a_i}, X_i)$. We need to ensure $G^X(S)$ is determined. The point is that since $\{a_i \mid i < \omega\} \in \mathrm{HOD}^M_{T,A,B}$, a slight modification of the proof of Lemma 5.22 shows that there are f_{a_i} such that $\langle f_{a_i} \mid i < \omega\rangle$ is 1-good. So $G^X(S)$ is determined. As in the proof of the completeness of E_a we have that I wins $G^X(S)$.

As in the proof of coherence there is a set $S_0(f_{a_1}, \dots, f_{a_n}, \dots) \in \mathscr{F}_X$ such that for all $\delta \in S_0(f_{a_1}, \dots, f_{a_n}, \dots)$,

$$\langle a_1^\delta, \dots, a_i^\delta, \dots\rangle \cong \langle a_1, \dots, a_i, \dots\rangle.$$

Fix $\delta \in S \cap S_0(f_{a_1}, \dots, f_{a_n}, \dots)$. Set

$$\begin{aligned} h_i : a_i &\to \kappa \\ (a_i)_j &\mapsto ((a_i)^\delta_{f_{a_i}})_j. \end{aligned}$$

Since $\delta \in S_0(f_{a_1}, \dots, f_{a_n}, \dots)$, the function

$$h = \bigcup_{i<\omega} h_i : \bigcup_{i<\omega} a_i \to \kappa$$

is well-defined. Since $\delta \in S(a_i, f_{a_i}, X_i)$, $h``a_i = (a_i)^\delta_{f_{a_i}} \in X_i$. However, h may not belong to $\mathrm{HOD}^M_{T,A,B}$. To see that there is such an h in $\mathrm{HOD}^M_{T,A,B}$ consider the tree $\mathscr{T}$ of attempts to build such a function. (The n^{th} level of T consists of approximations $h^* : \bigcup_{i<n} a_i \to \kappa$ and the order is by inclusion.) Thus $\mathscr{T} \in \mathrm{HOD}^M_{T,A,B}$ and the existence of h in V shows that $\mathscr{T}$ is ill-founded in V. But well-foundedness is absolute, so some such h must belong to $\mathrm{HOD}^M_{T,A,B}$. □

It remains to establish that

$$\mathrm{HOD}^M_{T,A,B} \models E_X \text{ is normal.}$$

This will follow from an analogue of the earlier strong normality theorems.

5.31 Definition. Assume M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for two moves. For $\alpha < \lambda$, let $f_\alpha : \omega^\omega \to \omega^\omega$ be a 1-good code of $\{\alpha\}$ (as in Lemma 5.22) and (as in Definition 5.25), for $\delta \in S_0(f_\alpha)$, let $\alpha^\delta_{f_\alpha}$ be the "reflected version" of α. We call the function

$$\begin{aligned} g_{f_\alpha} : S_0(f_\alpha) &\to \kappa \\ \delta &\mapsto \alpha^\delta_{f_\alpha} \end{aligned}$$

the *canonical function* associated to f_α.

Notice that the manner in which the ordinal $\alpha^\delta_{f_\alpha}$ is determined is different than in Section 4. In Section 4 we just chose $t \in Q_\alpha$ and let α^δ_t be unique such that $t \in Q^\delta_{\alpha^\delta_t}$. Notice also that g_{f_α} is 1-good since it is $\mathrm{OD}^M_{T,A,B,f_\alpha}$.

The statement and proof of strong normality are similar to before, only now we have to ensure that the objects are sufficiently good to guarantee the determinacy of the games defined in terms of them. The real parameters that arise in the proof of strong normality will now have to be generated using the technique of Lemma 5.22 and every time we use this technique we will sacrifice one degree of goodness. There will be finitely many such sacrifices and so it suffices to assume that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for n moves for some sufficiently large n. Furthermore, there is no loss in generality in making this assumption since in all of the applications of the Generation Theorem, one will be able to show without DC that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for $n < \omega$. As we shall see there will in fact only be two sacrifices of goodness. Thus, since we want our final object to be 1-good (to ensure that the games defined in terms of it are determined) it suffices to start with an object which is 3-good.

Theorem 5.32 (Strong Normality). *Suppose $g : \kappa \to \kappa$ is such that*

(1) *g is 3-good and*

(2) *I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$.*

Then there exists an $\alpha < \lambda$ such that

$$I \text{ wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\}),$$

where f_α is any 1-g-good code of α.

Proof. We begin with a few comments. First, note that since g is 3-good, by Lemma 5.22 we have that for each $\alpha < \lambda$ there is a 1-g-good code f_α of α (in fact, there is a 2-g-good code) and hence each game $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$ is determined. The only issue is whether I wins some such game.

Second, notice that α is uniquely specified. For suppose $f_{\hat\alpha}$ is a 1-g-good code of $\hat\alpha$ such that I wins the corresponding game. If $\alpha < \hat\alpha$, then

$$\{\delta \in S_0(f_\alpha) \cap S_0(f_{\hat\alpha}) \mid g_{f_\alpha}(\delta) < g_{f_{\hat\alpha}}(\delta)\} \in \mathscr{F}_X$$

and I wins $G^X(S)$ where S is this set. But then I cannot win both

$$G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$$

and

$$G^X(\{\delta \in S_0(f_{\hat{\alpha}}) \mid g(\delta) = g_{f_{\hat{\alpha}}}(\delta)\}).$$

Third, it will be useful at this point to both list the parameters that will arise in the proof and motivate the need for assuming that g is 3-good. In outline the proof will follow that of Theorem 4.12. The final game in the present proof (the one involving e_1) will be defined in terms of three parameters: g, f_η and e_0, corresponding respectively to f, y_η, and e_0 from Theorem 4.12. To ensure the determinacy of the final game we will need to take steps to ensure that (g, f_η, e_0) is 1-good. Now, the parameter e_0 will be obtained by applying the technique of Lemma 5.22 to the parameter (g, f_η) and so we will need to take steps to ensure that this parameter is 2-good. And the parameter f_η will in turn be obtained by applying the technique of Lemma 5.22 to the parameter g and so we have had to assume from the start that g is 3-good.

We now turn to the proof proper. Suppose $g : \kappa \to \kappa$ is such that

(1.1) g is 3-good and

(1.2) I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$.

Assume for contradiction that for each $\alpha < \lambda$ and for each 1-g-good code f_α of α,

(2.1) I does not win $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\})$,

and hence (since each such game is determined, as f_α is 1-g-good)

(2.2) I wins $G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) \neq g_{f_\alpha}(\delta)\})$.

Step 1. Let

$$\eta = \min\big(\{\beta < \lambda \mid \text{I wins } G^X(\{\delta \in S_0(f_\beta) \mid g(\delta) < g_{f_\beta}(\delta)\}) \text{ for each 1-}g\text{-good code } f_\beta \text{ of } \beta\}\big)$$

if such β exist; otherwise let $\eta = \lambda$. (So if there are such β then η is a limit ordinal.) Notice that

(3.1) whenever $\alpha < \eta$ and f_α is a 1-g-good code of α,

$$\text{I wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\}),$$

which is the desired situation. By Lemma 5.22, let

$$f_\eta \text{ be a 2-}g\text{-good code of } \eta.$$

For notational convenience, for $\delta \in S_0(f_\eta)$, let η_δ be $\eta^\delta_{f_\eta}$. By the definition of η, I wins $G^X(\{\delta \in S_0(f_\eta) \mid g(\delta) < g_{f_\eta}(\delta)\})$. Now update S_0 to be $S_0 \cap \{\delta \in S_0(f_\eta) \mid g(\delta) < g_{f_\eta}(\delta)\}$. We will work on this "large" set. Notice that S_0 is $\text{OD}^M_{T,A,B,g,f_\eta}$. If $\eta = \lambda$ then $\eta_\delta = \delta$ and we may omit mention of f_η in what follows.

For convenience let us write "$S \in \mu_X$" as shorthand for "I wins $G^X(S)$". To summarize:

(4.1) g is 3-good,

(4.2) (g, f_η) is 2-good (First Drop in Goodness),

(4.3) S_0 is $\text{OD}^M_{T,A,B,g,f_\eta}$,

(4.4) $S_0 \in \mu_X$ and for all $\delta \in S_0$, $g(\delta) < g_{f_\eta}(\delta)$, and

(4.5) for all $\alpha < \eta$ and for all 1-g-good codes f_α of α,

$$\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\} \in \mu_X.$$

Step 2. We now establish the "disjointness property".

Let

$$\begin{aligned} Z' = \big\{(x, \langle y, \sigma\rangle) \mid\ & x \in Q^\kappa_\alpha \text{ for some } \alpha < \eta, \\ & y \text{ codes a 1-}g\text{-good code } f_\alpha \text{ of } \alpha \\ & \qquad \text{such that } x \in \operatorname{ran}(f_\alpha \restriction B), \text{ and} \\ & \sigma \text{ is a winning strategy for I in} \\ & \qquad G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) > g_{f_\alpha}(\delta)\})\big\}. \end{aligned}$$

We have

(5.1) Z' is $\text{OD}^M_{T,A,B,g,f_\eta}$ and $Z' \subseteq Q^\kappa_{<\eta} \times \omega^\omega$, and

(5.2) for all $\alpha < \eta$,

$$Z' \cap (Q^\kappa_\alpha \times \omega^\omega) \neq \varnothing,$$

by (3.1).

Since (g, f_η) is 2-good the game in the proof of the Uniform Coding Lemma (Theorem 3.4) is determined. So *there is* an index $e \in \omega^\omega$ such that for all $\alpha < \eta$,

(6.1) $U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega)$ and

(6.2) $U^{(2)}_e(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing$.

The trouble is that we have no guarantee that such an index e has any degree of (g, f_η)-goodness; yet this is essential for the present proof since we shall go on to define games in terms of this index and we need some guarantee that these games are determined. As usual, we retreat from the reals we want to the good functions that capture them and this will lead to the second (and final) drop in goodness. Let

$$\begin{aligned} A_0 = \big\{x \in \omega^\omega \mid x_{\text{even}} \text{ is such that for all } \alpha < \eta \\ &(1)\ \ U^{(2)}_{x_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega) \text{ and} \\ &(2)\ \ U^{(2)}_{x_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing\big\}. \end{aligned}$$

So $A_0 \in \mathrm{OD}^M_{T,A,B,g,f_\eta}$. Now have I play A_0 in $(SG^B_{T,A,B,g,f_\eta})^M$ and let f_0 be II's response. Since (g, f_η) is 2-good, II's response f_0 is 1-(g, f_η)-good. Furthermore,

(7.1) $\forall x \in B\, \forall y \in \omega^\omega\, (f_0(x) * y \in A_0)$, which is to say,

(7.2) $\forall x \in B\, \forall y \in \omega^\omega\, (f_0(x) * y)_{\text{even}}$ is an index as in (6.1) and (6.2), hence

(7.3) $\forall \alpha < \eta$, $\bigcup_{x \in B} U^{(2)}_{(f_0(x)*0)_{\text{even}}}(Q^\kappa_{<\alpha}, Q^\kappa_\alpha)$ is as in (6.1) and (6.2).

The union in (7.3) is itself $\underset{\sim}{\Sigma}^1_1(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha)$ and so there is an $e_0 \in \omega^\omega$ which is definable from f_0 (and hence inherits the 1-(g, f_η)-goodness of f_0) such that

(8.1) (g, f_η, e_0) is 1-good (Second Drop in Goodness) and

(8.2) for all $\alpha < \eta$,

(1) $U^{(2)}_{e_0}(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \subseteq Z' \cap (Q^\kappa_\alpha \times \omega^\omega)$ and

(2) $U^{(2)}_{e_0}(B, Q^\kappa_{<\alpha}, Q^\kappa_\alpha) \neq \varnothing$.

Omitting Z', (8.2) is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_\eta, e_0\})$. So, for $\mathscr{F}_X$-almost all δ,

(8.3) for all $\alpha < \eta_\delta$,

(1) $U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \subseteq (Q^\delta_\alpha \times \omega^\omega)$ and

(2) $U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \neq \varnothing$.

The set S'_1 of such δ is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_\eta, e_0\})$. Let $S_1 = S'_1 \cap S_0$. Since $S'_1 \in \mu_X$ and $S_0 \in \mu_X$, it follows that $S_1 \in \mu_X$. Notice also that S_1 is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, g, f_\eta, e_0\})$. For $\delta \in S_1 \cup \{\kappa\}$ and $\alpha < \eta_\delta$, let

$$Z^\delta_\alpha = U^{(2)}_{e_0}(B, Q^\delta_{<\alpha}, Q^\delta_\alpha) \text{ and}$$
$$Z^\delta = \textstyle\bigcup_{\alpha<\eta_\delta} Z^\delta_\alpha.$$

Claim A (Disjointness Property). There is an $S_2 \subseteq S_1$ such that $S_2 \in \mu_X$ and for $\delta_1, \delta_2 \in S_2 \cup \{\kappa\}$ with $\delta_1 < \delta_2 \leqslant \kappa$,

$$Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing$$

for all $\alpha \in [g(\delta_1), \eta_{\delta_1})$ and $\beta \in [0, \eta_{\delta_2})$.

Proof. We begin by establishing a special case.

Subclaim. For μ_X-almost all δ,

$$Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing$$

for all $\alpha \in [g(\delta), \eta_\delta)$ and $\beta \in [0, \eta)$.

Proof. Let

$$G = \big\{\delta \in S_1 \mid Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing \text{ for all } \alpha \in [g(\delta), \eta_\delta) \text{ and } \beta \in [0, \eta)\big\}$$

be the set of "good points". Our aim is to show that $G \in \mu_X$. Note that G is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$. Since (g, f_η, e_0) is 1-good, $G^X(G)$ is determined. Assume for contradiction that $G \notin \mu_X$. Then, by determinacy, $\kappa \smallsetminus G \in \mu_X$. Since $S_1 \in \mu_X$, we have $(\kappa \smallsetminus G) \cap S_1 \in \mu_X$. Let σ' be a winning strategy for I in $G^X((\kappa \smallsetminus G) \cap S_1)$.

We get a contradiction much as before: We can "take control" of the games to produce a play z and an ordinal δ_0 such that

(9.1) z is a legal play for II against σ' and δ_0 is the associated ordinal and

(9.2) z is a legal play for II against each $\sigma \in (\mathrm{proj}_2(Z^{\delta_0}))_1$ and in each case δ_0 is the associated ordinal.

This will finish the proof: By (9.1) and the definition of G, there is an $\alpha_0 \in [g(\delta_0), \eta_{\delta_0})$ and a $\beta_0 \in [0, \eta)$ such that $Z^{\delta_0}_{\alpha_0} \cap Z^{\kappa}_{\beta_0} \neq \varnothing$. Fix $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0} \cap Z^{\kappa}_{\beta_0}$. Since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\kappa}_{\beta_0} \subseteq Z' \cap (Q^{\kappa}_{\beta_0} \times \omega^\omega)$ we have, by the definition of Z', $x_0 \in Q^{\kappa}_{\beta_0}$, y_0 codes a 1-g-good code f_{β_0} of β_0, $x_0 \in \mathrm{ran}(f_{\beta_0} \restriction B)$, and σ_0 is a winning strategy for I in $S(\{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\})$. Since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0}$, $\sigma_0 \in (\mathrm{proj}_2(Z^{\delta_0}))_1$. Now, by (9.2), z is a legal play for II against σ_0 with associated ordinal δ_0, and since σ_0 is a winning strategy for I in $S(\{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\})$, this implies

(10.1) $\delta_0 \in \{\delta \in S_0(f_{\beta_0}) \mid g(\delta) > g_{f_{\beta_0}}(\delta)\}$,

that is, $g(\delta_0) > g_{f_{\beta_0}}(\delta_0)$. We now argue

(10.2) $g_{f_{\beta_0}}(\delta_0) = \alpha_0$,

which is a contradiction since $\alpha_0 \geqslant g(\delta_0)$. Recall that by definition $g_{f_{\beta_0}}(\delta_0) = |f_{\beta_0}(x)|_{\leqslant \delta_0}$, where x is any element of B. Since we arranged $x_0 \in \mathrm{ran}(f_{\beta_0} \restriction B)$ and since $(x_0, \langle y_0, \sigma_0\rangle) \in Z^{\delta_0}_{\alpha_0}$, this implies that $g_{f_{\beta_0}}(\delta_0) = |f_{\beta_0}(x)|_{\leqslant \delta_0} = \alpha_0$, where x is any element of B. Thus, a play z as in (9.1) and (9.2) will finish the proof.

The play z is constructed as before:

Base Case. We have

(11.1) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_X$ and

(11.2) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\kappa))_1\, ((\sigma * y)_I)_0 \in U_X$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \sigma', e_0, f_\eta\})$-statement. So there is a $z_0 \in U_X$ such that $z_0 \leqslant_T \langle \sigma', e_0, f_\eta\rangle$ and for all δ if $z_0 \in U_\delta$ then

(11.3) $\forall y \in \omega^\omega\, ((\sigma' * y)_I)_0 \in U_\delta$ and

(11.4) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\delta))_1\, ((\sigma * y)_I)_0 \in U_\delta$.

$(n+1)^{\text{st}}$ Step. Assume we have defined $z_0, \ldots, z_n$ in such a way that

(12.1) $\forall y \in \omega^\omega \big(\forall i \leqslant n\, (y)_i = z_i \to ((\sigma' * y)_I)_{n+1} \in U_X\big)$ and

(12.2) $\forall y \in \omega^\omega\, \sigma \in (\mathrm{proj}_2(Z^\kappa))_1, \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_X\big)$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \sigma', e_0, f_\eta, z_0, \ldots, z_n\})$-statement. So there is a $z_{n+1} \in U_X$ such that $z_{n+1} \leqslant_T z_n$ and for all if $z_{n+1} \in U_\delta$ then

(12.3) $\forall y \in \omega^\omega \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma' * y)_I)_{n+1} \in U_\delta\big)$ and

(12.4) $\forall y \in \omega^\omega\, \forall \sigma \in (\mathrm{proj}_2(Z^\delta))_1 \big(\forall i \leqslant n\, (y)_i = z_i \rightarrow ((\sigma * y)_I)_{n+1} \in U_\delta\big)$.

Finally, let $z \in \omega^\omega$ be such that $(z)_i = z_i$ for all $i \in \omega$ and let δ_0 be least such that $(z)_i \in U_{\delta_0}$ for all $i \in \omega$. Notice that by our choice of z_n, for $n < \omega$, no DC is required to construct z. We have that for all $i \in \omega$,

(13.1) $((\sigma' * z)_I)_i \in U_{\delta_0}$ by (11.3) and (12.3) and

(13.2) $((\sigma * z)_I)_i \in U_{\delta_0}$ for all $\sigma \in (\mathrm{proj}_2(Z^{\delta_0}))_1$ by (11.4) and (12.4).

So we have (9.1) and (9.2), which is a contradiction. □

By the Subclaim,

(14.1) $\forall \delta \in G\, \forall \alpha \in [g(\delta), \eta_\delta)\, \forall \beta \in [0, \eta)\, (Z^\delta_\alpha \cap Z^\kappa_\beta = \varnothing)$.

This is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, e_0\rangle, g, G\})$-statement φ. Notice that since G is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$ and (g, f_η, e_0) is 1-good, it follows that (G, g, f_η, e_0) is 1-good. In particular, (G, g) is 1-good, and so Theorem 5.20 applies (taking $\langle f_\eta, e_0\rangle$ for the real t in the statement of that theorem) and we have that

(14.3) for $\mathscr{F}_X$-almost all δ_2,

  (1) $M \models \varphi[\langle f_\eta, e_0\rangle, F(\delta_2), \delta_2, g{\restriction}\delta_2, G \cap \delta_2]$, that is,
  (2) $\forall \delta_1 \in G \cap \delta_2\, \forall \alpha \in [g(\delta_1), \eta_{\delta_1})\, \forall \beta \in [0, \eta_{\delta_2})\, (Z^{\delta_1}_\alpha \cap Z^{\delta_2}_\beta = \varnothing)$.

Let S_2' be the set of such δ_2 in (14.3) and let $S_2 = S_2' \cap G$. Since $S_2' \in \mathscr{F}_X \subseteq \mu_X$ and $G \in \mu_X$, we have that $S_2 \in \mu_X$. Hence S_2 is as desired in Claim A. Also, S_2 is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$. □

Notice that two additional parameters have emerged, namely, G and S_2, but these do not lead to a drop in goodness since

(15.1) $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0,G,S_2} = \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0,G} = \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, and so

(15.2) (g, f_η, e_0, G, S_2) is 1-good.

STEP 3. We are now in a position to "compute g".

For $\delta \in S_2$, let

$$P^\delta = \bigcup \{ Z_\alpha^{\bar{\delta}} \mid \bar{\delta} \in S_2 \cap \delta \wedge \alpha \in [g(\bar{\delta}), \eta_{\bar{\delta}}) \}.$$

By (15.1), $P^\delta \in \mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$.

Claim B (TAIL COMPUTATION). There exists an index $e_1 \in \omega^\omega$ such that for all $\delta \in S_2$,

(1) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \subseteq Z^\delta_\alpha$ for all $\alpha \in [0, \eta_\delta)$,

(2) $U^{(2)}_{e_1}(P^\delta, Z^\delta_{g(\delta)}) = \varnothing$, and

(3) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \neq \varnothing$ for $\alpha \in (g(\delta), \eta_\delta)$.

Proof. As before it suffices to show (2) and (3′) $U^{(2)}_{e_1}(P^\delta, Z^\delta_\alpha) \cap Z^\delta_\alpha \neq \varnothing$ for $\alpha \in (g(\delta), \eta_\delta)$.

Let

$$G = \{ e \in \omega^\omega \mid \forall \delta \in S_2 \, (U^{(2)}_e(P^\delta, Z^\delta_{g(\delta)}) = \varnothing) \}.$$

Toward a contradiction assume that for each $e \in G$, (3′) in the claim fails for some δ and α. For each $e \in G$, let

$$\begin{aligned} \alpha_e = \ & \text{lexicographically least pair } (\delta, \alpha) \text{ such that} \\ & (1) \ \ \delta \in S_2, \\ & (2) \ \ g(\delta) < \alpha < \eta_\delta, \text{ and} \\ & (3) \ \ U^{(2)}_e(P^\delta, Z^\delta_\alpha) \cap Z^\delta_\alpha = \varnothing. \end{aligned}$$

Now play the game

$$\begin{array}{lccccccc} \mathrm{I} & x(0) & & x(1) & & x(2) & & \ldots \\ \mathrm{II} & & y(0) & & y(1) & & \ldots & \end{array}$$

where II wins iff $(x \in G \rightarrow (y \in G \wedge \alpha_y >_{\mathrm{lex}} \alpha_x))$.

The key point is that this payoff condition is $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, by (15.1), and hence, the game is determined, since (g, f_η, e_0) is 1-good.

The rest of the proof is exactly as before. □

From this point on there are no uses of determinacy that require further "joint goodness".

Claim C. There exists an $\alpha_0 < \eta$ such that

(1) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_{\alpha_0}) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_{\alpha}) \neq \varnothing$ for all $\alpha \in (\alpha_0, \eta)$, where

$$P^\kappa = \bigcup\{Z^\delta_\alpha \mid \delta \in S_2 \wedge \alpha \in [g(\delta), \eta_\delta)\}.$$

Proof. The statement that there is not a largest ordinal α_0 which is "empty" is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, e_0, e_1\rangle, g, G, S_2\})$. Since (g, f_η, e_0) is 1-good and G and S_2 are $\mathrm{OD}^M_{T,A,B,g,f_\eta,e_0}$, it follows that (g, G, S_2) is 1-good. Thus, the Reflection Theorem (Theorem 5.20) applies and we have that for $\mathscr{F}_X$-many δ, the statement reflects, which contradicts Claim B. □

Let α_0 be the unique ordinal as above and let f_{α_0} be a 1-g-good code of α_0 (which exists by Lemma 5.22). The statement

(16.2) $\forall x \in B\, f_{\alpha_0}(x) \in Q^\kappa_\alpha$ where α is such that

(1) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_{\alpha}) = \varnothing$ and

(2) $U^{(2)}_{e_1}(P^\kappa, Z^\kappa_{\beta}) \neq \varnothing$ for $\beta \in (\alpha, \eta_\delta)$.

is $\Sigma_1(M, \{X, \kappa, \mathbb{R}, \langle f_\eta, f_{\alpha_0}, e_0, e_1\rangle, g, G, S_2\})$. Since (g, G, S_2) is 1-good, the Reflection Theorem (Theorem 5.20) applies and hence for $\mathscr{F}_X$-almost all δ the statement reflects. Let $S'_3 \in \mathscr{F}_X$ be this set. Let $S_3 = S'_3 \cap S_2$. So $S_3 \in \mu_X$. By Claim B and Claim C, for $\delta \in S_3$ the ordinal α in question is $g(\delta)$. So I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \cap S_3 \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$ and hence I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$. This game is determined since f_{α_0} is 1-g-good.

To summarize:

(17.1) f_{α_0} is 1-g-good and

(17.2) I wins $G^X(\{\delta \in S_0(f_{\alpha_0}) \mid g(\delta) = g_{f_{\alpha_0}}(\delta)\})$,

which completes the proof of strong normality. □

Since every $g : \kappa \to \kappa$ in $\mathrm{HOD}^M_{T,A,B}$ is 3-good and since in the context of the main theorem we assume that M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for four moves, we have shown:

Corollary 5.33. *Suppose $g : \kappa \to \kappa$ is in $\mathrm{HOD}^M_{T,A,B}$ and such that I wins $G^X(\{\delta \in S_0 \mid g(\delta) < \lambda_\delta\})$. Then there exists an $\alpha < \lambda$ and a 1-g-good code f_α of α such that*

$$\text{I wins } G^X(\{\delta \in S_0(f_\alpha) \mid g(\delta) = g_{f_\alpha}(\delta)\}).$$

Lemma 5.34 (Normality). *In $\mathrm{HOD}^M_{T,A,B}$: If $a \in [\lambda]^{<\omega}$ and $f : [\kappa]^{|a|} \to \kappa$ is such that*

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a$$

for some $i \leqslant |a|$, then there is a $\beta < a_i$ such that

$$\{z \in [\kappa]^{|a \cup \{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$$

where k is such that β is the k^{th} element of $a \cup \{\beta\}$.

Proof. The proof just involves chasing through the definitions: Suppose $f : \kappa^{|a|} \to \kappa$ is a function in $\mathrm{HOD}^M_{T,A,B}$ such that for some $i \leqslant |a|$,

$$\{z \in [\kappa]^{|a|} \mid f(z) < z_i\} \in E_a.$$

Since M satisfies $\mathrm{ST}^B_{T,A,B}$-determinacy for four moves, f is 4-good. So, by Lemma 5.22, there is a 3-good code f_a of a. Hence

(1.1) I wins $G^X(\{\delta \in S_0(f_a) \mid f(a^\delta_{f_a}) < (a^\delta_{f_a})_i\})$.

Let

$$f^* : \kappa \to \kappa$$
$$\delta \mapsto \begin{cases} f(a^\delta_{f_a}) & \text{if } \delta \in S_0(f_a) \\ 0 & \text{otherwise.} \end{cases}$$

So $f^* \in \mathrm{HOD}^M_{T,A,B,f_a}$ and hence f^* is 3-good. By Theorem 5.32,

(1.2) there is an f_β such that

(1) f_β is a 1-f^*-good code of β,

(2) I wins $G^X(\{\delta \in S_0(f_\beta) \mid f^*(\delta) = g_{f_\beta}(\delta)\})$, and

(3) I wins $G^X(\{\delta \in S_0(f_\beta) \cap S_0(f_a) \mid f(a^\delta_{f_a}) = g_{f_\beta}(\delta)\})$.

Note that $\beta < a_i$ since if $\beta \geqslant a_i$ then for $\mathscr{F}_X$-almost all δ, $g_{f_\beta}(\delta) \geqslant (a^\delta_{f_a})_i$ and we get that I wins $G^X(\{\delta \in S_0(f_\beta) \cap S_0(f_a) \mid f(a^\delta_{f_a}) \geqslant (a^\delta_{f_a})_i\})$, which contradicts (1.1).

Let k be such that $\beta = (a \cup \{\beta\})_k$. Let $f_{a\cup\{\beta\}}$ be a 1-good code of $a \cup \{\beta\}$. Note that

(2.1) for $\mathscr{F}_X$-almost all δ,

$$\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_k = g_{f_\beta}(\delta)$$

and

(2.2) for $\mathscr{F}_X$-almost all δ,

$$\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_{a,a\cup\{\beta\}} = a^\delta_{f_a}$$

and, moreover, I wins on these sets (since the parameters in the definitions are 1-good). So (1.2)(3) yields

(3.1) I wins $G^X\big(\big\{\delta \in S_0(f_{a\cup\{\beta\}}) \mid f\big(\big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_{a,a\cup\{\beta\}}\big) = \big((a \cup \{\beta\})^\delta_{f_{a\cup\{\beta\}}}\big)_k\big\}\big)$

that is,

(3.2) $\{z \in [\kappa]^{|a\cup\{\beta\}|} \mid f(z_{a,a\cup\{\beta\}}) = z_k\} \in E_{a\cup\{\beta\}}$,

as desired. □

We are now in a position to take the "ultrapower" of $\mathrm{HOD}^M_{T,A,B}$ by E_X. It will be useful to recall this construction and record some basic facts concerning it. For further details see Steel's chapter in this Handbook.

Let

$$D = \{\langle a, f\rangle \in \mathrm{HOD}^M_{T,A,B} \mid a \in [\lambda]^{<\omega} \text{ and } f : [\kappa]^{|a|} \to \mathrm{HOD}^M_{T,A,B}\}.$$

We get an equivalence relation on D by letting

$$\langle a, f\rangle \sim_E \langle b, g\rangle \in D \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) = g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

Let $[a, f]$ be the elements of minimal rank of the equivalence class of $\langle a, f\rangle$. Let Ult be the structure with domain

$$\{[a, f] \mid \langle a, f\rangle \in D\}$$

and membership relation defined by

$$[a, f] \in_{E_X} [b, g] \leftrightarrow \{z \in [\kappa]^{|a\cup b|} \mid f(z_{a,a\cup b}) \in g(z_{b,a\cup b})\} \in E_{a\cup b}.$$

Since $\mathrm{HOD}^M_{T,A,B}$ satisfies AC, Łoś's theorem holds in the following form: For all formulas $\varphi(x_1, \ldots, x_n)$ and all elements $[a_1, f_1], \ldots, [a_n, f_n] \in \mathrm{Ult}$,

$$\begin{aligned}&\mathrm{Ult} \models \varphi\big[[a_1, f_1], \ldots, [a_n, f_n]\big] \leftrightarrow \\ &\quad \{z \in [\kappa]^{|b|} \mid \mathrm{HOD}^M_{T,A,B} \models \varphi[f_1(z_{a_1,b}), \ldots, f_n(z_{a_n,b})]\} \in E_b,\end{aligned}$$

where $b = \bigcup_{1\leqslant i\leqslant n} a_i$. It follows that

$$\begin{aligned}j'_E : \mathrm{HOD}^M_{T,A,B} &\to \mathrm{Ult}\\ x &\mapsto [\varnothing, c_x],\end{aligned}$$

where c_x is the constant function with value x, is an elementary embedding. The countable completeness of E_X ensures that Ult is well-founded and it is straightforward to see that it is extensional and set-like. So we can take the transitive collapse. Let

$$\pi : \mathrm{Ult} \to M_X$$

be the transitive collapse map and let

$$j_E : \mathrm{HOD}^M_{T,A,B} \to M_X$$

be the elementary embedding obtained by letting $j_E = \pi \circ j'_E$. The κ-completeness of each E_a, for $a \in [\lambda]^{<\omega}$, implies that j_E is the identity on $\mathrm{HOD}^M_{T,A,B} \cap V_\kappa$ and that κ is the critical point of j_E. Normality implies that for each $a \in [\lambda]^{<\omega}$, $\pi([a, z \mapsto z_i]) = a_i$, for each i such that $1 \leqslant i \leqslant |a|$. In particular, if $\alpha < \lambda$ then $\alpha = \pi([\{\alpha\}, z \mapsto \cup z])$. It follows that $\lambda \leqslant j_E(\kappa)$.

Lemma 5.35 (T-STRENGTH).

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \textit{There is a } T\textit{-strong cardinal.}$$

Proof. We already have that

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC},$$

by Lemma 5.16. It follows that there are arbitrarily large $\lambda < \Theta_M$ such that

$$\mathrm{HOD}^M_{T,A,B} \cap V_\lambda^{\mathrm{HOD}^M_{T,A,B}} = L_\lambda[A],$$

where $A \subseteq \lambda$ and $A \in \mathrm{HOD}^M_{T,A,B}$. Let λ be such an ordinal and let κ, j_E, etc. be as above. We have that $j_E(\kappa) \geqslant \lambda$ and it remains to show that

$$V_\lambda^{\mathrm{HOD}^M_{T,A,B}} \subseteq M_X$$

and

$$j_E(T \cap \kappa) \cap \lambda = T \cap \lambda.$$

The proof of each is the same. Let us start with the latter. We have to show that for all $\alpha < \lambda$,

$$\alpha \in j_E(T \cap \kappa) \leftrightarrow \alpha \in T.$$

We have

$$\begin{aligned} \alpha \in j_E(T \cap \kappa) &\leftrightarrow \pi([\{\alpha\}, z \mapsto \cup z]) \in \pi([\varnothing, c_{T\cap\kappa}]) \\ &\leftrightarrow [\{\alpha\}, z \mapsto \cup z] \in_{E_X} [\varnothing, c_{T\cap\kappa}] \\ &\leftrightarrow \{z \in [\kappa]^1 \mid \cup z \in T \cap \kappa\} \in E_{\{\alpha\}}. \end{aligned}$$

So we have to show that

$$\alpha \in T \leftrightarrow \{\{z\} \mid z \in T \cap \kappa\} \in E_{\{\alpha\}}.$$

Let $f_{\{\alpha\}}$ be a 1-good code of $\{\alpha\}$.

Assume $\alpha \in T$. We have to show that

$$\text{I wins } G^X\big(S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\})\big).$$

The key point is that the statement "for all $x \in B$, $|f_{\{\alpha\}}(x)|_{\leqslant\lambda} \in T$" is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_{\{\alpha\}}\})$-statement. So the set S of δ to which this statement reflects is in $\mathscr{F}_X$. Since $S \in \mathrm{OD}^M_{T,A,B,f_{\{\alpha\}}}$ and $f_{\{\alpha\}}$ is 1-good, $G^X(S)$ is determined and I wins. But

$$S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\}) = S_0(f_{\{\alpha\}}) \cap S$$

and so I wins this game as well.

Assume $\alpha \notin T$. We have to show that

$$\text{I does not win } G^X\big(S(\{\alpha\}, f_{\{\alpha\}}, \{\{z\} \mid z \in T \cap \kappa\})\big).$$

Again, the point is that the statement "for all $x \in B$, $|f_{\{\alpha\}}(x)|_{\leqslant\lambda} \notin T$" is a true $\Sigma_1(M, \{X, \kappa, \mathbb{R}, f_{\{\alpha\}}\})$-statement. So this statement reflects to $\mathscr{F}_X$-almost all δ, which implies that I cannot win the above game.

Exactly the same argument with 'A' in place of 'T' shows that

$$j_E(A \cap \kappa) \cap \lambda = A \cap \lambda,$$

and hence that

$$V_\lambda^{\mathrm{HOD}^M_{T,A,B}} = L_\lambda[A] \subseteq M_X,$$

which completes the proof. □

This completes the proof of the Generation Theorem □

5.4. Special Cases

We now consider a number of special instances of the Generation Theorem. In each case all we have to do is find appropriate values for the parameters Θ_M, T, A, and B. We begin by recovering the main result of Section 4.

Theorem 5.36. *Assume* ZF + AD*. Then*

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \textit{ is a Woodin cardinal.}$$

Proof. For notational convenience let $\Theta = \Theta^{L(\mathbb{R})}$. Our strategy is to meet the conditions of the Generation Theorem while at the same time arranging that $M = L_\Theta(\mathbb{R})[T, A, B]$ is such that

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta.$$

We will do this by taking care to ensure that the ingredients T, A, and B are in $\mathrm{HOD}^{L(\mathbb{R})}$ while at the same time packaging $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$ as part of T. It will then follow from the Generation Theorem that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal},$$

and by varying T the result follows.

To begin with let $\Theta_M = \Theta^{L(\mathbb{R})}$ and, for notational convenience, we continue to abbreviate this as Θ. By Theorem 3.9, Θ is strongly inaccessible in $\mathrm{HOD}^{L(\mathbb{R})}$. Also,

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta = \mathrm{HOD}^{L_\Theta(\mathbb{R})},$$

by Theorem 3.10. So we can let $H \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ code

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta.$$

Fix $T' \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ and let $T \in \mathscr{P}(\Theta) \cap \mathrm{HOD}^{L(\mathbb{R})}$ code T' and H. By Lemmas 3.7 and 3.8, there is an $\mathrm{OD}^{L(\mathbb{R})}$ sequence $A = \langle A_\alpha \mid \alpha < \Theta \rangle$ such that each A_α is a prewellordering of reals of length α. Let $B = \mathbb{R}$.

Let

$$M = L_\Theta(\mathbb{R})[T, A, B]$$

where Θ, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L(\mathbb{R})$ satisfies AD and M contains all reals. Moreover, since we have arranged that all of the ingredients T, A, and B are in $\mathrm{OD}^{L(\mathbb{R})}$ and also that T codes $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$, we have

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\Theta$$

and, since T' was arbitrary, the result follows as noted above. □

We can also recover the following approximation to Theorem 5.6.

Theorem 5.37. *Assume* ZF + $\mathrm{AC}_\omega(\mathbb{R})$. *Suppose* ST^B_X*-determinacy holds, where X is a set and B is non-empty and* OD_X. *Then*

$$\mathrm{HOD}_X \models \Theta_X \textit{ is a Woodin cardinal.}$$

Proof. Let $\Theta_M = \Theta_X$. Let $A = \langle A_\alpha \mid \alpha < \Theta_X \rangle$ be such that A_α codes the OD_X-least prewellordering of reals of length α. By Theorem 3.9, Θ_X is strongly inaccessible in HOD_X and so there exists an $H \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$ coding $\mathrm{HOD}_X \cap V_{\Theta_X}$. Let $T \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$ code H and some arbitrary $T' \in \mathscr{P}(\Theta_X) \cap \mathrm{HOD}_X$.

Let

$$M = L_{\Theta_M}(\mathbb{R})[T, A, B]$$

where Θ_M, T, A, and B are as above. Work in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Conditions (3)–(5) of the Generation Theorem are clearly met. For condition (2) note that by Lemma 3.7, $\Theta_X = \Theta^{\mathrm{HOD}_{\{X\}\cup\mathbb{R}}}$ and that by the arguments of Lemma 3.8 and Lemma 3.9, $\Theta^{\mathrm{HOD}_{\{X\}\cup\mathbb{R}}}$ is regular in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Thus, Θ_M is regular in $\mathrm{HOD}_{\{X\}\cup\mathbb{R}}$. Condition (6) follows from the fact that M is OD_X and M contains all of the reals. It remains to see that condition (1) can be met. For this we just have to see that Replacement holds in M. If Replacement failed in M then there would be a cofinal map $\pi : \omega^\omega \to \Theta_X$ that is definable from parameters in M, which in conjunction with A would lead to an OD_X surjection from ω^ω onto Θ_X, which is a contradiction. □

5.38 Remark. Work in ZF+DC. For μ a δ-complete ultrafilter on δ let E_μ be the (δ, λ)-extender derived from μ where $\lambda = j(\delta)$ (or $\lambda = \delta^\delta/\mu$) and j is the ultrapower map. We have the following corollary: Assume ZF + AD + DC. Then Θ_X is Woodin in HOD_X and this is witnessed by the collection of $E_\mu \cap \mathrm{HOD}_X$ where μ is a normal ultrafilter on some $\delta < \Theta_X$.

5.39 Remark. Theorem 5.6 cannot be directly recovered from the Generation Theorem and this is why we have singled it out for special treatment. However, it follows from the proof of the Generation Theorem, as can be seen by noting that in the case where one has full boldface determinacy the ultrafilters are actually in HOD_X by Kunen's theorem (Theorem 3.11).

4 Open Question. There are some interesting questions related to Theorem 5.37.

(1) Suppose $\Theta_X = \Theta_0$. Suppose ST_X^B-determinacy, where B is non-empty and OD_X. Is Θ_0 a Woodin cardinal in HOD?

(2) Suppose ST_X^B-determinacy, where X is a set and B is non-empty and OD_X. Is Θ_X a Woodin cardinal in HOD?

(3) In the AD^+ setting, every Θ_X is of the form Θ_α and there are constraints on this sequence. For example, each Θ_X must be of the form $\Theta_{\alpha+1}$. Does this constraint apply in the lightface setting?

Theorem 5.40. *Assume* ZF + AD. *Let S be a class of ordinals. Then for an S-cone of x,*

$$\mathrm{HOD}_S^{L[S,x]} \models \omega_2^{L[S,x]} \text{ is a Woodin cardinal.}$$

Proof. For an S-cone of x,

$$L[S,x] \models \mathrm{ZFC} + \mathrm{GCH \ below\ } \omega_1^V,$$

by Corollary 5.10, and, for all $n < \omega$,

$$L[S,x] \models \mathrm{ST}_S^B\text{-determinacy for } n \text{ moves},$$

where $B = [x]_S$, by Theorem 5.13. Let x be in this S-cone.

Let $\Theta_M = \omega_2^{L[S,x]}$. Since $L[S,x]$ satisfies GCH below ω_1^V and $L[S,x] = \mathrm{OD}_{S,x}^{L[S,x]}$, by Lemma 3.8 we have that

$$\omega_2^{L[S,x]} = \sup\{\alpha \mid \text{there is an } \mathrm{OD}_S^{L[S,x]} \text{ prewellordering of length } \alpha\},$$

in other words, $\omega_2^{L[S,x]} = (\Theta_S)^{L[S,x]}$. Let $A = \langle A_\alpha \mid \alpha < \omega_2^{L[S,x]}\rangle$ be such that A_α is the $\mathrm{OD}_S^{L[S,x]}$-least prewellordering of length α. Since $L[S,x] \models \mathrm{OD}_S$-determinacy, it follows (by Theorem 3.9) that $\omega_2^{L[S,x]}$ is strongly inaccessible in $\mathrm{HOD}_S^{L[S,x]}$. So there is a set $H \subseteq \omega_2^{L[S,x]}$ coding $\mathrm{HOD}_S^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}$. Let T' be in $\mathscr{P}(\omega_2^{L[S,x]}) \cap \mathrm{OD}_S^{L[S,x]}$ and let $T \in \mathscr{P}(\omega_2^{L[S,x]}) \cap \mathrm{OD}_S^{L[S,x]}$ code T and H. Let $B = [x]_S$.

Let

$$M = L_{\Theta_M}(\mathbb{R}^{L[S,x]})[T, A, B],$$

where Θ_M, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L[S,x]$ satisfies ST_S^B-determinacy for four moves, M is OD_S in $L[S,x]$ and M contains the reals of $L[S,x]$. Thus,

$$\mathrm{HOD}_{T,A,B}^M \models \mathrm{ZFC} + \text{ There is a } T\text{-strong cardinal.}$$

Since we have arranged that all of the ingredients T, A, and B are in $\mathrm{OD}^{L[S,x]}$ and also that T codes $\mathrm{HOD}^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}$, we have

$$\mathrm{HOD}_{T,A,B}^M = \mathrm{HOD}^{L[S,x]} \cap V_{\omega_2^{L[S,x]}}.$$

Since T' was arbitrary, the result follows. □

Theorem 5.41. *Assume* ZF + AD*. Then for an* S*-cone of* x*,*

$$\mathrm{HOD}_{S,\mathrm{HOD}_{S,x}} \models \omega_2^{\mathrm{HOD}_{S,x}} \text{ is a Woodin cardinal.}$$

Proof. This will follow from the next theorem which is more general. □

The next two theorems require some notation. Suppose Y is a set and $a \in H(\omega_1)$. For $x \in \omega^\omega$, the (Y,a)-*degree* of x is the set

$$[x]_{Y,a} = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,a,z} = \mathrm{HOD}_{Y,a,x}\}.$$

The (Y,a)-*degrees* are the sets of the form $[x]_{Y,a}$ for some $x \in \omega^\omega$. Define $x \leqslant_{Y,a} y$ to hold iff $x \in \mathrm{HOD}_{Y,a,y}$. A *cone of* (Y,a)-*degrees* is a set of the form $\{[y]_{Y,a} \mid y \geqslant_{Y,a} x_0\}$ for some $x_0 \in \omega^\omega$ and a (Y,a)-*cone of reals* is a set of the form $\{y \in \omega^\omega \mid y \geqslant_{Y,a} x_0\}$ for some $x_0 \in \omega^\omega$. The proof of the Cone Theorem (Theorem 2.9) generalizes to the present context. In the case where $a = \varnothing$ we speak of Y-*degrees*, etc.

Theorem 5.42. *Assume* ZF+AD. *Suppose* Y *is a set and* $a \in H(\omega_1)$. *Then for a* (Y,a)*-cone of* x,

$$\mathrm{HOD}_{Y,a,[x]_{Y,a}} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \text{ is a Woodin cardinal,}$$

where $[x]_{Y,a} = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,a,z} = \mathrm{HOD}_{Y,a,x}\}$.

Proof. By determinacy it suffices to show that the above statement holds for a Turing cone of x, which is what we shall do. The key issues in this case are getting a sufficient amount of GCH and strategic determinacy. To establish the first we need two preliminary claims. Recall that a set $A \subseteq \omega^\omega$ is *comeager* if and only if $\omega^\omega \smallsetminus A$ is meager.

Claim 1. Assume ZF + AD. Suppose that $\langle A_\alpha \mid \alpha < \gamma\rangle$ is a sequence of sets which are comeager in the space ω^ω, where either $\gamma \in \mathrm{On}$ or $\gamma = \mathrm{On}$, in which case the sequence is a definable proper class. Then $\bigcap_{\alpha<\gamma} A_\alpha$ is comeager.

Proof. Assume for contradiction that the claim fails and let γ be least such that there is a sequence $\langle A_\alpha \mid \alpha < \gamma\rangle$ the intersection of which is not comeager. By AD, $\bigcap_{\alpha<\gamma} A_\alpha$ has the property of Baire and so we may assume without loss of generality that $\bigcap_{\alpha<\gamma} A_\alpha$ is meager. So, every proper initial segment has comeager intersection while the whole sequence has meager intersection. We can now violate the Kuratowski-Ulam Theorem. (This is the analogue for category of Fubini's theorem. See [10, 5A.9].) Define f on the complement of $\bigcap_{\alpha<\gamma} A_\alpha$ as follows:

$$f(x) = \min(\{\alpha < \gamma \mid x \notin A_\alpha\}).$$

So if $y \in \bigcap_{\xi<\alpha} A_\xi$ then $f(y) > \alpha$. Since $\bigcap_{\alpha<\gamma} A_\alpha$ is meager, $\mathrm{dom}(f)$ is comeager. Consider the subset of the plane

$$Z = \{(x,y) \in \mathrm{dom}(f) \times \mathrm{dom}(f) \mid f(x) < f(y)\}.$$

For each $x \in \mathrm{dom}(f)$ the vertical section

$$Z_x = \{y \in \mathrm{dom}(f) \mid f(y) > f(x)\}$$

is comeager since it includes $\bigcap_{\alpha \leqslant f(x)} A_\alpha$ and for each $y \in \mathrm{dom}(f)$ the horizontal section

$$Z^y = \{x \in \mathrm{dom}(f) \mid f(x) < f(y)\}$$

is meager since its complement contains the comeager set $\bigcap_{\alpha<f(y)} A_\alpha$. Since Z has the property of Baire, this contradicts the Kuratowski-Ulam Theorem, the proof of which requires only $\mathrm{AC}_\omega(\mathbb{R})$, which follows from AD (Theorem 2.2). □

Claim 2. Assume ZF + AD. Suppose Y is a set, $a \in H(\omega_1)$ and $\mathbb{P} \in \mathrm{HOD}_{Y,a} \cap H(\omega_1)$ is a partial order. Then for comeager many $\mathrm{HOD}_{Y,a}$-generic $G \subseteq \mathbb{P}$,

$$\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G].$$

Proof. For each G we clearly have $\mathrm{HOD}_{Y,a}[G] \subseteq \mathrm{HOD}_{Y,a,G}$. We seek a set A that is comeager in the Stone space of $\mathbb{P}$ and such that for all $G \in A$, $\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G]$. We will do this by showing that for each $G \in A$ the latter model can compute the "ordinal theory" of the former model.

For every Σ_2-statement φ and finite sequence of ordinals $\vec{\xi}$ consider the statement $\varphi[\vec{\xi}, Y, a, G]$ about a generic G. Let $B^{\varphi,\vec{\xi},Y,a}$ be the associated collection of filters on $\mathbb{P}$ and let

$$\begin{aligned} P^{\varphi,\vec{\xi}} &= \{p \in \mathbb{P} \mid B^{\varphi,\vec{\xi},Y,a} \text{ is comeager in } O_p\} \text{ and} \\ N^{\varphi,\vec{\xi}} &= \{p \in \mathbb{P} \mid B^{\neg\varphi,\vec{\xi},Y,a} \text{ is comeager in } O_p\}, \end{aligned}$$

where O_p is the open set of generics containing p. These are the sets of conditions which "positively" and "negatively" decide $\varphi[\vec{\xi}, Y, a, G]$, respectively. So $P^{\varphi,\vec{\xi}} \cup N^{\varphi,\vec{\xi}}$ is predense. Now let

$$\begin{aligned} A_{\varphi,\vec{\xi}} =& \{G \subseteq \mathbb{P} \mid \varphi[\vec{\xi}, Y, a, G] \leftrightarrow G \cap P^{\varphi,\vec{\xi}} \neq \varnothing\} \\ & \cup \{G \subseteq \mathbb{P} \mid \neg\varphi[\vec{\xi}, Y, a, G] \leftrightarrow G \cap N^{\varphi,\vec{\xi}} \neq \varnothing\}. \end{aligned}$$

Each such set is comeager. We thus have a class size well-order of comeager sets and so, by the previous lemma,

$$A = \bigcap \{A_{\varphi,\vec{\xi}} \mid \varphi \text{ is a } \Sigma_2\text{-formula and } \vec{\xi} \in \mathrm{On}^{<\omega}\}$$

is comeager. But now we have that for all $G \in A$

$$\mathrm{HOD}_{Y,a,G} = \mathrm{HOD}_{Y,a}[G]$$

since the latter can compute all answers to questions involving the former—that is, questions of the form $\varphi[\vec{\xi}, Y, a, G]$ where φ is Σ_2—by checking whether G hits $P^{\varphi,\vec{\xi}}$ or $N^{\varphi,\vec{\xi}}$. (Notice that the restriction to Σ_2-formulas suffices (by reflection) since any statement about an initial segment of $\mathrm{HOD}_{Y,a,G}$ is Σ_2.) □

Claim 3. Assume ZF + AD. Suppose Y is a set and $a \in H(\omega_1)$. Then for a Turing cone of x,

$$\mathrm{HOD}_{Y,a,x} \models \text{GCH below } \omega_1^V.$$

Proof. It suffices to show that CH holds on a cone since given this the proof that GCH below ω_1^V holds on a cone goes through exactly as before.

Suppose for contradiction (by the Cone Theorem (Theorem 2.9)) that there is a real x_0 such that for all $x \geqslant_T x_0$,

$$\mathrm{HOD}_{Y,a,x} \models \neg\mathrm{CH}.$$

We will arrive at a contradiction by producing an $x \geqslant_T x_0$ with the feature that $\mathrm{HOD}_{Y,a,x} \models \mathrm{CH}$. As before x is obtained by forcing (in two steps) over HOD_{Y,a,x_0}. First, we get a HOD_{Y,a,x_0}-generic

$$G \subseteq \mathrm{Col}(\omega_1^{\mathrm{HOD}_{Y,a,x_0}}, \mathbb{R}^{\mathrm{HOD}_{Y,a,x_0}})$$

and then we use almost disjoint forcing to code G with a real. Viewing the generic g as a real, by the previous claim we have that for comeager many g,

$$\mathrm{HOD}_{Y,a,x_0,g} = \mathrm{HOD}_{Y,a,x_0}[g] \models \mathrm{CH},$$

and hence

$$\mathrm{HOD}_{Y,a,\langle x_0,g\rangle} \models \mathrm{CH},$$

which is a contradiction. □

Claim 4. Suppose Y is a set and $a \in H(\omega_1)$. Then for a Turing cone of x, for each $n < \omega$, II can play n moves of $SG^B_{Y,a,[x]_{Y,a}}$, where $B = [x]_{Y,a}$, and we demand in addition that II's moves belong to $\mathrm{HOD}_{Y,a,x}$, in other words, II can play n moves of the game

$$\begin{array}{lccccccc} \mathrm{I} & A_0 & & \cdots & & A_{n-1} & \\ \mathrm{II} & & f_0 & & \cdots & & f_{n-1} \end{array}$$

where we require, for $i + 1 < n$,

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^V_{Y,a,[x]_{Y,a}}$, $A_{i+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}^V_{Y,a,[x]_{Y,a},f_0,\ldots,f_i}$ and

(2) f_{i+1} is prestrategy for A_{i+1} that belongs to $\mathrm{HOD}_{Y,a,x}$ and is winning with respect to $[x]_{Y,a}$.

Proof. The proof of Theorem 5.13 actually establishes this stronger result. □

We are now in a position to meet the conditions of the Generation Theorem. For a Turing cone of x,

$$\mathrm{HOD}_{Y,a,x} \models \mathrm{ZFC} + \mathrm{GCH} \text{ below } \omega_1^V,$$

by Claim 3, and for all $n < \omega$,

$$\mathrm{ST}^B_{Y,a,[x]_{Y,a}}\text{-determinacy for } n \text{ moves}$$

holds in V where $B = [x]_{Y,a}$, by Claim 4. Let x be in this cone.

Let $\Theta_M = \omega_2^{\mathrm{HOD}_{Y,a,x}}$. Since $\mathrm{HOD}_{Y,a,x} \models \mathrm{ZFC} + \mathrm{GCH}$ below ω_1^V,

$$\Theta_M = \Theta.$$

Since every set is $\mathrm{OD}_{Y,a,x}$, and hence $\mathrm{OD}_{Y,a,[x]_Y,x}$,

$$\Theta = \Theta_{Y,a,[x]_Y},$$

by Lemma 3.8. Thus,

$$\omega_2^{\mathrm{HOD}_{Y,a,x}} = \Theta^{\mathrm{HOD}_{Y,a,x}}_{Y,a,[x]_{Y,a}}.$$

Letting $A = \langle A_\alpha \mid \alpha < \omega_2^{\mathrm{HOD}_{Y,a,x}} \rangle$ be such that A_α is the $\mathrm{OD}_{Y,a,[x]_{Y,a}}$-least prewellordering of length α we have that A is $\mathrm{OD}_{Y,a,[x]_{Y,a}}$. We also have that $\omega_2^{\mathrm{HOD}_{Y,a,x}}$ is strongly inaccessible in $\mathrm{HOD}_{Y,a,[x]_{Y,a}}$, by Theorem 3.9. So there is a set $H \subseteq \omega_2^{\mathrm{HOD}_{Y,a,x}}$ coding $\mathrm{HOD}_{Y,a,[x]_{Y,a}} \cap V_{\omega_2^{\mathrm{HOD}_{Y,a,x}}}$. Let T' be in $\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,a,x}}) \cap \mathrm{OD}_{Y,a,[x]_{Y,a}}$ and let $T \in \mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,a,x}}) \cap \mathrm{OD}_{Y,a,[x]_{Y,a}}$ code T' and H. Let $B = [x]_{Y,a}$.

Let

$$M = L_{\Theta_M}(\mathbb{R}^{\mathrm{HOD}_{Y,a,x}})[T, A, B],$$

where Θ_M, T, A, and B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met. Condition (6) follows from the fact that M is $\mathrm{OD}_{Y,a,[x]_{Y,a}}$ and we have arranged (in Claim 4) that all of II's moves in $SG^B_{Y,a,[x]_{Y,a}}$ are in M.

Thus,

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \text{ There is a } T\text{-strong cardinal},$$

and since we have arranged that

$$\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}_{Y,a,[x]_{Y,a}} \cap V_{\omega_2^{\mathrm{HOD}_{Y,a,x}}},$$

and T was arbitrary, the result follows. □

In the above theorem the degree notion $[x]_{Y,a}$ depends on the initial choice of a. However, later (in Section 6.2) we will want to construct models with many Woodin cardinals. A natural approach to doing this is to iteratively apply the previous theorem, starting off with $a = \varnothing$, increasing the degree of x to get that $\omega_2^{\mathrm{HOD}_{Y,x}}$ is a Woodin cardinal in $\mathrm{HOD}_{Y,[x]_Y}$, and then taking $a = [x]_Y$, increasing the degree of x yet again to get that $\omega_2^{\mathrm{HOD}_{Y,[x]_Y,x}}$ is a Woodin cardinal in $\mathrm{HOD}_{Y,[x]_Y,[x]_{Y,[x]_Y}}$, etc. This leads to serious difficulties since the degree notion is changing. We would like to keep the degree notion fixed as we supplement a and for this reason we need the following variant of the previous theorem.

Theorem 5.43. *Assume* ZF+AD*. Suppose Y is a set and $a \in H(\omega_1)$. Then for a Y-cone of x,*

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \textit{ is a Woodin cardinal,}$$

where $[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}$.

Proof. The proof is essentially the same as that of the previous theorem. Claims 1 to 3 are exactly as before. The only difference is that now in Claim 4 we have $[x]_Y$ in place of $[x]_{Y,a}$. The proof of this version of the claim is the same, as is that of the rest of the theorem. □

6. Definable Determinacy

We now use the Generation Theorem to derive the optimal amount of large cardinal strength from both lightface and boldface definable determinacy.

The main result concerning lightface definable determinacy is the following:

Theorem 6.1. *Assume* ZF+DC+Δ^1_2*-determinacy. Then for a Turing cone of x,*

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

When combined with the results mentioned in the introduction this has the consequence that the theories ZFC + OD-determinacy and ZFC + "There is a Woodin cardinal" are equiconsistent. In order to prove this theorem we will have to get into the situation of the Generation Theorem. The issue here is that Δ^1_2-determinacy does not imply that for a cone of x strategic determinacy holds in $L[x]$ with respect to the constructibility degree of x. Instead we will use a different basis set B, one for which we can establish ST^B-determinacy for four moves, using Δ^1_2-determinacy alone.

The main result concerning boldface definable determinacy is the following:

Theorem 6.2. *Assume* $\mathrm{ZF}+\mathrm{AD}$. *Suppose* Y *is a set. There is a generalized Prikry forcing* $\mathbb{P}_Y$ *through the* Y*-degrees such that if* $G \subseteq \mathbb{P}_Y$ *is* V*-generic and* $\langle [x_i]_Y \mid i<\omega\rangle$ *is the associated sequence, then*

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle, V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals.}$$

When combined with the results mentioned in the introduction this has the consequence that the theires ZFC + OD($\mathbb{R}$)-determinacy and ZFC + "There are ω-many Woodin cardinals" are equiconsistent. As an application we show that when conjoined with the Derived Model Theorem (Theorem 1.5 or, more generally, Theorem 8.12) this result enables one to reprove and generalize Kechris' theorem (Theorem 2.6).

6.1. Lightface Definable Determinacy

In this subsection we will work in the theory $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$-determinacy and examine the features of the model $L[x]$ for a Turing cone of reals x. Our aim is to show that for a Turing cone of x,

$$\mathrm{HOD}^{L[x]} \models \omega_2^{L[x]} \text{ is a Woodin cardinal.}$$

This will be done by showing that the conditions of the Generation Theorem can be met. We already know that this is true assuming full boldface determinacy in the background universe. But now we are working with a weak form of lightface definable determinacy and this presents new obstacles. The main difficulty is in showing that for a Turing cone of x,

$$L[x] \models \mathrm{ST}^B\text{-determinacy}$$

for an appropriate basis B. In the boldface setting we took our basis B to be the constructibility degree of x. But as we shall see (in Theorem 6.12) in our present setting one cannot secure this version of strategic determinacy. Nevertheless, it turns out that strategic determinacy holds for a different, smaller basis. This leads to the notion of *restricted strategic determinacy.*

We shall successively extract stronger and stronger forms of determinacy until we ultimately reach the version we need. The subsection closes with a series of limitative results, including results that motivate the need for strategic and restricted strategic determinacy.

Theorem 6.3 (Martin). *Assume* ZF + DC + Δ^1_2*-determinacy. Then* Σ^1_2*-determinacy.*

Proof. Consider $A = \{x \in \omega^\omega \mid \varphi(x)\}$ where φ is Σ^1_2. We have to show that A is determined. Our strategy is to show that if II (the Π^1_2 player) does not have a winning strategy for A then I (the Σ^1_2 player) has a winning strategy for A.

Assume that II does not have a winning strategy for A. First, we have to shift to a "local" setting where we can apply Δ^1_2-determinacy. For each $x \in \omega^\omega$,

$$L[x] \models \text{II does not have a winning strategy in } \{y \in \omega^\omega \mid \varphi(y)\}$$

(since otherwise, by Σ^1_3 upward absoluteness, II would have a winning strategy in V, contradicting our initial assumption) and so, by the Löwenheim-Skolem theorem, there is a countable ordinal λ such that

$$L_\lambda[x] \models \mathrm{T} + \text{II does not have a winning strategy in } \{y \in \omega^\omega \mid \varphi(y)\},$$

where T is some fixed sufficiently strong fragment of ZFC (such as $\mathrm{ZFC_N}$ where N is large or ZFC $-$ Replacement $+\ \Sigma_2$-Replacement). For $x \in \omega^\omega$, let

$$\lambda(x) = \mu\lambda\,(L_\lambda[x] \models \mathrm{T} + \text{II does not have a winning strategy in } \{y \in \omega^\omega \mid \varphi(y)\}).$$

For convenience let $A^x = \{y \in \omega^\omega \mid \varphi(y)\}^{L_{\lambda(x)}[x]}$.

Consider the game G

I	a, x
II	b

where I wins iff $a*b \in L_{\lambda(x)}[x]$ and $L_{\lambda(x)}[x] \models \varphi(a*b)$. Here Player I is to be thought of as choosing the playing field $L_{\lambda(x)}[x]$ in which the two players are to play an auxiliary round (via a and b) of the localized game A^x. The key point is that since $\lambda(x)$ is always defined this game is Δ^1_2 and hence determined.

We claim that I has a winning strategy in G (and so I wins each round of the localized games A^x) and, furthermore, that (by ranging over these rounds and applying upward Σ^1_2-absoluteness) this winning strategy yields a winning strategy for I in A.

Assume for contradiction (by Δ^1_2-determinacy) that II has a winning strategy τ_0 in G. For each $x \geqslant_T \tau_0$, in $L_{\lambda(x)}[x]$ we can derive a winning strategy τ^x for II in A^x as follows: For $a \in (\omega^\omega)^{L_{\lambda(x)}[x]}$, let $(a*\tau^x)_{II} = b$ where b is such that $(\langle a, x\rangle * \tau_0)_{II} = b$. Since τ_0 is a winning strategy for II in G and we have arranged that $a*b \in L_{\lambda(x)}[x]$, II must win in virtue of the second clause, which means that $a*b \notin A^x$. Thus, $L_{\lambda(x)}[x] \models$ "τ^x is a winning strategy for II in A^x", which is a contradiction.

Thus I has a winning strategy σ_0 in G. Consider the derived strategy σ such that for $b \in \omega^\omega$, $(\sigma*b)_I = a$ where a is such that $(\sigma_0*b)_I = \langle a, x\rangle$. Since σ_0 is a winning strategy for I in G, $\sigma*b \in L_{\lambda(x)}[x]$ and $L_{\lambda(x)}[x] \models \varphi(\sigma*b)$ and so, by upward Σ^1_2-absoluteness, $V \models \varphi(\sigma*b)$. Thus, σ is a winning strategy for I in A. □

6.4 Remark.

(1) The above proof relativizes to a real parameter to show that $\Delta^1_2(x)$-determinacy implies $\Sigma^1_2(x)$-determinacy.

(2) A similar but more elaborate argument shows that if Δ^1_2-determinacy holds and for every real x, $x^\#$ exists, then $\mathrm{Th}(L[x])$ is constant for a Turing cone of x. See [7].

Theorem 6.5 (Martin). *Assume* ZF + DC + Δ^1_2-*determinacy. If I has a winning strategy in a* Σ^1_2-*game then I has a* Δ^1_3 *strategy.*

Proof. Consider $A = \{x \in \omega^\omega \mid \varphi(x)\}$ where φ is Σ^1_2. Our strategy is to show that if II (the Π^1_2 player) does not win A then I (the Σ^1_2 player) wins A via a Δ^1_3 strategy.

Assume that II does not have a winning strategy in A. For $x \in \omega^\omega$, let $\lambda(x)$, A^x, G, and σ_0 be as in the previous proof. Since σ_0 is a winning strategy for I in G, for $x \geqslant_T \sigma_0$, in $L_{\lambda(x)}[x]$ we can derive a winning strategy

σ^x for I in A^x as follows: For $x \geqslant_T \sigma_0$ and $b \in (\omega^\omega)^{L_{\lambda(x)}[x]}$ let $(\sigma * b)_I = a$ where a is such that $(\sigma_0 * b)_I = \langle a, x \rangle$.

Next we show that there is an $x_0 \geqslant_T \sigma_0$ such that for all $x \geqslant_T x_0$, $L_{\lambda(x)}[x] \models \Delta^1_2$-determinacy. Let $\langle \varphi_n, \psi_n \rangle$ enumerate the pairs of Σ^1_2-formulas and let $A_\varphi = \{x \in \omega^\omega \mid \varphi(x)\}$. Using DC let z_n be such that z_n codes a winning strategy for A_{φ_n} if $A_{\varphi_n} = \omega^\omega \smallsetminus A_{\psi_n}$ (i.e. A_{φ_n} is Δ^1_2); otherwise let $z_n = \langle 0, 0, \ldots \rangle$. Finally, let x_0 code $\langle z_n \mid n < \omega \rangle$. Thus, for $x \geqslant_T x_0$,

$$L_{\lambda(x)}[x] \models \text{I has a } \Sigma^1_4 \text{ strategy in } A^x$$

by the Third Periodicity Theorem of Moschovakis.

(For a proof of Third Periodicity see Jackson's chapter in this Handbook. The statement of Third Periodicity typically involves boldface determinacy. However, the proof shows that lightface Δ^1_2 determinacy suffices to get Σ^1_4 winning strategies for Σ^1_2 games that I wins. To see this note that $\text{Scale}(\Sigma^1_2)$ holds in ZF + DC. Furthermore, we also have the determinacy of the Σ^1_2 games (denoted $G^n_{s,t}$ in Jackson's chapter) that are used to define the prewellorderings and ultimately the definable strategies. It follows that these prewellorderings and strategies are $\Game\Sigma^1_2 \subseteq \Sigma^1_4$. (Notice that if we had $\undertilde{\Delta}^1_2$-determinacy then we could flip the quantifiers and conclude that $\Game\Sigma^1_2 = \Pi^1_3$ and hence get Δ^1_3 strategies. However, in our present lightface setting some more work is required.))

For $x \geqslant_T x_0$, let $\hat{\sigma}^x$ be the Σ^1_4-strategy for I in A^x. For a Turing cone of x the formula $\varphi(y, z)$ defining this strategy is constant. We can now "freeze out" the value of $\hat{\sigma}^x$ on a Turing cone of x. The key point is that the function $x \mapsto L_{\lambda(x)}[x]$ is Δ^1_2. So, for each $s \in \omega^{2n}$ and $m \in \omega$ the statement

$$L_{\lambda(x)}[x] \models \varphi(s, m)$$

is Δ^1_2. Thus, for each $s \in \omega^{2n}$, the m such that $L_{\lambda(x)}[x] \models \varphi(s, m)$ is fixed for a Turing cone of x. Since there are only countably many $s \in \omega^{2n}$ this means that the value of $\hat{\sigma}^x$ is fixed on a Turing cone of x. Finally, letting

$$\begin{aligned} \sigma(s) = m &\leftrightarrow \exists x_0 \forall x \geqslant_T x_0 \, (L_{\lambda(x)}[x] \models \varphi(s, m)) \\ &\leftrightarrow \forall x_0 \exists x \geqslant_T x_0 \, (L_{\lambda(x)}[x] \models \varphi(s, m)) \end{aligned}$$

(where we have used Δ^1_2-determinacy to flip the quantifiers) we have that σ is a Δ^1_3 winning strategy for I in A. □

Kechris and Solovay showed (in [6]) that under ZF+DC+Δ^1_2-determinacy there is a real x_0 that "enforces" OD-determinacy in the following sense: For all $x \geqslant_T x_0$, $L[x] \models$ OD-determinacy. We will need the following strengthening of this result, which involves a stronger notion of "enforcement". We need the following definition: An ordinal λ is *additively closed* (*a.c.*) iff for all $\alpha, \beta < \lambda$, $\alpha + \beta < \lambda$.

Theorem 6.6. *Assume* ZF + DC + Δ^1_2*-determinacy. Then there is a real* x_0 *such that for all additively closed* $\lambda > \omega$*, and for all reals* x*, if* $x_0 \in L_\lambda[x]$*, then* $L_\lambda[x] \models$ OD*-determinacy.*

Proof. Some preliminary remarks are in order. First, for λ additively closed, $L_\lambda[x]$ might satisfy only a very weak fragment of ZFC; so the statement "$L_\lambda[x] \models$ OD-determinacy" is to be taken in the following external sense: For each $\xi < \lambda$ and for each formula φ, $L_\lambda[x] \models$"Either I or II has a winning strategy for $\{z \in \omega^\omega \mid \varphi(z, \xi)\}$". The point is that this statement makes sense even when $\{z \in \omega^\omega \mid \varphi(z, \xi)\}$ is a proper class from the point of view of $L_\lambda[x]$. Second, the key feature of additively closed $\lambda > \omega$, is that if $y \in L_\lambda[x]$ then $L_\lambda[y] \subseteq L_\lambda[x]$. This is true since additively closed ordinals $\lambda > \omega$ are such that $\alpha + \lambda = \lambda$ for all $\alpha < \lambda$ and so if y is constructed at stage α, then $L_\lambda[x]$ still has λ-many remaining stages in which to "catch up" and construct everything in $L_\lambda[y]$. Third, the proof of the theorem is a "localization" of the proof of Theorem 5.12.

Assume for contradiction that for every real x_0 there is an additively closed $\lambda > \omega$ and a real x such that $x_0 \in L_\lambda[x]$ and $L_\lambda[x] \not\models$ OD-determinacy. So, for every real x_0 there is an additively closed $\lambda > \omega$ and a real $x' \geqslant_T x_0$ such that $L_\lambda[x'] \not\models$ OD-determinacy (since we can take $x' = \langle x, x_0 \rangle$ where x and x_0 are as in the first statement) and hence, by the Löwenheim-Skolem theorem,

$$\forall x_0 \in \omega^\omega\, \exists x \geqslant_T x_0\, \exists \lambda\, \lambda \text{ is a.c.} \wedge \omega < \lambda < \omega_1 \wedge \\ L_\lambda[x] \not\models \text{OD-determinacy},$$

where 'a.c.' abbreviates 'additively closed'. Since the condition on x in this statement is Σ^1_2 and since we have Σ^1_2-determinacy (by Theorem 6.3)

$$\exists x_0 \in \omega^\omega\, \forall x \geqslant_T x_0\, \exists \lambda\, \lambda \text{ is a.c.} \wedge \omega < \lambda < \omega_1 \wedge \\ L_\lambda[x] \not\models \text{OD-determinacy}$$

by the Cone Theorem (Theorem 2.9). Let

$$\lambda(x) = \begin{cases} \mu\lambda\,(\omega < \lambda < \omega_1 \wedge \lambda \text{ is a.c.} \wedge L_\lambda[x] \not\models \text{OD-det}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Notice that for a Turing cone of x

$$\lambda(x) \text{ is defined}$$

and that there are x_0 of arbitrarily large Turing degree such that for all $x \geqslant_T x_0$

$$\lambda(x) \geqslant \lambda(x_0).$$

To see this last point it suffices to observe that otherwise (by Σ^1_2-determinacy and the Cone Theorem (Theorem 2.9)) there would be an infinite descending sequence of ordinals. This point will be instrumental below in ensuring that Player II can "steer into the right model".

For each x such that $\lambda(x)$ is defined let (φ_x, ξ_x) be lexicographically least such that

$$L_{\lambda(x)}[x] \models \{z \in \omega^\omega \mid \varphi_x(z, \xi_x)\} \text{ is not determined}$$

and let $A^x = \{z \in \omega^\omega \mid \varphi_x(z, \xi_x)\}$. (However, note that since A^x might be a proper class from the point of view of $L_{\lambda(x)}[x]$, when we write '$L_{\lambda(x)}[x] \models a \in A^x$' we really mean '$L_{\lambda(x)}[x] \models \varphi_x(a, \xi_x)$'.)

Consider the game

$$\begin{array}{ll} \text{I} & a, b \\ \text{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d\rangle$, I wins iff $\lambda(p)$ is defined and $L_{\lambda(p)}[p] \models$ "$a * d \in A^p$". This game is Σ^1_2, hence determined.

(Notice that in contrast to the proof of Theorem 5.12 we cannot include x_0 in p since we need our game to be lightface definable. However, in the plays of interest we will have one player fold in x_0. This will ensure that the first clause of the winning condition is satisfied and so the players are to be thought of as cooperating to determine the model $L_{\lambda(p)}[p]$ and simultaneously playing an auxiliary game (via a and d) on the least non-determined OD set of this model, namely, A^x.)

We will arrive at a contradiction by showing that neither player can win.

Case 1: I has a winning strategy σ_0.

Let $x_0 \geqslant_T \sigma_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. We claim that $L_{\lambda(x_0)}[x_0] \models$ "I has a winning strategy σ in A^{x_0}", which is a contradiction. The strategy σ is the strategy derived by playing the main game according to σ_0 while having II feed in x_0 for c and playing some auxiliary play $d \in L_{\lambda(x_0)}[x_0]$; that is, $(\sigma{*}d)_I = a$ where a is such that $(\sigma_0{*}\langle x_0, d\rangle)_I = \langle a, b\rangle$:

$$\begin{array}{ll} \text{I} & a, b \\ \text{II} & x_0, d. \end{array}$$

Let $p = \langle a, b, x_0, d\rangle$. Since we have ensured that $p \geqslant_T x_0$ we know that $\lambda(p)$ is defined and, since σ_0 is winning for I, I must win in virtue of the first clause and so $L_{\lambda(p)}[p] \models$ "$a{*}d \in A^p$". It remains to see that II has managed to "steer into the right model", that is, that

$$L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0]$$

and hence

$$A^p = A^{x_0}.$$

Since $x_0 \geqslant_T \sigma_0$ and $d \in L_{\lambda(x_0)}[x_0]$ we have that $p \in L_{\lambda(x_0)}[x_0]$ and

$$L_{\lambda(x_0)}[p] = L_{\lambda(x_0)}[x_0]$$

(where for the left to right inclusion we have used that $\lambda(x_0)$ is additively closed). Furthermore, by arrangement, $\lambda(p) \geqslant \lambda(x_0)$ since $p \geqslant_T x_0$. But $\lambda(p)$ is the *least* additively closed λ such that $\omega < \lambda < \omega_1$ and $L_\lambda[p] \not\models$ OD-determinacy. Thus, $\lambda(p) = \lambda(x_0)$ and

$$L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0].$$

So $L_{\lambda(x_0)}[x_0] \models$ "$\sigma{*}d \in A^{x_0}$". Since this is true for any $d \in L_{\lambda(x_0)}[x_0]$, this means that $L_{\lambda(x_0)}[x_0] \models$ "σ is a winning strategy for I in A^{x_0}", which is a contradiction.

Case 2: II has a winning strategy τ_0.

Let $x_0 \geqslant_T \tau_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. For $a \in L_{\lambda(x_0)}[x_0]$ let $(a{*}\tau)_{II}$ where d is such that $(\langle a, x_0\rangle{*}\tau_0)_{II} = \langle c, d\rangle$. Since $p \geqslant_T x_0$, II must win in virtue of the second clause. The rest of the argument is exactly as above. So we have that $L_{\lambda(x_0)}[x_0] \models$ "τ is a winning strategy for II in A^{x_0}", which is a contradiction. □

6.7 Remark. The proof relativizes to a real parameter to show ZF + DC + $\Sigma^1_2(x)$-determinacy implies that there is a real enforcing (in the strong sense of Theorem 6.6) OD_x-determinacy.

Corollary 6.8 (Kechris and Solovay). *Assume* ZF. *Suppose* $L[x] \models \Delta^1_2$-*determinacy, where* $x \in \omega^\omega$. *Then* $L[x] \models$ OD-*determinacy.*

Proof. This follows by reflection. ☐

We will now extract an even stronger form of determinacy from Δ^1_2-determinacy. We begin by recalling some definitions. The *strategic game with respect to the basis* B is the game SG^B

$$\begin{array}{llllllll} \mathrm{I} & A_0 & \cdots & A_n & \cdots & & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}$, $A_{n+1} \in \mathscr{P}(\omega^\omega) \cap \mathrm{OD}_{f_0,\ldots,f_n}$ and

(2) f_n is a prestrategy for A_n that is winning with respect to B,

and II wins iff he can play all ω rounds. We say that *strategic determinacy holds with respect to the basis* B (ST^B-*determinacy*) if II wins SG^B.

In the context of $L[S,x]$ we dropped reference to the basis B since it was always understood to be $\{y \in \omega^\omega \mid L[S,y] = L[S,x]\}$. In our present lightface setting we will have to pay more careful attention to B since (as we will see in Theorem 6.12) Δ^1_2-determinacy is insufficient to ensure that for a Turing cone of x, $L[x] \models \mathrm{ST}^B$-determinacy, where $B = \{y \in \omega^\omega \mid L[y] = L[x]\}$. We will now be interpreting strategic determinacy in the local setting of models $L_\lambda[x]$ where $x \in \omega^\omega$ and λ is a countable ordinal and the relevant basis will be of the form $C \cap \{y \in \omega^\omega \mid L_\lambda[y] = L_\lambda[x]\}$ where C is a Π^1_2 set of $L_\lambda[x]$. It is in the attempt to "localize" the proof of Theorem 5.14 that the need for the Π^1_2 set becomes manifest. The issue is one of "steering into the right model" and can be seen to first arise in the proof of Claim 3 below.

Let RST-determinacy abbreviate the statement "There is a Π^1_2 set C such that C contains a Turing cone and ST^B-determinacy holds where $B = C \cap \{y \in \omega^\omega \mid L[y] = V\}$". Here 'R' stands for 'restrictive'. We will be interpreting this notion over models $L_\lambda[x]$ that do not satisfy full Replacement. In such a case it is to be understood that the statement involves the Σ_1 definition of ordinal definability.

Theorem 6.9. *Assume* ZF + DC + Δ^1_2*-determinacy. Then for a Turing cone of* x,

$$L[x] \models \text{RST-}\textit{determinacy}.$$

Proof. We will actually prove something stronger: Assume ZF + $V = L[x]$ + Δ^1_2-determinacy for some $x \in \omega^\omega$. Let T be the theory ZFC − Replacement + Σ_2-Replacement. Then there is a real z_0 such that if $L_\lambda[z]$ is such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models$ T then $L_\lambda[z] \models$ RST-determinacy. The theorem follows by reflection.

Assume for contradiction that for every real z_0 there is a real $z \geqslant_T z_0$ and an ordinal λ such that $L_\lambda[z] \models \mathrm{T} + \neg$RST-determinacy. The preliminary step is to reduce to a local setting where we can apply Δ^1_2-determinacy. By the Löwenheim-Skolem theorem

$$\forall z_0 \in \omega^\omega \, \exists z \geqslant_T z_0 \, \exists \lambda < \omega_1 \, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}).$$

Since the condition on z in this statement is Σ^1_2 and since we have OD-determinacy (by Corollary 6.8) it follows (by the Cone Theorem (Theorem 2.9)) that

$$\exists z_0 \in \omega^\omega \, \forall z \geqslant_T z_0 \, \exists \lambda < \omega_1 \, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}).$$

For $z \in \omega^\omega$, let

$$\lambda(z) = \begin{cases} \mu\lambda \, (L_\lambda[z] \models \mathrm{T} + \neg\text{RST-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Thus, if $\lambda(z)$ is defined, then for every $(\Pi^1_2)^{L_{\lambda(z)}[z]}$ set C that contains a Turing cone, I wins the game

$$\begin{array}{llllll} \mathrm{I} & A_0 & \cdots & A_n & \cdots & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathrm{OD}^{L_{\lambda(z)}[z]}$, $A_{n+1} \in \mathrm{OD}^{L_{\lambda(z)}[z]}_{f_0,\ldots,f_n}$, and

(2) f_n is a prestrategy for A_n that is winning with respect to $C \cap \{y \in \omega^\omega \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$.

We now need to specify a particular $(\Pi^1_2)^{L_{\lambda(z)}[z]}$ set since (i) we want to get our hands on a canonical winning strategy σ^z for I and (ii) we need to solve the "steering problem". The naïve approach would be to forget about the Π^1_2 sets and just work with $\{y \in \omega^\omega \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$. The trouble is that for an element y of this set we might have $\lambda(y) < \lambda(z)$ and yet (when we implement the proof of Theorem 5.13) we will need to ensure that $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and thus $A^y = A^z$ and for this we require that $\lambda(y) = \lambda(z)$. So we will need to intersect with a set C that "holds up the value of $\lambda(y)$". A good candidate is the following: For each z such that $\lambda(z)$ is defined let

$$C_z = \{y \in \omega^\omega \mid \lambda(y) \text{ is undefined}\}^{L_{\lambda(z)}[z]}.$$

This is a $(\Pi^1_2)^{L_{\lambda(z)}[z]}$-set. It contains z (since in $L_{\lambda(z)}[z]$ the ordinal $\lambda(z)$ is certainly undefined). We would like to ensure that it contains the cone above z.

Claim 1. For a Turing cone of z,

(1) $\lambda(z)$ is defined,

(2) for all reals $y \in L_{\lambda(z)}[z]$, if $y \geqslant_T z$ then $\lambda(y) = \lambda(z)$.

Proof. We have already proved (1). Assume for contradiction that (2) does not hold on a Turing cone. Then (by OD-determinacy) for every real z there is a real $z' \geqslant_T z$ such that $\lambda(z')$ is defined and in $L_{\lambda(z')}[z']$ there is a real z'' such that $z'' \geqslant_T x'$ and $\lambda(z'') < \lambda(z)$. But then, for each $n < \omega$, we can successively choose $z_{n+1} \geqslant_T z_n$ such that $\lambda(z_{n+1}) < \lambda(z_n)$, which is a contradiction. ☐

For each z as in Claim 1 we now have that C_z contains the Turing cone above z (since, by (2) of Claim 1, $\lambda(y) = \lambda(z)$ and so $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and again in $L_{\lambda(z)}[z]$ the ordinal $\lambda(y) = \lambda(z)$ is undefined). Letting

$$B_z = \{y \in C_z \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$$

we have that

$$\text{I wins } (SG^{B_z})^{L_{\lambda(z)}[z]}.$$

Moreover, since we have arranged that $L_{\lambda(z)}[z] \models \mathrm{T}$, Player I has a canonical strategy $\sigma^z \in \mathrm{HOD}^{L_{\lambda(z)}[z]}$. (This is because, since $L_{\lambda(z)}[z] \models \mathrm{T}$, the $\mathrm{OD}^{L_{\lambda(z)}[z]}$ sets of reals are sets (and not proper classes) in $L_{\lambda(z)}[z]$. So the tree on which

$(SG^{B_z})^{L_{\lambda(z)}[z]}$ is played is an element of $\mathrm{HOD}^{L_{\lambda(z)}[z]}$.) Notice also that σ^z depends only on the model $L_{\lambda(z)}[z]$, in the sense that if $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ then $\sigma^y = \sigma^z$.

Our aim is to obtain a contradiction by defeating σ^z for some z in the Turing cone of Claim 1. We will do this by constructing a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that I must win via $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ and, for a Turing cone of z, the winning strategies give rise to prestrategies $f_0^z, f_1^z, \ldots, f_n^z, \ldots$ that constitute a non-losing play against σ^z in the game $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Step 0. Consider (in $L[x]$) the game G_0

$$\begin{array}{lcl} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is either 1 or 2 and, letting $p = \langle a, b, c, d\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A_0^p$", where $A_0^p = \sigma^p(\varnothing)$.

In the plays of interest we will ensure that p is in the cone of Claim 1. So clause (1) of the winning condition will be automatically satisfied and the decisive factor will be whether in $L_{\lambda(p)}[p]$ Player ϵ wins the auxiliary round (via a and d) of A_0^p. This game is Σ_2^1 (for Player I), hence determined.

Claim 2. I has a winning strategy σ_0 in G_0.

Proof. Assume for contradiction that I does not have a winning strategy in G_0. Then, by Σ_2^1-determinacy, II has a winning strategy τ_0 in G_0. Let $z_0 \geqslant_T \tau_0$ be such that for all $z \geqslant_T z_0$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models$ OD-determinacy (by Theorem 6.6).

Consider $A_0^{z_0} = \sigma^{z_0}(\varnothing)$. Since $L_{\lambda(z_0)}[z_0] \models$ OD-determinacy, $L_{\lambda(z_0)}[z_0] \models$ "$A_0^{z_0}$ is determined". We will use τ_0 to show that neither player can win this game. Suppose for contradiction that $L_{\lambda(z_0)}[z_0] \models$ "σ is a winning strategy for I in $A_0^{z_0}$". Run G_0 according to τ_0, having Player I (falsely) predict that

Player I wins the auxiliary game, while steering into $L_{\lambda(z_0)}[z_0]$ by playing $b = z_0$ and using σ to respond to τ_0 on the auxiliary play:

$$\begin{array}{lll} \mathrm{I} & 1 & (\sigma * d)_I, z_0 \\ \mathrm{II} & & c, d \end{array}$$

We have to see that Player I has indeed managed to steer into $L_{\lambda(z_0)}[z_0]$, that is, we have to see that $L_{\lambda(p)}[p] = L_{\lambda(z_0)}[z_0]$, where $p = \langle (\sigma * d)_I, z_0, c, d\rangle$. Since $\sigma, z_0, \tau_0 \in L_{\lambda(z_0)}[z_0]$ and $\lambda(z_0)$ is additively closed, we have $L_{\lambda(z_0)}[p] = L_{\lambda(z_0)}[z_0]$. But $\lambda(p) = \lambda(z_0)$ since z_0 satisfies Claim 1. Thus, $L_{\lambda(p)}[p] = L_{\lambda(z_0)}[z_0]$ and hence $A_0^p = A_0^{z_0}$. Finally, since τ_0 is a winning strategy for II in G_0 and $\epsilon = 1$, we have that $L_{\lambda(p)}[p] \models$ "$\sigma * d \notin A_0^p$", and hence $L_{\lambda(z_0)}[z_0] \models$ "$\sigma * d \notin A_0^{z_0}$", which contradicts the assumption that σ is a winning strategy for I. Similarly, we can use τ_0 to defeat any strategy τ for II in $A_0^{z_0}$. □

Since the game is Σ^1_2 for Player I, Player I has a Δ^1_3-strategy σ_0, by Theorem 6.5.

Claim 3. For every real $z \geqslant_T \sigma_0$ in the Turing cone of Claim 1, there is a prestrategy f_0^z such that f_0^z is definable in $L_{\lambda(z)}[z]$ from σ_0 and f_0^z is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Proof. Fix $z \geqslant_T \sigma_0$ as in Claim 1. Consider $A_0^z = \sigma^z(\varnothing)$. Let f_0^z be the prestrategy derived from σ_0 in $L_{\lambda(z)}[z]$ by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L_{\lambda(z)}[z]}$ let $f_0^z(y)$ be such that for $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$, $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. f_0^z is clearly definable in $L_{\lambda(z)}[z]$ from σ_0. We claim that f_0^z is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

To motivate the need for the Π^1_2 set, let us first see why f_0^z need not be a prestrategy for II in A_0^z that is winning with respect to $\{y \in (\omega^\omega)^{L_{\lambda(z)}[z]} \mid L_{\lambda(z)}[y] = L_{\lambda(z)}[z]\}$. Consider such a real y and an auxiliary play $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$. By definition $f_0^z(y)$ is such that $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. Assume first that $\epsilon = 1$. Since σ_0 is a winning strategy for I in G_0, $f_0^z(y) * d = a * d \in A_0^p$ where $p = \langle a, b, y, d\rangle$. What we need, however, is that $f_0^z(y) * d = a * d \in A_0^z$. The trouble is that we may have $L_{\lambda(p)}[p] = L_{\lambda(y)}[y] \subsetneq L_{\lambda(z)}[z]$ because although $L_{\lambda(z)}[y] = L_{\lambda(z)}[z]$ we might have $\lambda(y) < \lambda(z)$. And if this is indeed the case then we cannot conclude that $A_0^p = A_0^z$. If $\epsilon = 0$ then $f_0^z(y) * d = a * d \notin A_0^p$ but again what we need is that $f_0^z(y) * d = a * d \notin A_0^z$ and the same problem arises.

The above problem is solved by demanding in addition that $y \in C_z$, since then $\lambda(y) = \lambda(z)$ and so $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f_0^z(y)*d = a*d \in A_0^p = A_0^z$" as desired. Thus f_0^z is a non-losing first move for II against σ_z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. □

Step n+1. Assume that we have defined (in $L[x]$) games $G_0, \ldots, G_n$ with winning strategies $\sigma_0, \ldots, \sigma_n \in \mathrm{HOD}$ such that for all $z \geqslant_T \langle \sigma_0, \ldots, \sigma_n \rangle$ in the Turing cone of Claim 1 there are prestrategies $f_0^z, \ldots, f_n^z$ such that f_i^z is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \ldots, \sigma_i$ (for all $i \leqslant n$) and $f_0^z, \ldots, f_n^z$ is a non-losing partial play for II in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Consider (in $L[x]$) the game G_{n+1}

$$\begin{array}{lcc} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is 1 or 2 and, letting $p = \langle a, b, c, d, \sigma_0, \ldots, \sigma_n \rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a*d \in A_{n+1}^p$", where A_{n+1}^p is I's response via σ^p to II's partial play $f_0^p, \ldots, f_n^p$

If p satisfies condition (1) then, since $p \geqslant_T \langle \sigma_0, \ldots, \sigma_n \rangle$, we have, by the induction hypothesis, prestrategies $f_0^p, \ldots, f_n^p$ such that f_i^p is definable in $L_{\lambda(p)}[p]$ from $\sigma_0, \ldots, \sigma_i$ (for all $i \leqslant n$) and $f_0^p, \ldots, f_n^p$ is a non-losing partial play for II in $(SG^{B_p})^{L_{\lambda(p)}[p]}$. Thus, condition (2) in the definition of the game makes sense.

This game is $\Sigma_2^1(\sigma_0, \ldots, \sigma_n)$ (for Player I) and hence determined (since $\sigma_0, \ldots, \sigma_n \in \mathrm{HOD}$ and we have OD-determinacy, by Theorem 6.6).

Claim 4. I has a winning strategy σ_{n+1} in G_{n+1}.

Proof. Assume for contradiction that I does not have a winning strategy. Then, by OD-determinacy, II has a winning strategy τ_{n+1}. Let $z_{n+1} \geqslant_T \langle \tau_{n+1}, \sigma_0, \ldots, \sigma_n \rangle$ be such that for all $z \geqslant_T z_{n+1}$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_{n+1} \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models \mathrm{OD}_{\sigma_0, \ldots, \sigma_n}$-determinacy (by the relativized version of Theorem 6.6).

It follows that

$$L_{\lambda(z_{n+1})}[z_{n+1}] \models A^{z_{n+1}}_{n+1} \text{ is determined},$$

where $A^{z_{n+1}}_{n+1} = \sigma^{z_{n+1}}(\langle f^{z_{n+1}}_0, \ldots, f^{z_{n+1}}_n \rangle)$. This is because $A^{z_{n+1}}_{n+1}$ is an element of $\text{HOD}^{L_{\lambda(z_{n+1})}[z_{n+1}]}(\sigma_0, \ldots, \sigma_n)$ (as all of the ingredients $\sigma^{z_{n+1}}$, $f^{z_{n+1}}_0$, $\ldots$, $f^{z_{n+1}}_n$ used to define $A^{z_{n+1}}_{n+1}$ are in this model) and we arranged that $L_{\lambda(z_{n+1})}[z_{n+1}]$ satisfies $\text{OD}_{\sigma_0,\ldots,\sigma_n}$-determinacy.

[The enforcement of the parameterized version of OD-determinacy in (2) appears to be necessary. The point is that even though, in Step 1 for example, σ_0 is Δ^1_3 and $\sigma_0 \in L_{\lambda(z)}[z]$ we have no guarantee that in $L_{\lambda(z_1)}[z_1]$, σ_0 satisfies this definition. If we did then we would have that A^z_1 is in $\text{HOD}^{L_{\lambda(z_1)}[z_1]}$ and hence just enforce OD-determinacy.]

We will use τ_{n+1} to show that neither player can win this game. The argument is exactly as in Step 0 except with the subscripts '0' replaced by 'n+1': Suppose for contradiction that $L_{\lambda(z_{n+1})}[z_{n+1}] \models$ "σ is a winning strategy for I in $A^{z_{n+1}}_{n+1}$". Run G_{n+1} according to τ_{n+1}, having Player I (falsely) predict that Player I wins the auxiliary game, while steering into $L_{\lambda(z_{n+1})}[z_{n+1}]$ by playing $b = z_{n+1}$ and using σ to respond to τ_{n+1} on the auxiliary play:

$$\begin{array}{lll} \text{I} & 1 & (\sigma * d)_I, z_{n+1} \\ \text{II} & & c, d \end{array}$$

We have to see that Player I has indeed managed to steer into $L_{\lambda(z_{n+1})}[z_{n+1}]$, that is, we have to see that $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$, where p is the set $\langle (\sigma * d)_I, z_{n+1},\ c, d\rangle$. Since $\sigma, z_{n+1}, \tau_{n+1} \in L_{\lambda(z_{n+1})}[z_{n+1}]$ and $\lambda(z_{n+1})$ is additively closed, we have $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$. Since $p \geqslant_T z_{n+1}$ and z_{n+1} satisfies the condition of Claim 1, $\lambda(p) = \lambda(z)$, and so $L_{\lambda(p)}[p] = L_{\lambda(z_{n+1})}[z_{n+1}]$ and hence $A^p_{n+1} = A^{z_{n+1}}_{n+1}$. Finally, since τ_{n+1} is a winning strategy for II in G_{n+1} and $\epsilon = 1$, we have that $L_{\lambda(p)}[p] \models$ "$\sigma * d \notin A^p_{n+1}$", and hence $L_{\lambda(z_{n+1})}[z_{n+1}] \models$ "$\sigma * d \notin A^{z_{n+1}}_{n+1}$", which contradicts the assumption that σ is a winning strategy for I. Similarly, we can use τ_{n+1} to defeat any strategy τ for II in $A^{z_{n+1}}_{n+1}$. □

Since the game is $\Sigma^1_2(\sigma_0, \ldots, \sigma_n)$ for Player I, Player I has a $\Delta^1_3(\sigma_0, \ldots, \sigma_n)$ strategy σ_{n+1}, by the relativized version of Theorem 6.5.

Claim 5. For every real $z \geqslant_T \langle \sigma_0, \ldots, \sigma_n \rangle$ as in Claim 1, there is a pre-strategy f^z_{n+1} that is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \ldots, \sigma_{n+1}$ and such that $f^z_0, \ldots, f^z_{n+1}$ is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$.

Proof. The proof is just like the proof of Claim 3. Fix $z \geqslant_T \langle \sigma_0, \ldots, \sigma_n \rangle$ as in Claim 1 and consider $A^z_{n+1} = \sigma^z(\langle f^z_0, \ldots, f^z_n \rangle)$. Let f^z_{n+1} be the prestrategy derived from σ_{n+1} in $L_{\lambda(z)}[z]$ by extracting the response in the auxiliary game, that is, for $y \in (\omega^\omega)^{L_{\lambda(z)}[z]}$ let $f^z_{n+1}(y)$ be such that for $d \in (\omega^\omega)^{L_{\lambda(z)}[z]}$, $f^z_{n+1}(y) * d = a * d$ where a is such that $(\sigma_{n+1} * \langle y, d \rangle)_I = \langle \epsilon, a, b \rangle$. Clearly, f^z_{n+1} is definable in $L_{\lambda(z)}[z]$ from $\sigma_0, \ldots, \sigma_{n+1}$. We claim that f^z_{n+1} is a non-losing first move for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. Again the point is that for $y \in B_z$, $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$, hence $A^y_{n+1} = A^z_{n+1}$. Thus, $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f^z_{n+1}(y) * d = a * d \in A^p_{n+1} = A^z_{n+1}$" as desired. Hence $\langle f^z_0, \ldots, f^z_{n+1} \rangle$ is a non-losing play for II against σ^z in $(SG^{B_z})^{L_{\lambda(z)}[z]}$. □

Finally, letting z^∞ be such that $z^\infty \geqslant_T z_n$ for all n and z^∞ is as in Claim 1, we have that $f^{z^\infty}_0, \ldots, f^{z^\infty}_n, \ldots$ defeats σ^{z^∞} in $(SG^{B_{z^\infty}})^{L_{\lambda(z^\infty)}[z^\infty]}$, which is a contradiction. □

Theorem 6.10. *Assume* ZF + DC + Δ^1_2*-determinacy. Then for a Turing cone of x,*

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \omega_2^{L[x]} \textit{ is a Woodin cardinal.}$$

Proof. For a Turing cone of x, $L[x] \models$ RST-determinacy, by Theorem 6.9. Let x be in this cone. We have to meet the conditions of the Generation Theorem. Let $\Theta_M = \omega_2^{L[x]}$. Since $L[x]$ satisfies GCH and $L[x] = \mathrm{OD}_x^{L[x]}$,

$$\omega_2^{L[x]} = \sup\{\alpha \mid \text{there is an } \mathrm{OD}^{L[x]} \text{ prewellordering of length } \alpha\},$$

in other words, $\omega_2^{L[x]} = (\Theta_0)^{L[x]}$. Let $A = \langle A_\alpha \mid \alpha < \omega_2^{L[x]} \rangle$ be such that A_α is the $\mathrm{OD}^{L[x]}$-least prewellordering of length α. Since $L[x] \models$ OD-determinacy, it follows (by Theorem 3.9) that $\omega_2^{L[x]}$ is strongly inaccessible in $\mathrm{HOD}^{L[x]}$. So there is a set $H \subseteq \omega_2^{L[x]}$ coding $\mathrm{HOD}^{L[x]} \cap V_{\omega_2^{L[x]}}$. Let T' be in $\mathscr{P}(\omega_2^{L[x]}) \cap \mathrm{OD}^{L[x]}$ and let $T \in \mathscr{P}(\omega_2^{L[x]}) \cap \mathrm{OD}^{L[x]}$ code T' and H. Let B be as in the statement of RST-determinacy.

Let

$$M = (L_{\Theta_M}(\mathbb{R})[T, A, B])^{L[x]},$$

where Θ_M, T, A, B are as above. Conditions (1)–(5) of the Generation Theorem are clearly met and condition (6) follows since $L[x]$ satisfies RST-determinacy, M is OD in $L[x]$ and M contains the reals of $L[x]$. Thus,

$$\mathrm{HOD}^M_{T,A,B} \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal.}$$

Since, by arrangement, $\mathrm{HOD}^M_{T,A,B} = \mathrm{HOD}^{L[x]} \cap V_{\omega_2^{L[x]}}$, it follows that

$$\mathrm{HOD}^{L[x]} \models \mathrm{ZFC} + \text{There is a } T\text{-strong cardinal.}$$

Since T' was arbitrary, the theorem follows. □

We close with four limitative results. The first result motivates the need for the notion of strategic determinacy by showing that strategic determinacy does not follow trivially from OD-determinacy in the sense that for some OD basis there are OD prestrategies.

Theorem 6.11. *Assume* ZF. *Then for each non-empty* OD *set* $B \subseteq \omega^\omega$, *there is an* OD *set* $A \subseteq \omega^\omega$ *such that there is no* OD *prestrategy in* A *which is winning with respect to the basis* B.

Proof. Assume for contradiction that there is a set $B \subseteq \omega^\omega$ which is OD and such that for all OD sets $A \subseteq \omega^\omega$ there is an OD prestrategy f_A in A which is winning with respect to B. We may assume OD-determinacy since if OD-determinacy fails then the theorem trivially holds (as clearly one cannot have a prestrategy which is winning with respect to a non-empty basis for a non-determined game).

We shall need to establish three claims.

Claim 1. Assume ZF. Then

$$\bigcap \big\{A \subseteq \omega^\omega \mid A \in \mathrm{OD}, A \text{ is Turing invariant, and } A \text{ contains a Turing cone}\big\} = \varnothing.$$

Proof. For each $\alpha < \omega_1$, let

$$A_\alpha = \{z \in \omega^\omega \mid \exists x, y \in \omega^\omega \text{ such that } x \equiv_T y \leqslant_T z \text{ and } x \text{ codes } \alpha\}.$$

Notice that A_α is OD, Turing invariant, and contains a Turing cone. But clearly

$$\bigcap_{\alpha<\omega_1} A_\alpha = \varnothing$$

since otherwise there would be a real z which recursively encodes all countable ordinals. □

Claim 2. Assume ZF + OD-determinacy. Then

$$\text{HOD} \models \text{There is a countably complete ultrafilter on } \omega_1^V.$$

Proof. Since we are not assuming $\mathrm{AC}_\omega(\mathbb{R})$, the proof of Theorem 2.12 does not directly apply. To see this note that ω_1^V may not be regular in V—in fact, we do not even know whether ω_1^V is regular in HOD. Nevertheless, we will be able to implement some of the previous arguments by dropping into an appropriate model of $\mathrm{AC}_\omega(\mathbb{R})$. In the case of countable completeness an additional change will be required since without $\mathrm{AC}_\omega(\mathbb{R})$ we cannot choose countably many strategies as we did in the earlier proof. Let

$$\mu = \{S \subseteq \omega_1^V \mid S \in \text{HOD and I has a winning strategy in } G(S)\},$$

where $G(S)$ is the game from Theorem 2.12.

Subclaim 1. $\text{HOD} \models \mu \cap \text{HOD}$ is an ultrafilter.

Proof. It is clear that $\omega_1^V \in \mu$ and $\varnothing \notin \mu$. It is also clear that if $S \in \mu$ and $S' \in \text{HOD} \cap \mathscr{P}(\omega_1^V)$ and $S \subseteq S'$ then $S' \in \mu$.

Suppose that $S \in \text{HOD} \cap \mathscr{P}(\omega_1^V)$ and that II has a winning σ strategy in $G(S)$. We claim that I has a winning strategy in $G(\omega_1^V \smallsetminus S)$. Suppose for contradiction that I does not have a winning strategy. Then, by OD-determinacy, II has a winning strategy σ'. Now work in $L[\sigma, \sigma']$. Using $\undertilde{\Sigma}^1_1$-boundedness, by the usual arguments, one can construct a play x for I which is legal against both σ and σ' and in each case has the same associated ordinal $\alpha < \omega_1^{L[\sigma,\sigma']}$. This is a contradiction.

We now show that if $S_1, S_2 \in \mu$ then $S_1 \cap S_2 \in \mu$. Let σ_1 be a winning strategy for I in $G(S_1)$ and let σ_2 be a winning strategy for I in $G(S_2)$. Suppose for contradiction that $S_1 \cap S_2 \notin \mu$. Since $S_1 \cap S_2$ is OD, $G(S_1 \cap S_2)$ is determined and so II has a winning strategy in $G(S_1 \cap S_2)$, which implies that I has a winning strategy σ in $G(\omega_1^V \smallsetminus (S_1 \cap S_2))$. Work in $L[\sigma_1, \sigma_2, \sigma]$. The strategy σ_1 witnesses (by the usual argument using $\undertilde{\Sigma}^1_1$-boundedness) that $S_1 \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club. Likewise, σ_2 witnesses that $S_2 \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club and σ witnesses that $(\omega_1^V \smallsetminus (S_1 \cap S_2)) \cap \omega_1^{L[\sigma_1,\sigma_2,\sigma]}$ contains a club. This contradiction completes the proof of Subclaim 1. □

Subclaim 2. $\text{HOD} \models \mu \cap \text{HOD}$ is countably complete.

Proof. Suppose for contradiction that the subclaim fails. Let $\langle S_i \mid i < \omega\rangle \in$ HOD be such that for each $i < \omega$, $S_i \in \mu$ and $\bigcap_{i<\omega} S_i = \varnothing$. Consider the game

$$\begin{array}{lccccccc} \text{I} & i & & y(0) & & y(1) & & \dots \\ \text{II} & & x(0) & & x(1) & & \dots & \end{array}$$

where II wins if and only if $x{*}y$ is a winning play for I in $G(S_i)$. The idea is that Player I begins by specifying a set S_i in our fixed sequence and then the two players play an auxiliary round of $G(S_i)$, with Player I playing as Player II and Player II playing as Player I.

Notice that this game is OD, hence determined. We claim that II has a winning strategy. Suppose for contradiction that I has a winning strategy σ. In the first move the strategy σ produces a fixed k. Since σ is winning for I, for each $x \in \omega^\omega$, $x{*}\sigma$ is a win for II in $G(S_k)$. But this is impossible since $S_k \in \mu$ and so I has a winning strategy τ_k in $G(S_k)$; thus, by following τ_k in the auxiliary game, II (playing as I) can defeat σ.

Let τ be a winning strategy for II. Work in $L[\tau]$. We claim that in $L[\tau]$, τ witnesses that for all $i < \omega$, $S_i \cap \omega_1^{L[\tau]}$ contains a club. For our purposes we just need a single $\alpha \in \bigcap_{i<\omega} S_i$. The point is that Player I can play any i as the first move and then use $\underset{\sim}{\Sigma}^1_1$-boundedness to produce a real y such that for all $i < \omega$, $i^\frown y$ is a legal play and in each case the ordinal produced in the auxiliary game is some fixed $\alpha < \omega_1^{L[\tau]}$. This contradiction completes the proof of Subclaim 2. □

Thus,

$$\text{HOD} \models \mu \cap \text{HOD is a countably complete ultrafilter on } \omega_1^V,$$

which completes the proof. □

It follows that ZF + OD-determinacy proves that

$$\text{HOD} \models \exists\kappa \leqslant \omega_1^V\ (\kappa \text{ is a measurable cardinal}),$$

(as witnessed by letting κ be the completeness of the ultrafilter), and hence that $\mathbb{R} \cap \text{HOD}$ is countable. Let $\alpha < \omega_1$ be the length of the canonical well-ordering of $\mathbb{R} \cap \text{HOD}$. Let t code α. Then in HOD_t there is a real y^* such that for all $z \in \mathbb{R} \cap \text{HOD}$, $z \leqslant_T y^*$. Let y^* be such a real.

Claim 3. Suppose $z \in B$. Suppose A is OD, A is Turing invariant, and A contains a Turing cone. Then A contains the Turing cone above $\langle y^*, z\rangle$.

Proof. By our original supposition for contradiction recall that we let f_A be an OD prestrategy which is winning with respect to B. Since A contains a Turing cone f_A must be winning for Player I. This means that for all $z \in B$, for all $y \in \omega^\omega$, $f_A(z) * y \in A$. Now let $y \geqslant_T \langle y^*, z\rangle$. We wish to show that $y \in A$. The point is that

$$y \equiv_T f_A(z) * y \in A$$

and since A is Turing invariant, this implies that $y \in A$. □

Claim 3 contradicts Claim 1, which completes the proof. □

The second result motivates the need for restricted strategic determinacy by showing that $V = L[x] + \Delta^1_2$-determinacy does not imply ST^B-determinacy, where B is the constructibility degree of x. Thus, in Theorem 6.9 it was necessary to drop down to a restricted form of strategic determinacy. It also follows from the theorem that something close to $\undertilde{\Delta}^1_2$-determinacy is required to establish that ST^B-determinacy holds with respect to the constructibility degree of x since the statement "$\undertilde{\Delta}^1_2$-determinacy" is equivalent to the statement "for every $y \in \omega^\omega$ there is an inner model M such that $y \in M$ and $M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$"

Theorem 6.12. *Assume* ZF + $V = L[x]$ *for some* $x \in \omega^\omega$. *Suppose* ST^B-*determinacy, where* $B = \{y \in \omega^\omega \mid L[y] = L[x]\}$. *Suppose there exists an* $\alpha > \omega_1^{L[x]}$ *such that* $L_\alpha[x] \models \mathrm{ZFC}$. *Then for every* $y \in \omega^\omega$ *there is a transitive model* M *such that* $y \in M$ *and* $M \models \mathrm{ZFC} +$ *There is a Woodin cardinal.*

Proof. Let

$$A_0 = \{y \in \omega^\omega \mid \neg\exists M\,(M \text{ is transitive} \wedge y \in M \wedge \\ M \models \mathrm{ZFC} + \text{There is a Woodin cardinal})\}.$$

Suppose for contradiction that $A_0 \neq \varnothing$. Let $t \in A_0$. It follows that A_0 contains a Turing cone of reals. Let Player I play A_0 in SG^B and let f_0 be II's response. Since Player I can win a round of A_0 by playing t, f_0 is winning for I with respect to B, that is, for all $y \in B$, $f_0(y) \in A_0$. We will arrive at a contradiction by constructing a real $y \in B$ such that $f_0(y) \notin A_0$.

We claim that

$$\mathrm{HOD}^{L_\alpha[x]}_{f_0} \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

First note that

$$L[x] \models \mathrm{ST}^B_{f_0}\text{-determinacy}.$$

Since $L_\alpha[x]$ is ordinal definable in $L[x]$ (as $\alpha > \omega_1^{L[x]}$ and so $L_\alpha[x] = L_\alpha[x']$ for any real x' such that $V = L[x']$) it follows that

$$L_\alpha[x] \models \mathrm{ST}^B_{f_0}\text{-determinacy}.$$

Thus, by the relativized version of Theorem 6.10,

$$\mathrm{HOD}^{L_\alpha[x]}_{f_0} \models \mathrm{ZFC} + \text{There is a Woodin cardinal}.$$

Therefore $f_0 \notin A_0$.

By $\Sigma^1_2(f_0)$-absoluteness, $L[f_0]$ satisfies that there is a countable transitive model M such that $f_0 \in M$ and

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal}.$$

Since $L_\alpha[x] \models$ "$f_0^\#$ exists" (by the effective version of Solovay's Theorem (Theorem 2.15) there is a countable ordinal λ such that $L_\lambda[f_0]$ satisfies ZFC+ "M is countable". In $L_\lambda[f_0]$ let P be a perfect set of reals that are Cohen generic over M. Since P is perfect in $L_\lambda[f_0]$ there is a path $c \in [P]$ which codes x in the sense that $c \geqslant_T x$.

Our desired real y is $\langle f_0, c\rangle$. To see that $\langle f_0, c\rangle \in B$ note that $L_\lambda(\langle f_0, c\rangle)$ can compute x and hence $L_{\omega_1}[\langle f_0, c\rangle] = L_{\omega_1}[x]$. To see that $f_0(\langle f_0, c\rangle) \notin A_0$ note that since c is Cohen generic over M, the model $M[c]$ is a transitive model containing $f_0(\langle f_0, c\rangle)$ satisfying ZFC + "There is a Woodin cardinal". This is a contradiction. □

The third result shows that Martin's "lightface form" of Third Periodicity (Theorem 6.3) does not generalize to higher levels. In fact, the result shows that even ZFC + OD-determinacy (assuming consistency of course) does not imply that for every Σ^1_4 game which Player I wins, Player I has a Δ^1_5 strategy (or even an OD strategy). The reason that the "lightface form" of Third Periodicity holds at the level of Σ^1_2 but not beyond is that in Third Periodicity boldface determinacy is used to get scales but in ZF + DC we get Scale($\utilde{\Sigma}^1_2$) for free.

Theorem 6.13. *Assume* ZF + $V = L[x]$ + OD-*determinacy for some* $x \in \omega^\omega$. *There is a* Π^1_2 *set of reals which contains a Turing cone but which does not contain a member in* HOD.

Proof. Consider the set

$$A = \{y \in \omega^\omega \mid \text{for all additively closed } \lambda < \omega_1, \text{ for all } z \geqslant_T y, \text{ if } x \in \mathrm{OD}^{L_\lambda[z]} \text{ then } x \leqslant_T y\}.$$

This is a Π^1_2 set. Notice that each $y \in A$ witnesses that $\mathbb{R} \cap \mathrm{HOD}^{L[z]}$ is countable for each $z \geqslant_T y$.

Claim 1. A contains a Turing cone.

Proof. For $y \in \omega^\omega$ and α such that $\omega < \alpha < \omega_1$, let $R_{\alpha,y}$ be the set of reals which are ordinal definable in $L_\alpha[y]$ and let $<_{\alpha,y}$ the canonical well-ordering of $R_{\alpha,y}$, where we arrange that $<_{\alpha,y}$ is an initial segment of $<_{\alpha',y}$ when $\alpha < \alpha'$. For $y \in \omega^\omega$, let $R_y = \bigcup\{R_{\alpha,y} \mid \omega < \alpha < \omega_1\}$ and let $<_y$ be the induced order on R_y (where we order first by α and then by $<_{\alpha,y}$). Let z^y_α be the α^{th} real in $<_y$ and let ϑ_y be the ordertype of $<_y$. Notice that $R_{\alpha,y}$, R_y, $<_{\alpha,y}$, $<_y$, z^y_α and ϑ_y depend only on the Turing degree of y.

Our strategy is to "freeze out" the values of R_y and $<_y$ on a Turing cone of y. For $\alpha < \omega_1$, the set

$$A_\alpha = \{y \in \omega^\omega \mid \vartheta_y > \alpha\}$$

is OD and hence, by OD-determinacy, either it or its complement contains a Turing cone. Moreover, if A_α contains a Turing cone and $\bar{\alpha} < \alpha$ then $A_{\bar{\alpha}}$ contains a Turing cone. Thus,

$$A' = \{\alpha < \omega_1 \mid A_\alpha \text{ contains a Turing cone}\}$$

is an initial segment of ω_1. For each $\alpha \in A'$, and for each $y \in \omega^\omega$, the statement "$\vartheta_y > \alpha$ and $z^y_\alpha(n) = m$" is an OD-statement about y. So, by OD-determinacy, the value of z^y_α is fixed for a Turing cone of y. We write z_α for this stable value. It follows that $\langle z_\alpha \mid \alpha \in A'\rangle$ is a definable well-ordering of reals and hence, by OD-determinacy, A' must be countable (by the effective version of Solovay's theorem (Theorem 2.15) and the argument in the Claim in Theorem 5.9). Let $\vartheta = \sup\{\alpha + 1 \mid \alpha \in A'\}$. Finally, let $R_\infty = \{z_\alpha \mid \alpha < \vartheta\}$ and $<_\infty = \{(z_\alpha, z_\beta) \mid \alpha < \beta < \vartheta\}$. We claim that for a Turing cone of y, $R_y = R_\infty$. To see this let $y \in L[x]$ be such that $x \leqslant_T y$ (so, in particular $L[x] = L[y]$) and y belongs to all of the (countably many) cones fixing z_α for $\alpha \in A'$. Then $\vartheta_y = \vartheta$ and $R_y = R_\infty$. (In fact, $R_y = \mathbb{R}^{\mathrm{HOD}}$.)

Let z_0 be such that for all $z \geqslant_T z_0$, $R_z = R_{z_0} = R_\infty$. Since R_∞ is countable, we can choose $y_0 \geqslant_T z_0$ such that $R_\infty \leqslant_T y_0$. Then for all $z \geqslant_T y_0$, $R_z = R_\infty \leqslant_T y_0$, that is, $y_0 \in A$. Likewise, if $y \geqslant_T y_0$, then $y \in A$. □

Claim 2. $A \cap \mathrm{HOD}^{L[x]} = \varnothing$.

Proof. Suppose for contradiction that $y \in A \cap \mathrm{HOD}^{L[x]}$. Since $y \in A$, y witnesses that R_z is countable for all $z \geqslant_T y$. Let z be such that

$$R_x = R_z.$$

Then since $y \in \mathrm{HOD}^{L[x]}$,

$$\mathrm{HOD}^{L[x]} \models \mathbb{R} \text{ is countable},$$

which is impossible. □

This completes the proof. □

The final result is a refinement of a theorem of Martin ([4, Theorem 13.1]). It shows that $\mathrm{ZF} + \mathrm{DC} + \Delta^1_2$-determinacy implies that for a Turing cone of x, $\mathrm{HOD}^{L[x]}$ has a Δ^1_3 well-ordering of reals and hence that for a Turing cone of x, Δ^1_2-determinacy fails in $\mathrm{HOD}^{L[x]}$.

Theorem 6.14. *Assume* $\mathrm{ZF} + V = L[x] + \Delta^1_2$*-determinacy, for some* $x \in \omega^\omega$*. Then in* HOD *there is a* Δ^1_3*-well-ordering of the reals.*

Proof. For $y \in \omega^\omega$ and α such that $\omega < \alpha < \omega_1$, let $R_{\alpha,y}$, $<_{\alpha,y}$, z^y_α, R_y, $<_y$, and ϑ_y be as in the proof of Theorem 6.13. Let A' and R_∞ be as in the proof of Theorem 6.13. The argument of Claim 1 of Theorem 6.13 shows that $R_\infty = \mathbb{R}^{\mathrm{HOD}}$: To see this let $x' \in L[x]$ be such that $x \leqslant_T x'$ (so, in particular $L[x] = L[x']$) and x' belongs to all of the (countably many) cones fixing z_α for $\alpha \in A'$. Then $\vartheta_{x'} = \vartheta$ and $R_\infty = R_{x'} = \mathbb{R} \cap \mathrm{HOD}^{L[x']} = \mathbb{R}^{\mathrm{HOD}}$.

Notice that

$$\exists y_0 \forall y \geqslant_T y_0 \forall \omega < \alpha < \omega_1 \, (R_{\alpha,y} \subseteq R_\infty \wedge <_{\alpha,y} \trianglelefteq <_\infty),$$

where $\trianglelefteq$ denotes ordering by initial segment; x' as above is such a y_0. Since R_∞ and $<_\infty$ are countable they can be coded by a real. Let y_0 be the base of the above cone and let a be a real coding $\langle y_0, R_\infty, <_\infty \rangle$. The statement "$a$ codes $\langle y_0, R_\infty, <_\infty \rangle$ and for all $y \geqslant_T y_0$, for all $\alpha < \omega_1$, $R_{\alpha,y} \subseteq R_\infty$ and $<_{\alpha,y} \trianglelefteq <_\infty$" is a Π^1_2 truth about a. Writing $\psi(a)$ for this statement we have the following Π^1_3-definitions (in $L[x]$) of $\omega^\omega \cap \mathrm{HOD}$ and $<_\infty$:

$$z \in R_\infty \leftrightarrow \forall a \, [\, a \text{ codes } (z, R, <) \wedge \psi(a) \rightarrow z \in R \,]$$

and

$$z_0 <_\infty z_1 \leftrightarrow \forall a\,[\,a \text{ codes } (z, R, <) \wedge \psi(a) \to z_0 < z_1\,]\,.$$

We now look at things from the point of view of HOD. Fix $\xi < \vartheta$. We claim that ξ is countable in HOD. Consider the game

$$\begin{array}{ll} \text{I} & a, b \\ \text{II} & c \end{array}$$

where I wins iff there is an $\alpha < \omega_1$ such that $z_\xi \in R_{\alpha,\langle b,c\rangle}$ and a codes the ordertype of $<_{\alpha,\langle b,c\rangle}\restriction z_\xi$. This game is $\Sigma^1_2(z_\xi)$ (for Player I) and since $z_\xi \in \text{HOD}$ the game is determined. Moreover, I must win (since I can play $b = y_0$ and an a coding ξ). By (the relativized version of) Theorem 6.5, Player I has a winning strategy $\sigma \in \text{HOD}$. It follows that ξ is less than the least admissible relative to σ, which in turn is countable in HOD.

Thus, we can let z be a real in HOD coding $<_\infty\restriction z_\xi$. Consider the game $G(z, z_\xi)$

$$\begin{array}{ll} \text{I} & a \\ \text{II} & b \end{array}$$

where I wins iff there exists an α such that $z_\xi \in R_{\alpha,\langle a,b\rangle}$ and $<_{\alpha,\langle a,b\rangle}\restriction z_\xi = z$. This game is $\Sigma^1_2(\langle z, z_\xi\rangle)$, hence determined. Moreover, I must win. So I has a winning strategy $\sigma_\xi \in \text{HOD}$.

Finally, notice the following: If $y \geqslant_T \sigma_\xi$ then $y \equiv_T \sigma_\xi * y$ and

$$\forall\alpha\,(\omega < \alpha < \omega_1 \wedge z_\xi \in R_{\alpha,y} \to <_{\alpha,y}\restriction z_\xi = <_\infty\restriction z_\xi).$$

So the following is a Σ^1_3-calculation of $<_\infty$ in HOD:

$$\begin{aligned} x <_\infty y \;\leftrightarrow\; & \exists a \in \omega^\omega \text{ coding } (y_0, <, z) \text{ such that} \\ & < \text{ is a linear ordering of its domain, } \mathrm{dom}(<), \\ & x, y, z \in \mathrm{dom}(<), \\ & x < y \text{ and } y < z, \text{ and} \\ & \forall y' \geqslant_T y_0\, \forall\alpha\,(\omega < \alpha < \alpha_1 \wedge z \in R_{\alpha,y'} \\ & \qquad\qquad\qquad \to <_{\alpha,y'}\restriction z = <\restriction z). \end{aligned}$$

This completes the proof, since clearly a Σ^1_3 total ordering is also Π^1_3. □

Putting everything together we have that ZF + DC + Δ^1_2-determinacy implies that for a Turing cone of x, $\text{HOD}^{L[x]}$ is an inner model with a Woodin cardinal and a Δ^1_3 well-ordering of reals. It follows that Δ^1_2-determinacy fails in $\mathrm{HOD}^{L[x]}$ for a Turing cone of x.

Some interesting questions remain. For example: Does $\mathrm{HOD}^{L[x]}$ satisfy GCH, for a Turing cone of x? Is $\mathrm{HOD}^{L[x]}$ a fine-structural model, for a Turing cone of x? We will return to this topic in Section 8.

6.2. Boldface Definable Determinacy

In this section we will work in ZF + AD. Our aim is to extract the optimal amount of large cardinal strength from boldface determinacy by constructing a model of ZFC that contains ω-many Woodin cardinals.

We shall prove a very general theorem along these lines. Our strategy is to iteratively apply Theorem 5.43. Recall that this theorem states that under ZF + AD, for a Y-cone of x,

$$\mathrm{HOD}_{Y,a,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,a,x}} \text{ is a Woodin cardinal,}$$

where

$$[x]_Y = \{z \in \omega^\omega \mid \mathrm{HOD}_{Y,z} = \mathrm{HOD}_{Y,x}\}.$$

We start by taking a to be the empty set. By Theorem 5.43, there exists an x_0 such that for all $x \geqslant_Y x_0$,

$$\mathrm{HOD}_{Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x}} \text{ is a Woodin cardinal.}$$

To generate a model with two Woodin cardinals we would like to apply Theorem 5.43 again, this time taking a to be $[x_0]_Y$. This gives us an $x_1 \geqslant_Y x_0$ such that for all $x \geqslant_Y x_1$,

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ is a Woodin cardinal}$$

and we would like to argue that

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ are Woodin cardinals.}$$

But there are two difficulties in doing this. First, in the very least, we need to ensure that

$$\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}}$$

and this is not immediate. Second, in moving to the larger model we need to ensure that we have not collapsed the first Woodin cardinal; a sufficient condition for this is that

$$\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} = \mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y},$$

but again this is not immediate. It turns out that both difficulties can be overcome by taking x to be of sufficiently high "Y-degree". This will be the content of an elementary observation and a "preservation" lemma. Once these two hurdles are overcome we will be able to generate models with n Woodin cardinals for each $n < \omega$. We shall then have to take extra steps to ensure that we can preserve ω-many Woodin cardinals. This will be achieved by shooting a Prikry sequence through the Y-degrees and proving an associated "generic preservation" lemma.

6.15 Remark. It is important to note that in contrast to Theorem 5.42 here the degree notion in Theorem 5.43 does not depend on a and this is instrumental in iteratively applying the theorem to generate several Woodin cardinals. In contexts such as $L(\mathbb{R})$ where HOD "relativizes" in the sense that $\mathrm{HOD}_a = \mathrm{HOD}[a]$, one could also appeal to Theorem 5.42, since in such a case $\mathrm{HOD}_{Y,a,[x]_{Y,a}} = \mathrm{HOD}_{Y,a,[x]_Y}$. Our reason for not taking this approach is twofold. First, it would take us too far afield to give the argument that $\mathrm{HOD}_a = \mathrm{HOD}[a]$ in, for example, $L(\mathbb{R})$. Second, it is of independent interest to work in a more general setting.

We shall be working with the "Y-degrees"

$$\mathscr{D}_Y = \{[x]_Y \mid x \in \omega^\omega\}.$$

Let μ_Y be the cone filter over $\mathscr{D}_Y$. As noted earlier, the argument of Theorem 2.9 shows that μ_Y is an ultrafilter. Also, by Theorem 2.8 we know that μ_Y is countably complete.

Lemma 6.16 (Preservation Lemma). *Assume* ZF + AD. *Suppose Y is a set, $a \in H(\omega_1)$, and $\alpha < \omega_1$. Then for a Y-cone of x,*

$$\mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a,[x]_Y} = \mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a}.$$

Proof. The right-to-left direction is immediate. Suppose for contradiction that the left-to-right direction fails. For sufficiently large $[x]_Y$, let

$$f([x]_Y) = \text{ least } Z \in \mathscr{P}(\alpha) \cap \mathrm{HOD}_{Y,a,[x]_Y} \smallsetminus \mathrm{HOD}_{Y,a},$$

where the ordering is the canonical ordering of $\mathrm{OD}_{Y,a,[x]_Y}$. This function is defined for a Y-cone of x and it is $\mathrm{OD}_{Y,a}$. Let $Z_0 \in \mathscr{P}(\alpha)$ be such that

$$\xi \in Z_0 \text{ iff } \xi \in f([x]_Y) \text{ for a } Y\text{-cone of } x.$$

Since α is countable and since μ_Y is countably complete $Z_0 = f([x]_Y)$ for sufficiently large x. Thus, $Z_0 \in \mathrm{HOD}_{Y,a}$, which is a contradiction. □

We are now in a position to iteratively apply Theorem 5.43 to generate a model with n Woodin cardinals.

Step 0. By Theorem 5.43, let x_0 be such that for all $x \geqslant_Y x_0$,

$$\mathrm{HOD}_{Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x}} \text{ is a Woodin cardinal.}$$

Step 1. Recall that ω_1^V is strongly inaccessible in any inner model of ZFC, by the Claim of Theorem 5.9. It follows that $\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_1^V$ and so when we choose $x_1 \geqslant_Y x_0$ we may assume that x_1 codes $\omega_2^{\mathrm{HOD}_{Y,x_0}}$. Thus, there exists an $x_1 \geqslant_Y x_0$ such that for all $x \geqslant_Y x_1$,

$$\omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}},$$

and, by the Preservation Lemma (taking a to be $[x_0]_Y$),

$$\mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} = \mathscr{P}(\omega_2^{\mathrm{HOD}_{Y,x_0}}) \cap \mathrm{HOD}_{Y,[x_0]_Y},$$

and, by Theorem 5.9 (taking a to be $[x_0]_Y$),

$$\mathrm{HOD}_{Y,[x_0]_Y,[x]_Y} \models \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x}} \text{ is a Woodin cardinal.}$$

It follows that

$$\mathrm{HOD}_{Y,[x_0]_Y,[x_1]_Y} \models \omega_2^{\mathrm{HOD}_{Y,x_0}} < \omega_2^{\mathrm{HOD}_{Y,[x_0]_Y,x_1}} \text{ are Woodin cardinals.}$$

Step n+1. It is useful at this stage to introduce a piece of notation: For $x_0 \leqslant_Y \cdots \leqslant_Y x_{n+1}$, let

$$\delta_0(x_0) = \omega_2^{\mathrm{HOD}_{Y,x_0}}$$

and

$$\delta_{n+1}(x_0, \ldots, x_{n+1}) = \omega_2^{\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,x_{n+1}}}.$$

Suppose that we have chosen $x_0 \leqslant_Y x_1 \leqslant_Y \cdots \leqslant_Y x_n$ such that

$$\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle} \models \delta_0(x_0) < \cdots < \delta_n(x_0,\ldots,x_n) \text{ are Woodin cardinals.}$$

Again, since ω_1^V is strongly inaccessible in any inner model of ZFC, it follows that each of these ordinals is countable in V and so when we choose $x_{n+1} \geqslant_Y x_n$ we may assume that x_{n+1} collapses these ordinals. Thus, there exists an $x_{n+1} \geqslant_Y x_n$ such that for all $x \geqslant_Y x_{n+1}$,

$$\delta_n(x_0,\ldots,x_n) < \delta_{n+1}(x_0,\ldots,x_n,x)$$

and, by the Preservation Lemma (taking a to be $\langle [x_0]_Y,\ldots,[x_n]_Y\rangle$),

$$\mathscr{P}(\delta_n(x_0,\ldots,x_n)) \cap \mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y} = \mathscr{P}(\delta_n(x_0,\ldots,x_n)) \cap \mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle}$$

and, by Theorem 5.43 (taking a to be $\langle [x_0]_Y,\ldots,[x_n]_Y\rangle$),

$$\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y} \models \delta_{n+1}(x_0,\ldots,x_n,x) \text{ is a Woodin cardinal.}$$

It follows that

$$\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle,[x]_Y} \models \delta_0(x_0) < \cdots < \delta_{n+1}(x_0,\ldots,x_n,x) \text{ are Woodin cardinals.}$$

We now need to ensure that when we do the above stacking for ω-many stages, the Woodin cardinals $\delta_n(x_0,\ldots,x_n)$ are preserved in the final model. This is not immediate since, for example, if we are not careful then the reals $x_0, x_1, \ldots$ might code up a real that collapses $\sup_{n<\omega} \delta_n(x_0,\ldots,x_n)$. To circumvent this difficulty we implement the construction relative to a "Prikry sequence" of degrees $[x_0]_Y, [x_1]_Y, \ldots$.

6.17 Definition (The forcing $\mathbb{P}_Y$). Assume ZF + AD. Suppose Y is a set. Let $\mathscr{D}_Y$ and μ_Y be as above. The conditions of $\mathbb{P}_Y$ are of the form $\langle [x_0]_Y,\ldots,[x_n]_Y, F\rangle$ where $F : \mathscr{D}_Y^{<\omega} \to \mu_Y$. The ordering on $\mathbb{P}_Y$ is:

$$\langle [x_0]_Y,\ldots,[x_n]_Y,[x_{n+1}]_Y,\ldots,[x_m]_Y, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle [x_0]_Y,\ldots,[x_n]_Y, F\rangle$$

if and only if

(1) $[x_{i+1}]_Y \in F(\langle [x_0]_Y, \ldots, [x_i]_Y \rangle)$ for all $i \geqslant n$ and

(2) $F^*(p) \subseteq F(p)$ for all $p \in \mathscr{D}_Y^{<\omega}$.

The point of the following lemma is to avoid appeal to DC.

Lemma 6.18. *Assume* ZF + AD. *Suppose φ is a formula in the forcing language and $\langle p, F \rangle \in \mathbb{P}_Y$. Then there is an F^* such that $\langle p, F^* \rangle \leqslant_{\mathbb{P}_Y} \langle p, F \rangle$ and $\langle p, F^* \rangle$ decides φ. Moreover, F^* is uniformly definable from $\langle p, F \rangle$ and φ.*

Proof. Fix φ a formula and $\langle p, F \rangle \in \mathbb{P}_Y$. Let us use '$p$' and '$q$' for "lower parts" of conditions—that is, finite sequences of $\mathscr{D}_Y$—'F' and 'G' for the corresponding "upper parts", and 'a' for elements of $\mathscr{D}_Y$. Write $q \trianglerighteq p$ to indicate that p is an initial segment of q. Set

$$\begin{aligned} Z_0 &= \{\, q \mid q \trianglerighteq p \text{ and } \exists G \ \langle q, G \rangle \leqslant_{\mathbb{P}_Y} \langle p, F \rangle \text{ and } \langle q, G \rangle \Vdash \varphi \,\}, \\ Z_{\alpha+1} &= \{\, q \mid \{\, a \mid q^\frown a \in Z_\alpha \,\} \in \mu_Y \,\}, \text{ and} \\ Z_\lambda &= \bigcup_{\alpha<\lambda} Z_\alpha \text{ for } \lambda \text{ a limit.} \end{aligned}$$

Let $D_q^\alpha = \{a \mid q^\frown a \in Z_\alpha\}$. So $Z_{\alpha+1} = \{q \mid D_q^\alpha \in \mu_Y\}$. We claim that for each α,

(1) if $q \in Z_\alpha$, then $D_q^\alpha \in \mu_Y$, and hence

(2) $Z_a \subseteq Z_{\alpha+1}$.

The proof is by induction on α: For $\alpha = 0$ suppose $q \in Z_0$ and let G witness this. So $G(q) \in \mu_Y$. Notice that for each $a \in G(q)$,

$$\langle q^\frown a, G \rangle \leqslant_{\mathbb{P}_Y} \langle q, G \rangle$$

and so $q^\frown a \in Z_0$, i.e. $G(q) \subseteq \{a \mid q^\frown a \in Z_0\} = D_q^0$ and so $D_q^0 \in \mu_Y$ and $Z_0 \subseteq Z_1$. Assume (1) holds for $\alpha + 1$. It follows that $Z_{\alpha+1} \subseteq Z_{\alpha+2}$. Suppose $q \in Z_{\alpha+2}$. Then, by the definition of $Z_{\alpha+2}$, $D_q^{\alpha+1} \in \mu_Y$. However, since $Z_{\alpha+1} \subseteq Z_{\alpha+2}$, it follows that $D_q^{\alpha+1} \subseteq D_q^{\alpha+2}$. So $D_q^{\alpha+2} \in \mu_Y$. For λ a limit ordinal suppose $q \in Z_\lambda$. Then $q \in Z_\alpha$ for some $\alpha < \lambda$. So, by the induction hypothesis, $D_q^\alpha \in \mu_Y$. Since $Z_\alpha \subseteq Z_\lambda$, $D_q^\alpha \subseteq D_q^\lambda$ and so $D_q^\lambda \in \mu_Y$.

Now define a ranking function $\rho : \mathscr{D}_Y^{<\omega} \to \mathrm{On} \cup \{\infty\}$ by

$$\rho(q) = \begin{cases} \text{least } \alpha \text{ such that } q \in Z_\alpha & \text{if there is such an } \alpha \\ \infty & \text{otherwise.} \end{cases}$$

We begin by noting the following three persistence properties which will aid us in shrinking F so as to decide φ. First, if $\rho(q) = \infty$ then set

$$A_q = \{ a \mid \rho(q^\frown a) = \infty \}$$

and notice that $A_q \in \mu_Y$ since otherwise we would have $\{ a \mid \rho(q^\frown a) \in \mathrm{On} \} \in \mu_Y$ (as μ_Y is an ultrafilter) and letting $\beta = \sup\{ \rho(q^\frown a) \mid \rho(q^\frown a) \in \mathrm{On} \}$ we would have $q \in Z_{\beta+1}$, a contradiction. Second, if $\rho(q) \in \mathrm{On} \smallsetminus \{ 0 \}$ then set

$$B_q = \{ a \mid \rho(q^\frown a) < \rho(q) \}$$

and notice that $B_q \in \mu_Y$ since $\rho(q)$ is clearly a successor, say $\alpha + 1$, and $q \in Z_\alpha$ and so (by our claim) $D^\alpha_q \in \mu_Y$; but $D^\alpha_q \subseteq B_q$. Third, if $\rho(q) = 0$ then set

$$C_q = \{ a \mid \rho(q^\frown a) = 0 \}$$

and notice that $C_q \in \mu_Y$ since clearly $q \in Z_0$ and so, by the claim, $D^0_q \in \mu_Y$; but $D^0_q \subseteq C_q$.

Now either $\rho(p) = \infty$ or $\rho(p) \in \mathrm{On}$.

Claim 1. If $\rho(p) = \infty$ then there is an F^* such that $\langle p, F^* \rangle \leqslant_{\mathbb{P}_Y} \langle p, F \rangle$ and $\langle p, F^* \rangle \Vdash \neg\varphi$.

Proof. Define F^* as follows:

$$F^*(q) = \begin{cases} F(q) \cap A_q & \text{if } \rho(q) = \infty \\ F(q) & \text{otherwise.} \end{cases}$$

Suppose that it is not the case that $\langle p, F^* \rangle \Vdash \neg\varphi$. Then $\exists \langle q, G \rangle \leqslant_{\mathbb{P}_Y} \langle p, F^* \rangle$ such that $\langle q, G \rangle \Vdash \varphi$. But then q is such that $\rho(q) = 0$. However, F^* witnesses that in fact $\rho(q) = \infty$: Suppose $q = p^\frown a_0{}^\frown \cdots {}^\frown a_k$. Since $\rho(p) = \infty$ and $a_0 \in F^*(p)$, we have that $a_0 \in A_p$ and so $\rho(p^\frown a_0) = \infty$. Continuing in this manner, we get that $\rho(q) = \infty$. This is a contradiction. □

Claim 2. If $\rho(p) \in \mathrm{On}$ then there is an F^* such that $\langle p, F^* \rangle \leqslant_{\mathbb{P}_Y} \langle p, F \rangle$ and $\langle p, F^* \rangle \Vdash \varphi$.

Proof. Define F^* as follows:

$$F^*(q) = \begin{cases} F(q) \cap B_q & \text{if } \rho(q) \in \mathrm{On} \smallsetminus \{0\} \\ F(q) \cap C_q & \text{if } \rho(q) = 0 \\ F(q) & \text{otherwise.} \end{cases}$$

We claim that $\langle p, F^*\rangle \Vdash \varphi$. Assume not. Then $\exists\langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F^*\rangle$ such that $\langle q, G\rangle \Vdash \neg\varphi$. Since $\langle q, G\rangle \leqslant_{\mathbb{P}_Y} \langle p, F^*\rangle$ and $\rho(p) \in \mathrm{On}$ we have that $\rho(q) \in \mathrm{On}$ (by an easy induction using the definition of F^*). We may assume that $\langle q, G\rangle$ is chosen so that $\rho(q)$ is as small as possible. But then $\rho(q) = 0$ as otherwise there is an a such $\langle q^\frown a, G\rangle \leqslant_{\mathbb{P}_Y} \langle q, G\rangle$, $\langle q^\frown a, G\rangle \Vdash \neg\varphi$ and $\rho(q^\frown a) < \rho(q)$, contradicting the minimality of $\rho(q)$. Now since $\rho(q) = 0$, $q \in Z_0$ and so $\exists G'\langle q, G'\rangle \Vdash \varphi$. But this is a contradiction since $\langle q, G'\rangle$ is compatible with $\langle q, G\rangle$ and $\langle q, G\rangle \Vdash \neg\varphi$. □

This completes the proof of the lemma. □

We can now obtain the following "generic preservation" lemma.

Lemma 6.19 (Generic Preservation Lemma). *There exists an F such that if $G \subseteq \mathbb{P}_Y$ is V-generic and $\langle \varnothing, F\rangle \in G$ and $\langle [x_i]_Y \mid i < \omega\rangle$ is the generic sequence associated to G, then, for all $i < \omega$,*

$$\mathscr{P}(\delta_i(x_0, \ldots, x_i))^{V[G]} \cap \mathrm{HOD}^{V[G]}_{Y,\langle [x_j]_Y \mid j<\omega\rangle, V} = \mathscr{P}(\delta_i(x_0, \ldots, x_i)) \cap \mathrm{HOD}^V_{Y,\langle [x_0]_Y, \ldots, [x_i]_Y\rangle}.$$

Proof. We need the following extension of Lemma 6.18: Suppose $\langle \varphi_\xi \mid \xi < \alpha\rangle$ is a countable sequence of formulas in the forcing language (evaluated in a rank initial segment) and $\langle p, F\rangle \in \mathbb{P}_Y$ is a condition. Then there is an F^* such that $\langle p, F^*\rangle \leqslant_{\mathbb{P}_Y} \langle p, F\rangle$ and $\langle p, F^*\rangle$ decides φ_ξ, for each $\xi < \alpha$, and F^* is uniformly definable from $\langle \varphi_\xi \mid \xi < \alpha\rangle$ and $\langle p, F\rangle$. For each $\xi < \alpha$, let F_ξ be as in Lemma 6.18 (where it is denoted F^*). Letting F^* be the "intersection" of the F_α—i.e., such that $F^*(q) = \bigcap_{\alpha<\beta} F_\alpha(q)$ for each $q \in D_Y^{<\omega}$—we have that $\langle p, F^*\rangle$ decides φ_ξ for each $\xi < \alpha$ and that F^* is uniformly definable from $\langle \varphi_\xi \mid \xi < \alpha\rangle$ and $\langle p, F\rangle$.

Suppose $a \in H(\omega_1)$ and $\alpha < \omega_1$. We claim that we can definably associate with a and α a function $F_{a,\alpha} : \mathscr{D}_Y^{<\omega} \to \mu_Y$ such that $\langle \varnothing, F_{a,\alpha}\rangle$ forces

$$\mathscr{P}(\alpha)^{V[G]} \cap \mathrm{HOD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle, V} = \mathscr{P}(\alpha) \cap \mathrm{HOD}^V_{Y,a}.$$

Let φ be the formula in the forcing language that expresses the displayed statement. By Lemma 6.18 there is an $\mathrm{OD}_{Y,a}$ condition $\langle \varnothing, G\rangle$ deciding φ. Suppose for contradiction that this condition forces $\neg\varphi$. Since right-to-left inclusion holds trivially (as we are including V as a parameter) it must be that the left-to-right inclusion fails. Let $A \subseteq \alpha$ be $<\mathrm{OD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle,V}$-least such that

$$A \in \mathrm{HOD}^{V[G]}_{Y,a,\langle [x_i]_Y \mid i<\omega\rangle,V} \setminus \mathrm{HOD}^V_{Y,a}.$$

Now, for each $\xi < \alpha$ let φ_ξ be the statement expressing "$\xi \in A$". In an $\mathrm{OD}_{Y,a}$ fashion we can successively shrink $\langle \varnothing, G\rangle$ to decide each φ_ξ. But then A is $\mathrm{OD}_{Y,a}$ and hence in $\mathrm{HOD}^V_{Y,a}$, which is a contradiction.

We now define a "master function" $F : \mathscr{D}^{<\omega}_Y \to \mu_Y$ such that for all $\langle [x_0]_Y, \ldots, [x_n]_Y\rangle \in \mathscr{D}^{<\omega}_Y$,

$$F(\langle [x_0]_Y, \ldots, [x_n]_Y\rangle) = F_{\langle [x_0]_Y\rangle, \delta_0(x_0)} \cap \cdots \cap F_{\langle [x_0]_Y, \ldots, [x_n]_Y\rangle, \delta_n(x_0,\ldots,x_n)}.$$

Suppose $\langle \varnothing, F\rangle \in G$. Suppose $\langle p, H\rangle \in G$ and $p\restriction i+1 = \langle [x_0]_Y, \ldots, [x_i]_Y\rangle$. It follows that $\langle p, H \wedge F\rangle \in G$, where, by definition, $H \wedge F$ is such that $(H \wedge F)(q) = H(q) \cap F(q)$ for each q. But

$$\begin{aligned}\langle p, H\wedge F\rangle &\leqslant_{\mathbb{P}_Y} \langle p\restriction i+1, H\wedge F\rangle \\ &\leqslant_{\mathbb{P}_Y} \langle p\restriction i+1, H \wedge F_{p\restriction i+1, \delta_i(x_0,\ldots,x_i)}\rangle \\ &\leqslant_{\mathbb{P}_Y} \langle \varnothing, F_{p\restriction i+1, \delta_i(x_0,\ldots,x_i)}\rangle,\end{aligned}$$

(using the definition of F for the second line) and so $\langle \varnothing, F_{p\restriction i+1,\delta_i(x_0,\ldots,x_i)}\rangle \in G$. Finally, by the definition of $F_{p\restriction i+1,\delta_i(x_0,\ldots,x_i)}$, this condition forces

$$\begin{aligned}\mathscr{P}(\delta_i(x_0,\ldots,x_i))^{V[G]} &\cap \mathrm{HOD}^{V[G]}_{Y,\langle [x_0]_Y,\ldots,[x_i]_Y\rangle, \langle [x_i]_Y \mid i<\omega\rangle, V} \\ &= \mathscr{P}(\delta_i(x_0,\ldots,x_i)) \cap \mathrm{HOD}^V_{Y,\langle [x_0]_Y,\ldots,[x_i]_Y\rangle},\end{aligned}$$

which completes the proof. □

Theorem 6.20. *Assume* ZF + AD*. Then there is a condition* $\langle \varnothing, F\rangle \in \mathbb{P}_Y$ *such that if* $G \subseteq \mathbb{P}_Y$ *is* V*-generic and* $\langle \varnothing, F\rangle \in G$*, then*

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle,V} \models \mathrm{ZFC} + \textit{There are } \omega\textit{-many Woodin cardinals,}$$

where $\langle [x_i]_Y \mid i < \omega\rangle$ *is the sequence associated with* G*.*

Proof. Let $\langle \varnothing, F, \rangle$ be the condition from the Generic Preservation Lemma (Lemma 6.19). We claim that

$$\mathrm{HOD}^{V[G]}_{Y,\langle [x_i]_Y \mid i<\omega\rangle,V} \models \delta_0(x_0) < \cdots < \delta_n(x_0,\ldots,x_n) < \cdots$$

are Woodin cardinals.

By the Generic Preservation Lemma it suffices to show that for each $n < \omega$

$$\mathrm{HOD}_{Y,\langle [x_0]_Y,\ldots,[x_n]_Y\rangle} \models \delta_0(x_0) < \cdots < \delta_n(x_0,\ldots,x_n) \text{ are Woodin cardinals,}$$

which follows by genericity and the argument for the finite case. □

As an interesting application of this theorem in conjunction with the Derived Model Theorem (Theorem 8.12), we obtain Kechris' theorem that under ZF + AD, DC holds in $L(\mathbb{R})$. This alternate proof is of interest since it is entirely free of fine structure and it easily generalizes.

Theorem 6.21 (Kechris). *Assume* ZF + AD. *Then* $L(\mathbb{R}) \models \mathrm{DC}$.

Proof Sketch. Work in ZF + AD + $V = L(\mathbb{R})$. Let $Y = \varnothing$ and let $N = \mathrm{HOD}^{V[G]}_{\langle [x_i]_Y \mid i<\omega\rangle, V}$ where G and $[x_i]_Y$ are as in the above theorem. By genericity, the Woodin cardinals δ_i of N have ω_1^V as their supremum. By Vopěnka's theorem (see the proof of Theorem 7.8 below for the statement and a sketch of the proof), each $x \in \mathbb{R}^V$ is N-generic for some $\mathbb{P} \in N \cap V_{\omega_1^V}$. Thus, $N(\mathbb{R}^V)$ is a symmetric extension of N. The derived model of $N(\mathbb{R}^V)$ (see Theorem 8.12 below) satisfies $\mathrm{DC}_{\mathbb{R}}$ and therefore DC since $N \models \mathrm{AC}$. Furthermore $N(\mathbb{R}^V)$ contains $L(\mathbb{R})$ and cannot contain more since then $L(\mathbb{R})$ would have forced its own sharp. (This follows from AD^+ theory: Assume $\mathrm{ZF} + \mathrm{DC}_{\mathbb{R}} + \mathrm{AD} + V = L(\mathscr{P}(\mathbb{R}))$. Suppose $A \subseteq \mathbb{R}$. Then either $V = L(A, \mathbb{R})$ or $A^\#$ exists. See Definition 8.10 below.) Thus, $L(\mathbb{R}) \models \mathrm{AD} + \mathrm{DC}$. □

7. Second-Order Arithmetic

The statement that all Δ^1_2 sets are determined is really a statement of second-order arithmetic. So a natural question is whether the construction culminating in Section 6.1 can be implemented in this more limited setting. In this section we show that a variant of the construction can be carried out in this context. We break the construction into two steps. First, we show that a variant of the above construction can be carried out with respect to an object smaller than $\omega_2^{L[x]}$, one that is within the reach of second-order arithmetic. Second, we show that this version of the construction can be carried out in the weaker theory of second-order arithmetic.

The need to alter the previous construction is made manifest in the following result:

Theorem 7.1. *Assume* ZF + $V = L[x] + \Delta^1_2$*-determinacy, for some* $x \in \omega^\omega$*. Suppose* N *is such that*

(1) $\mathrm{On} \subseteq N \subseteq \mathrm{HOD}^{L[x]}$ *and*

(2) $N \models \delta$ *is a Woodin cardinal.*

Then $\delta \geqslant \omega_2^{L[x]}$.

However, it turns out that $\omega_1^{L[x]}$ can be a Woodin cardinal in an inner model that overspills $\mathrm{HOD}^{L[x]}$.

Theorem 7.2. *Assume* ZF + $V = L[x] + \Delta^1_2$*-determinacy, for some* $x \in \omega^\omega$*. Then there exists an* $N \subseteq L[x]$ *such that*

$$N \models \mathrm{ZFC} + \omega_1^{L[x]} \text{ is a Woodin cardinal.}$$

Moreover, this result is optimal.

Theorem 7.3. *Assume* ZF + Δ^1_2*-determinacy. Then there is a real* x *such that*

(1) $L[x] \models \Delta^1_2$*-determinacy, and*

(2) *for all* $\alpha < \omega_1^{L[x]}$*,* α *is not a Woodin cardinal in any inner model* N *such that* $\mathrm{On} \subseteq N$.

In Section 7.1 we prove Theorem 7.2. More precisely, we prove the following:

Theorem 7.4. *Assume* ZF + DC + Δ^1_2*-determinacy. Then for a Turing cone of* x,

$$\mathrm{HOD}^{L[x]}_{[x]_T} \models \mathrm{ZFC} + \omega_1^{L[x]} \text{ is a Woodin cardinal.}$$

This involves relativizing the previous construction to the Turing degree of x, replacing the notions that concerned reals (for example, winning strategies) with relativized analogues that concern only those reals in the Turing degree of x.

In Section 7.2 we show that the relativized construction goes through in the setting of second-order arithmetic.

Theorem 7.5. *Assume that* $\mathrm{PA}_2 + \Delta^1_2$*-determinacy is consistent. Then* ZFC + "On *is Woodin" is consistent.*

Here PA_2 is the standard axiomatization of second-order arithmetic (without AC). The statement that On is Woodin is to be understood schematically. Alternatively, one could work with the conservative extension GBC of ZFC and the analogous conservative extension of PA_2. This would enable one to fuse the schema expressing that On is Woodin into a single statement.

7.1. First Localization

To prove Theorem 7.4 we have to prove an analogue of the Generation Theorem where ω_2 is replaced by ω_1. The two main steps are (1) getting a suitable notion of strategic determinacy and (2) getting definable prewellorderings for all ordinals less than ω_1.

For $x \in \omega^\omega$ we "relativize" our previous notions to the Turing degree of x. The *relativized reals* are $R_x = \{y \in \omega^\omega \mid y \leqslant_T x\}$. Fix $A \subseteq R_x$. A *relativized strategy for I* is a function $\sigma : \bigcup_{n<\omega} \omega^{2n} \to \omega$ such that $\sigma \in R_x$. A relativized strategy σ for I is *winning in A* iff for all $y \in R_x$, $\sigma * y \in A$. The corresponding notions for II are defined similarly. A *relativized prestrategy* is a continuous function f such that (the code for) f is in R_x and for all $y \in R_x$, $f(y)$ is a relativized strategy for either I or II. We say that a relativized prestrategy f is *winning in A for I (II) with respect to* $B \subseteq R_x$ if in addition we have that for all $y \in B$, $f(y)$ is a relativized winning strategy for I (II) in A. (In our present setting our basis B will always be $[x]_T$.) We say that a set $A \subseteq R_x$ is *determined in the relativized sense* if either I or II has a relativized winning strategy for A. Let OD-$[x]_T$-determinacy be the statement that for every $\mathrm{OD}_{[x]_T}$ subset of R_x either Player I or Player II has a relativized winning strategy.

The *strategic game relativized to* $[x]_T$ is the game SG-$[x]_T$

$$\begin{array}{lcccccc} \mathrm{I} & A_0 & \cdots & A_n & \cdots & & \\ \mathrm{II} & & f_0 & \cdots & f_n & \cdots \end{array}$$

where we require

(1) $A_0 \in \mathscr{P}(R_x) \cap \mathrm{OD}_{[x]_T}$, $A_{n+1} \in \mathscr{P}(R_x) \cap \mathrm{OD}_{[x]_T, f_0, \ldots, f_n}$ and

(2) f_n is a relativized prestrategy that is winning in A_n with respect to $[x]_T$,

and II wins if and only if II can play all ω rounds. We say that *strategic determinacy relativized to* $[x]_T$ *holds* (ST-$[x]_T$-*determinacy*) if II wins SG-$[x]_T$.

We caution the reader that in the context of the relativized notions we are dealing only with *definable* versions of relativized determinacy such as OD-$[x]_T$-determinacy and SG-$[x]_T$-determinacy. In fact, *full* relativized determinacy can never hold. But as we shall see both OD-$[x]_T$-determinacy and SG-$[x]_T$-determinacy can hold.

Theorem 7.6. *Assume* ZF + DC + Δ^1_2*-determinacy. Let* T *be the theory* ZFC − Replacement + Σ_2-Replacement*. There is a real* x_0 *such that for all reals* x *and for all ordinals* λ *if* $x_0 \in L_\lambda[x]$ *and* $L_\lambda[x] \models \mathrm{T}$*, then* $L_\lambda[x] \models$ OD-$[x]_T$*-determinacy.*

Proof. The proof is similar to that of Theorem 6.6. Assume for contradiction that for every real x_0 there is an ordinal λ and a real x such that $x_0 \in L_\lambda[x]$ and $L_\lambda[x] \models \mathrm{T} + \neg\mathrm{OD}\text{-}[x]_T$-determinacy, where T = ZFC − Replacement + Σ_2-Replacement. As before, by the Löwenheim-Skolem theorem and Σ^1_2-determinacy the ordinal

$$\lambda(x) = \begin{cases} \mu\lambda\,(L_\lambda[x] \models \mathrm{T} + \neg\mathrm{OD}\text{-}[x]_T\text{-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise} \end{cases}$$

is defined for a Turing cone of x. For each x such that $\lambda(x)$ is defined, let $A^x \subseteq R_x$ be the $(\mathrm{OD}_{[x]_T})^{L_{\lambda(x)}[x]}$-least counterexample.

Consider the game

$$\begin{array}{ll} \mathrm{I} & a, b \\ \mathrm{II} & c, d \end{array}$$

where, letting $p = \langle a, b, c, d\rangle$, I wins iff $\lambda(p)$ is defined and $L_{\lambda(p)}[p] \models$ "$a * d \in A^p$", where a and d can be thought of as strategies. This game is Σ^1_2, hence determined.

We arrive at a contradiction by showing that neither player can win.

Case 1: I has a winning strategy σ_0.

Let $x_0 \geqslant_T \sigma_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined. We claim that $L_{\lambda(x_0)}[x_0] \models$ "I has a relativized winning strategy σ in A^{x_0}", which is a contradiction. The relativized strategy σ is derived as follows: Given $d \restriction n \in \omega^n$ have II play $x_0 \restriction n, d \restriction n$ in the main game. Let $a \restriction n, b \restriction n$ be σ_0's response along the way and let $a(n)$ be σ_0's next move. Then set $\sigma(d \restriction n) = a(n)$. (Clearly, σ is continuous, and the real $a = \sigma(d)$ it defines is to be thought of as coding a strategy for Player I.) This strategy σ is clearly recursive in σ_0, hence it is a relativized strategy.

It remains to show that for every $d \in R_{x_0}$, $\sigma * d \in A^{x_0}$. The point is that for $d \in R_{x_0}$, $p \equiv_T x_0$, where $p = \langle a, b, x_0, d\rangle$ is the play obtained by letting $\langle a, b\rangle = (\sigma_0 * \langle x_0, d\rangle)_I$. It follows that $\lambda(p) = \lambda(x_0)$ and hence $L_{\lambda(p)}[p] = L_{\lambda(x_0)}[x_0]$ and $A^p = A^{x_0}$. Thus, $L_{\lambda(x_0)}[x_0] \models$ "$\sigma(d) * d \in A^{x_0}$". So $L_{\lambda(x_0)}[x_0] \models$ "σ is a relativized winning strategy for I in A^{x_0}".

Case 2: II has a winning strategy τ_0.

Let $x_0 \geqslant_T \tau_0$ be such that for all $x \geqslant_T x_0$, $\lambda(x)$ is defined and $\lambda(x) \geqslant \lambda(x_0)$. Given $a \restriction (n+1) \in \omega^{n+1}$ have I play $a \restriction (n+1), x_0 \restriction (n+1)$ in the main game. Let $c \restriction n, d \restriction n$ be τ_0's response along the way. Then set $\tau(a \restriction n) = d(n)$. This strategy is clearly recursive in τ_0, hence it is a relativized strategy, and, as above, $L_{\lambda(x_0)}[x_0] \models$ "τ is a relativized winning strategy for II in A^{x_0}". □

Theorem 7.7. *Assume* $\mathrm{ZF} + \mathrm{DC} + \Delta^1_2$*-determinacy. Then for a Turing cone of* x,

$$L[x] \models \text{ST-}[x]_T\text{-}determinacy.$$

Proof. The proof is a straightforward variant of the proof of Theorem 6.9. In fact it is simpler. We note the major changes.

As before we assume that $V = L[x]$ and show that there is a real z_0 with the feature that if $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$, then $L_\lambda[z] \models \text{ST-}[x]_T$-determinacy.

Assume for contradiction that this fails. For $z \in \omega^\omega$, let

$$\lambda(z) = \begin{cases} \mu\lambda\,(L_\lambda[z] \models \mathrm{T} + \neg\text{ST-}[x]_T\text{-determinacy}) & \text{if such a } \lambda \text{ exists} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The following is immediate.

Claim 1. For a Turing cone of z, $\lambda(z)$ is defined.

For each z in the cone of Claim 1 Player I has a canonical strategy σ^z that depends only on the Turing degree of z, the point being that if $y \equiv_T z$ then $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$.

As before our aim is to obtain a contradiction by defeating σ^z for some z in the Turing cone of Claim 1. We do this by constructing a sequence of games $G_0, G_1, \ldots, G_n, \ldots$ such that I must win via $\sigma_0, \sigma_1, \ldots, \sigma_n, \ldots$ and, for a cone of z, the winning strategies give rise to prestrategies $f^z_0, f^z_1, \ldots, f^z_n, \ldots$ that constitute a non-losing play against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Step 0. Consider (in $L[x]$) the game G_0

$$\begin{array}{llll} \text{I} & \epsilon & a, b \\ \text{II} & & c, d \end{array}$$

where ϵ is either 1 or 2 and, letting $p = \langle a, b, c, d \rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A_0^p$", where $A_0^p = \sigma^p(\varnothing)$.

Claim 2. I has a winning strategy σ_0 in G_0.

Proof. Assume for contradiction that I does not have a winning strategy in G_0. Then, by Σ^1_2-determinacy, II has a winning strategy τ_0 in G_0. Let $z_0 \geqslant_T \tau_0$ be such that for all $z \geqslant_T z_0$,

(1) z satisfies the conditions of Claim 1 and

(2) if λ and z are such that $z_0 \in L_\lambda[z]$ and $L_\lambda[z] \models \mathrm{T}$ then $L_\lambda[z] \models$ OD-$[x]_T$-determinacy (by Theorem 7.6).

Consider $A_0^{z_0} = \sigma^{z_0}(\varnothing)$. Since $L_{\lambda(z_0)}[z_0] \models$ OD-$[x]_T$-determinacy, assume without loss of generality that $L_{\lambda(z_0)}[z_0] \models$ "σ is a relativized winning strategy for I in $A_0^{z_0}$". We use τ_0 to defeat this relativized strategy. Run G_0 according to τ_0, having Player I (falsely) predict that Player I wins the auxiliary game, while steering into $L_{\lambda(z_0)}[z_0]$ by playing $b = z_0$ and using σ to respond to τ_0 on the auxiliary play:

$$\begin{array}{llll} \text{I} & 1 & (\sigma * d)_I, z_0 \\ \text{II} & & c, d \end{array}$$

The point is that $p \equiv_T z_0$ (since $\sigma, \tau_0 \in R_{z_0}$) and so $\lambda(p) = \lambda(z_0)$. Thus the "steering problem" is immediately solved and we have a contradiction as before. □

Since the game is Σ^1_2 for Player I, Player I has a Δ^1_3 strategy σ_0, by Theorem 6.5.

Claim 3. For every real $z \geqslant_T \sigma_0$ there is a prestrategy f_0^z such that f_0^z is recursive in σ_0 as in Claim 1 and f_0^z is a non-losing first move for II against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Proof. Fix $z \geqslant_T \sigma_0$ as in Claim 1 and consider $A_0^z = \sigma^z(\varnothing)$. Let f_0^z be the prestrategy derived from σ_0 as follows: Given $y \restriction n$ and $d \restriction n$ have II play $y \restriction n, d \restriction n$ in G_0. Let $\epsilon, a \restriction n, b \restriction n$ be σ_0's response along the way and let $a(n)$ be σ_0's next move. Then let $f_0^z(y \restriction n) = a(n)$. We have that f_0^z is recursive in $\sigma_0 \leqslant_T z$ and for $y \in [z]_T$, $f_0^z(y) \in R_z$. It remains to see that for $y \in [z]_T$, $f_0^z(y)$ is a relativized winning strategy for I in A_0^z. The point is that since $y \in [z]_T$, $\lambda(y) = \lambda(z)$ and so $L_{\lambda(y)}[y] = L_{\lambda(z)}[z]$ and $A_0^y = A_0^z$. For $d \in R_z$, by definition $f_0^z(y) * d = a * d$ where a is such that $(\sigma_0 * \langle y, d\rangle)_I = \langle \epsilon, a, b\rangle$. So, letting $p = \langle a, b, y, d\rangle$ we have $p \equiv_T y$. Thus, $\epsilon = 1$ iff $L_{\lambda(z)}[z] \models$ "$f_0^z(y) * d \in A_0^z$". □

Step n+1. Assume that we have defined (in $L[x]$) games $G_0, \dots, G_n$ with winning strategies $\sigma_0, \dots, \sigma_n \in \mathrm{HOD}$ such that for all $z \geqslant_T \langle \sigma_0, \dots, \sigma_n\rangle$ as in Claim 1 there are prestrategies $f_0^z, \dots, f_n^z$ such that f_i^z is recursive in $\langle \sigma_0, \dots, \sigma_i\rangle$ (for all $i \leqslant n$) and $f_0^z, \dots f_n^z$ is a non-losing partial play for II in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Consider (in $L[x]$) the game G_{n+1}

$$\begin{array}{lcc} \mathrm{I} & \epsilon & a, b \\ \mathrm{II} & & c, d \end{array}$$

where ϵ is 1 or 2 and, letting $p = \langle a, b, c, d, \sigma_0, \dots, \sigma_n\rangle$, I wins iff

(1) p satisfies the condition on z in Claim 1 (so σ^p makes sense) and

(2) $\epsilon = 1$ iff $L_{\lambda(p)}[p] \models$ "$a * d \in A_{n+1}^p$", where A_{n+1}^p is I's response via σ^p to II's partial play $f_0^p, \dots, f_n^p$

This game is $\Sigma_2^1(\sigma_0, \dots, \sigma_n)$ (for Player I) and hence determined (since $\sigma_0, \dots, \sigma_n \in \mathrm{HOD}$ and we have OD-determinacy).

Claim 4. I has a winning strategy σ_{n+1} in G_{n+1}.

Proof. The proof is as before, only now we use the relativized version of Theorem 7.6 to enforce $\mathrm{OD}_{\sigma_0,\dots,\sigma_n}$-$[x]_T$-determinacy. □

Since the game is $\Sigma_2^1(\sigma_0, \dots, \sigma_n)$ for Player I, Player I has a $\Delta_3^1(\sigma_0, \dots, \sigma_n)$ strategy σ_{n+1}, by the relativized version of Theorem 6.5.

Claim 5. For every real $z \geqslant_T \langle \sigma_0, \dots, \sigma_n\rangle$ there is a prestrategy f_{n+1}^z such that f_{n+1}^z is recursive in $\langle \sigma_0, \dots, \sigma_{n+1}\rangle$ and $f_0^z, \dots, f_{n+1}^z$ is a non-losing first move for II against σ^z in $(SG\text{-}[x]_T)^{L_{\lambda(z)}[z]}$.

Proof. The proof is just like the proof of Claim 3. □

Finally, letting z^∞ as in Claim 1 be such that $z^\infty \geqslant_T z_n$ for all n we have that $f_0^{z^\infty}, \ldots, f_n^{z^\infty}, \ldots$ defeats σ^{z^∞} in $(SG\text{-}[x]_T)^{L_{\lambda(z^\infty)}[z^\infty]}$, which is a contradiction. □

Theorem 7.8. *Assume* ZF + DC. *Then for every* $x \in \omega^\omega$ *and for every* $\alpha < \omega_1^{L[x]}$ *there is an* $\mathrm{OD}_{[x]_T}$ *surjection* $\rho : [x]_T \to \alpha$.

Proof. First we need to review Vopěnka's theorem. Work in $L[x]$ and let $d = [x]_T$. Let

$$\mathbb{B}'_d = \{A \subseteq d \mid A \in \mathrm{OD}_d\},$$

ordered under $\subseteq$. There is an OD_d isomorphism π between $(\mathbb{B}'_d, \subseteq)$ and a partial ordering $(\mathbb{B}_d, \leqslant)$ in HOD_d.

Claim 1. $(\mathbb{B}_d, \leqslant)$ is complete in HOD_d and every real in d is HOD_d-generic for $\mathbb{B}_d$.

Proof. For completeness consider $S \subseteq \mathbb{B}_d$ in HOD_d. We have to show that $\bigvee S$ exists. Let $S' = \pi^{-1}[S]$. Then $\bigvee S' = \bigcup S' \in \mathbb{B}'_d$ as this set is clearly OD_d. So $\bigvee S = \pi(\bigvee S')$.

Now consider $z \in d$. Let $G'_z = \{A \in \mathbb{B}'_d \mid z \in A\}$ and let $G_z = \pi[G'_z]$. We claim that G_z is HOD_d-generic for $\mathbb{B}_d$. Let $S \subseteq \mathbb{B}_d$ be a maximal antichain. So $\bigvee S = 1$. Let $S' = \pi^{-1}[S]$. Note $\bigvee S' = d$. Thus there exists a $b \in S$ such that $z \in \pi(b)$. So G_z is HOD_d-generic for $\mathbb{B}_d$. Now the map $f : \omega \to \mathbb{B}_d$, defined by $f(n) = \pi(\{x \in d \mid n \in x\})$, is in HOD_d. Moreover, $n \in z$ iff $f(n) \in G_z$. Thus $z \in \mathrm{HOD}_d[G_z]$. □

Notice that $\mathrm{HOD}_d[G] = L[x]$ for every $G = G_z$ that is HOD_d-generic (where $z \in d$) since such a generic adds a real in $[x]_T$. Thus, if $\mathrm{HOD}_d \neq L[x]$ then $\mathbb{B}_d$ is non-trivial. This is a key difference between our present setting and that of Vopěnka's—in general our partial order does not have atoms.

If $L[x] = \mathrm{HOD}_d$ then clearly for each $\alpha < \omega_1^{L[x]}$ there exists a surjection $\rho : d \to \alpha$ such that $\rho \in \mathrm{OD}_d$. So we may assume that $L[x] \neq \mathrm{HOD}_d$. Thus, for every $z \in d$

$$\omega_1^{L[x]} = \omega_1^{\mathrm{HOD}_d[G_z]}.$$

Claim 2. Assume ZFC. Suppose λ is an uncountable regular cardinal, $\mathbb{B}$ is a complete Boolean algebra, and $V^{\mathbb{B}} \models \lambda = \omega_1$. Then for every $\alpha < \lambda$ there is an antichain in $\mathbb{B}$ of size $|\alpha|$.

Proof. If λ is a limit cardinal then since $\mathbb{B}$ collapses all uncountable cardinals below λ it cannot be $\bar{\lambda}$-c.c. for any uncountable cardinal $\bar{\lambda} < \lambda$.

Suppose $\lambda = \bar{\lambda}^+$. We need to show that there is an antichain of size $\bar{\lambda}$. If $\bar{\lambda} > \omega$ then this is immediate since $\mathbb{B}$ collapses $\bar{\lambda}$ and so it cannot be $\bar{\lambda}$-c.c. So assume $\bar{\lambda} = \omega$. There must be an antichain of size ω since not every condition in $\mathbb{B}$ is above an atom. □

Letting $\lambda = \omega_1^{L[x]}$, we are in the situation of the claim. So, for every $\alpha < \lambda$ there is an antichain S_α in $\mathbb{B}$ of size $|\alpha|$. Letting $S'_\alpha = \pi^{-1}[S_\alpha]$ we have that S'_α is an OD_d subset of $\mathbb{B}'_d$ consisting of pairwise disjoint OD_d subsets of d. Picking an element from each set we get an OD_d-surjection $\rho : d \to \alpha$. □

Theorem 7.9. *Assume* $\mathrm{ZF}+\mathrm{DC}+\Delta^1_2$*-determinacy. Then for a Turing cone of* x*,*

$$\mathrm{HOD}^{L[x]}_{[x]_T} \models \omega_1^{L[x]} \textit{ is a Woodin cardinal.}$$

Proof. For a Turing cone of x, $L[x] \models \mathrm{OD}_{[x]_T}$-determinacy (by the relativized version of Theorem 6.6) and $L[x] \models \mathrm{ST}\text{-}[x]_T$-determinacy (by Theorem 7.7). Let x be in this cone and work in $L[x]$. Let $d = [x]_T$. Since $L[x] \models \mathrm{OD}_{[x]_T}$-determinacy, $\omega_1^{L[x]}$ is strongly inaccessible in HOD_d. Let $H \subseteq \omega_1^{L[x]}$ code $\mathrm{HOD}_d \cap V_{\omega_1^{L[x]}}$. Fix $T \in \mathscr{P}(\omega_1^{L[x]}) \cap \mathrm{OD}_d$ and let $T_0 \subseteq \omega_1^{L[x]}$ code T and H. Let $A = \langle A_\alpha \mid \alpha < \omega_1^{L[x]} \rangle$ be such that A_α is an OD_d prewellordering of length greater than or equal to α (by Theorem 7.8). Let $B = d$. Consider the structure

$$M = \left(L_{\omega_1^{L[x]}}(\mathbb{R})[T_0, A, B]\right)^{L[x]}.$$

We claim that

$$\mathrm{HOD}^M \models \text{There is a } T_0\text{-strong cardinal},$$

which completes the proof as before. The reason is that we are in the situation of the Generation Theorem, except with $\omega_1^{L[x]}$ replacing $\omega_2^{L[x]}$ and $\mathrm{ST}\text{-}[x]_T$-determinacy replacing ST^B-determinacy. The proof of the Generation Theorem goes through unchanged. One just has to check that all of the operations we performed before (which involved definability in various parameters) are in fact recursive in the relevant parameters. □

7.2. Second Localization

We now wish to show that the above construction goes through when we replace $\mathrm{ZF} + \mathrm{DC}$ with PA_2. Notice that if we had $\underset{\sim}{\Delta}^1_2$-determinacy then this would be routine.

Theorem 7.10. *Assume* $\mathrm{PA}_2 + \underset{\sim}{\Delta}^1_2$*-determinacy. Then for all reals* x*, there is a model* N *such that* $x \in N$ *and*

$$N \models \mathrm{ZFC} + \textit{There is a Woodin cardinal.}$$

Proof. Working in PA_2 if one has $\underset{\sim}{\Delta}^1_2$-determinacy then for every $x \in \omega^\omega$, $x^\#$ exists. It follows that for all $x \in \omega^\omega$, there is an ordinal $\alpha < \omega_1$ such that $L_\alpha[x] \models \mathrm{ZFC}$. Using $\underset{\sim}{\Delta}^1_2$-determinacy one can find a real x_0 enforcing OD-determinacy. Thus we have a model $L_{\alpha_0}[x_0]$ satisfying $\mathrm{ZFC}+V=L[x_0]+$OD-determinacy and this puts us in the situation of Theorem 6.10. □

The situation where one only has Δ^1_2-determinacy is bit more involved.

Theorem 7.11. *Assume that* $\mathrm{PA}_2 + \Delta^1_2$*-determinacy is consistent. Then* ZFC + "On *is Woodin" is consistent.*

Proof Sketch. First we pass to a theory that more closely resembles the theory used to prove Theorem 7.9. In PA_2 one can simulate the construction of $L_{\omega_1}[x]$. Given a model M of PA_2 and a real $x \in M$, there is a definable set of reals A coding the elements of $L_{\omega_1}[x]$. One can then show that the "inner model" $L_{\omega_1}[x]$ satisfies $\mathrm{ZFC}-$Power Set$+V=L[x]$ (using, for example Comprehension to get Replacement). Thus, $\mathrm{ZFC}-$ Power Set $+ V=L[x]$ is a conservative extension of PA_2.

Next we need to arrange a sufficient amount of definable determinacy. The most natural way to secure Δ^1_2-determinacy is to let x encode winning strategies for all Δ^1_2 games. However, this approach is unavailable to us since we have not included AC in PA_2 and, in any case, we wish to work with OD-determinacy (understood schematically). For this we simultaneously run (an elaboration of) the proof of Theorem 6.6 while defining $L_{\omega_1}[x]$. In this way, for any model M of PA_2, there is a real x and an associated definable set of reals A which codes a model $L_{\omega_1}[x]$ satisfying $\mathrm{ZFC} -$ Power Set $+ V=L[x] +$ OD-determinacy.

Working in $\mathrm{ZFC} -$ Power Set $+ V=L[x] +$ OD-determinacy we wish now to show that

$$\mathrm{HOD}_{[x]_T} \models \mathrm{ZFC} + \text{On is Woodin.}$$

So we have to localize the construction of the previous section to the structure $\langle L_{\omega_1}[x], [x]_T\rangle$. The first step is to show that

$$\langle L_{\omega_1}[x], [x]_T\rangle \models \text{ST-}[x]_T\text{-determinacy for } n \text{ moves,}$$

for each n. Here by ST-$[x]_T$-determinacy we mean what we meant in the previous section. However, there is a slight metamathematical issue that arises when we work without Powerset, namely, at each stage of the game the potential moves for Player I are a proper class from the point of view of $\langle L_{\omega_1}[x], [x]_T\rangle$. So in quantifying over these moves we have to use the first-order definition of OD in $\langle L_{\omega_1}[x], [x]_T\rangle$. The winning condition for the n-move version of the game is first-order over $\langle L_{\omega_1}[x], [x]_T\rangle$ but since the complexity of the definition increases as n increases the full game is not first-order over $\langle L_{\omega_1}[x], [x]_T\rangle$. This is why we have had to restrict to the n-move version.

The proof of this version of the theorem is just like that of Theorem 7.7, only now one has to keep track of definability and verify that there is no essential use of Powerset (for example, in the proof of Third Periodicity). The proof of Theorem 7.8 goes through as before. Finally, as in the proof of Theorem 7.9, the proof of the Generation Theorem gives a structure M such that

$$\text{HOD}^M_{[x]_T} \models \text{ZFC} + \text{On is } T\text{-strong,}$$

for an arbitrary $\text{OD}^{\langle L_{\omega_1}[x], [x]_T\rangle}_{[x]_T}$ class T of ordinals, which implies the final result. □

This raises the following question: Are the theories $\text{PA}_2 + \Delta^1_2$-determinacy and ZFC + "On is Woodin" equiconsistent? We turn to this and other more general issues in the next section.

8. Further Results

In this section we place the above results in a broader setting by discussing some results that draw on techniques that are outside the scope of this chapter. The first topic concerns the intimate connection between axioms of definable determinacy and large cardinal axioms (as mediated through inner models). The second topic concerns the surprising convergence between two very different approaches to inner model theory—the approach based on generalizations of L and the approach based on HOD. In both cases the relevant material on inner model theory can be found in Steel's chapter in this Handbook.

8.1. Large Cardinals and Determinacy

The connection between axioms of definable determinacy and inner models of large cardinals is even more intimate than indicated by the above results. We have seen that certain axioms of definable determinacy imply the existence of inner models of large cardinal axioms. For example, assuming $\mathrm{ZFC} + \underset{\sim}{\Delta}^1_2$-determinacy, for each $x \in \omega^\omega$, there is an inner model M such that $x \in M$ and

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

And, assuming $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$, in $L(\mathbb{R})$ there is an inner model M such that

$$M \models \mathrm{ZFC} + \text{There is a Woodin cardinal.}$$

In many cases these implications can be reversed—axioms of definable determinacy are actually *equivalent* to axioms asserting the existence of inner models of large cardinals. We discuss what is known about this connection, starting with a low level of boldface definable determinacy and proceeding upward. We then turn to lightface determinacy, where the situation is more subtle. It should be emphasized that our concern here is not merely with consistency strength but rather with outright equivalence (over ZFC).

Theorem 8.1. *The following are equivalent*:

(1) $\underset{\sim}{\Delta}^1_2$*-determinacy.*

(2) *For all $x \in \omega^\omega$, there is an inner model M such that $x \in M$ and $M \models$ There is a Woodin cardinal.*

Theorem 8.2. *The following are equivalent*:

(1) PD (*Schematic*).

(2) *For every $n < \omega$, there is a fine-structural, countably iterable inner model M such that $M \models$ There are n Woodin cardinals.*

Theorem 8.3. *The following are equivalent*:

(1) $\mathrm{AD}^{L(\mathbb{R})}$.

(2) *In $L(\mathbb{R})$, for every set S of ordinals, there is an inner model M and an $\alpha < \omega_1^{L(\mathbb{R})}$ such that $S \in M$ and $M \models \alpha$ is a Woodin cardinal.*

Theorem 8.4. *The following are equivalent*:

(1) $\mathrm{AD}^{L(\mathbb{R})}$ *and* $\mathbb{R}^{\#}$ *exists.*

(2) $M_\omega^{\#}$ *exists and is countably iterable.*

Theorem 8.5. *The following are equivalent*:

(1) *For all* $\mathbb{B}$, $V^{\mathbb{B}} \models \mathrm{AD}^{L(\mathbb{R})}$.

(2) $M_\omega^{\#}$ *exists and is fully iterable.*

The above examples concern *boldface* definable determinacy. The situation with *lightface* definable determinacy is more subtle. For example, assuming ZFC + Δ^1_2-determinacy, must there exist an $\alpha < \omega_1$ and an inner model M such that α is a Woodin cardinal in M? In light of Theorem 8.1 one would expect that this is indeed the case. However, since Theorem 8.1 also holds in the context of PA_2 one would then expect that the theories $\mathrm{PA}_2+\Delta^1_2$-determinacy and PA_2 + "There is an $\alpha < \omega_1$ and an inner model M such that $M \models \alpha$ is a Woodin cardinal" are *equivalent*, and yet this expectation is in conflict with the expectation that the theories $\mathrm{PA}_2 + \Delta^1_2$-determinacy and ZFC + "On is Woodin" are *equiconsistent*. In fact, this seems likely, but the details have not been fully checked. We state a version for third-order Peano arithmetic, PA_3, and second-order ZFC. But first we need a definition and some preliminary results.

8.6 Definition. A partial order $\mathbb{P}$ is *δ-productive* if for all δ-c.c. partial orders $\mathbb{Q}$, the product $\mathbb{P} \times \mathbb{Q}$ is δ-c.c.

Theorem 8.7. *In the fully iterable, 1-small, 1-Woodin Mitchell-Steel model the extender algebra built using all extenders on the sequence which are strong to their length is δ-productive.*

This is a warm-up since in the case of interest we do not have iterability. It is unknown if iterability is necessary.

Theorem 8.8. *Suppose δ is a Woodin cardinal. Then there is a proper class inner model $N \subseteq V$ such that*

(1) $N \models \delta$ *is a Woodin cardinal and*

(2) $N \models$ *There is a complete* δ*-c.c. Boolean algebra* $\mathbb{B}$ *such that*

$$N^{\mathbb{B}} \models \Delta^1_2\text{-}determinacy.$$

Let ZFC_2 be second-order ZFC.

Theorem 8.9. *The following are equiconsistent*:

(1) $\mathrm{PA}_3 + \Delta^1_2$*-determinacy.*

(2) $\mathrm{ZFC}_2 + \mathrm{On}$ *is Woodin.*

We now turn from theories to models and discuss the manner in which one can pass back and forth between models of infinitely many Woodin cardinals and models of definable determinacy at the level of $\mathrm{AD}^{L(\mathbb{R})}$ and beyond. We have already dealt in detail with one direction of this—the transfer from models of determinacy to models with Woodin cardinals—and the other direction—the transfer from models with Woodin cardinals to models of determinacy—was briefly discussed in the introduction, but the situation is much more general. To proceed at the appropriate level of generality we need to introduce a potential strengthening of AD.

A set $A \subseteq \omega^\omega$ is $^\infty$*-borel* if there is a set $S \subseteq \mathrm{On}$, an ordinal α, and a formula φ such that

$$A = \{y \in \omega^\omega \mid L_\alpha[S, y] \models \varphi[S, y]\}.$$

It is fairly straightforward to show that to say that A is $^\infty$-borel is equivalent to saying that it has a "transfinite borel code". Notice that under AC every set of reals is $^\infty$-borel.

8.10 Definition. Assume $\mathrm{ZF}+\mathrm{DC}_{\mathbb{R}}$. The theory AD^+ consists of the axioms:

(1) Every set $A \subseteq \omega^\omega$ is $^\infty$-borel.

(2) Suppose $\lambda < \Theta$ and $\pi : \lambda^\omega \to \omega^\omega$ is a continuous surjection. Then for each $A \subseteq \omega^\omega$ the set $\pi^{-1}[A]$ is determined.

8.11 Conjecture. AD implies AD^+.

It is known that the failure of this implication has strong consistency strength. For example, $\mathrm{AD} + \neg\mathrm{AD}^+$ proves $\mathrm{Con}(\mathrm{AD}_{\mathbb{R}})$.

The following theorem—the *Derived Model Theorem*—is a generalization of Theorem 1.5, mentioned in the introduction.

Theorem 8.12. *Suppose that δ is a limit of Woodin cardinals. Suppose that $G \subseteq \mathrm{Col}(\omega, {<}\delta)$ is V-generic and let $\mathbb{R}_G = \cup\{\mathbb{R}^{V[G\restriction\alpha]} \mid \alpha < \delta\}$. Let Γ_G be the set of $A \subseteq \mathbb{R}_G$ such that*

(1) $A \in V(\mathbb{R}_G)$,

(2) $L(A, \mathbb{R}_G) \models \mathrm{AD}^+$.

Then $L(\Gamma_G, \mathbb{R}_G) \models \mathrm{AD}^+$.

There is a "converse" to the Derived Model Theorem, the proof of which is a generalization of the proof of Theorem 6.20.

Theorem 8.13. *Assume AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$. There is a partial order $\mathbb{P}$ such that if H is $\mathbb{P}$-generic over V then there is an inner model $N \subseteq V[H]$ such that*

(1) $N \models \mathrm{ZFC}$,

(2) ω_1^V *is a limit of Woodin cardinals in N,*

(3) *there is a g which is $\mathrm{Col}(\omega, {<}\omega_1^V)$-generic over N and such that*

 (a) $\mathbb{R}^V = \mathbb{R}_g$,

 (b) $\Gamma_g = \mathscr{P}(\mathbb{R})^V$,

 where $\mathbb{R}_g$ and Γ_g are as in the previous theorem with N in the role of V.

Thus, there is an intimate connection between models with infinitely many Woodin cardinals and models of definable determinacy at the level of $\mathrm{AD}^{L(\mathbb{R})}$ and beyond. Moreover, the link is even tighter in the case of fine-structural inner models with Woodin cardinals. For example, if one first applies the Derived Model Theorem to M_ω (the Mitchell-Steel model for ω-many Woodin cardinals) and then applies the "converse" theorem to the resulting derived model $L(\mathbb{R}^*)$ then one recovers the original model M_ω.

8.2. HOD-Analysis

There is also an intimate connection between the two approaches to inner model theory mentioned in the introduction—the approach based on generalizations of L and the approach based on HOD.

As mentioned in the introduction, the two approaches have opposing advantages and disadvantages. The disadvantage of the first approach is that the problem of actually defining the models that can accommodate large cardinals—*the inner model problem*—is quite a difficult problem. However, the advantage is that once the inner model problem is solved at a given level of large cardinals the inner structure of the models is quite transparent and so these models are suitable for extracting the large cardinal content inherent in a given statement. The advantage of the approach based on HOD is that this model is trivial to define and it can accommodate virtually every large cardinal. The disadvantage—*the tractability problem*—is that in general the inner structure of HOD is about as tractable as that of V and so it is not generally suitable for extracting the large cardinal content from a given statement.

Nevertheless, we have taken the approach based on HOD and we have found that $\mathrm{AD}^{L(\mathbb{R})}$ and Δ^1_2-determinacy are able to overcome (to some extent) the tractability problem for their natural models, $L(\mathbb{R})$ and $L[x]$ for a Turing (or constructibility) cone of x. For example, we have seen that under $\mathrm{AD}^{L(\mathbb{R})}$,

$$\mathrm{HOD}^{L(\mathbb{R})} \models \Theta^{L(\mathbb{R})} \text{ is a Woodin cardinal,}$$

and that under Δ^1_2-determinacy, for a Turing cone of reals x,

$$\mathrm{HOD}^{L[x]} \models \omega_2^{L[x]} \text{ is a Woodin cardinal.}$$

Despite this progress, much of the structure of HOD in these contexts is far from clear. For example, it is unclear whether under Δ^1_2-determinacy, for a Turing cone of reals x, $\mathrm{HOD}^{L[x]}$ satisfies GCH, something that would be immediate in the case of "L-like" inner models.

Since the above results were first proved, Mitchell and Steel developed the fine-structural version of the "L-like" inner models at the level of Woodin cardinals. These models have the form $L[\vec{E}]$ where $\vec{E}$ is a sequence of (partial) extenders and (as noted above) their inner structure is very well understood—for example, they satisfy GCH and many of the other combinatorial properties that hold in L. A natural question, then, is whether there is any connection between these radically different approaches, that is, whether HOD as computed in $L(\mathbb{R})$ under $\mathrm{AD}^{L(\mathbb{R})}$ or in $L[x]$, for a Turing cone of x, under Δ^1_2-determinacy, bears any resemblance to the $L[\vec{E}]$ models. The remainder of this section is devoted to this question. We begin with $\mathrm{HOD}^{L(\mathbb{R})}$ and its generalizations (where a good deal is known) and then turn to $\mathrm{HOD}^{L[x]}$ (where the central question is open). Again, the situation with lightface determinacy is more subtle.

The theorems concerning $\mathrm{HOD}^{L(\mathbb{R})}$ only require $\mathrm{AD}^{L(\mathbb{R})}$ but they are simpler to state under the stronger assumption that $\mathrm{AD}^{L(\mathbb{R})}$ holds in all generic extensions of V. By Theorem 8.5, this assumption is equivalent to the statement that $M_\omega^\#$ exists and is fully iterable.

The first hint that $\mathrm{HOD}^{L(\mathbb{R})}$ is a fine-structural model is the remarkable fact that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathbb{R} = M_\omega \cap \mathbb{R}.$$

The agreement between $\mathrm{HOD}^{L(\mathbb{R})}$ and M_ω fails higher up but $\mathrm{HOD}^{L(\mathbb{R})}$ agrees with an iterate of M_ω at slightly higher levels. More precisely, letting N be the result of iterating M_ω by taking the ultrapower ω_1^V-many times using the (unique) normal ultrafilter on the least measurable cardinal, we have that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap \mathscr{P}(\omega_1^V) = N \cap \mathscr{P}(\omega_1^V).$$

Steel improved this dramatically by showing that

$$\mathrm{HOD}^{L(\mathbb{R})} \cap V_{(\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}}$$

is the direct limit of a directed system of iterable fine-structural inner models.

Theorem 8.14 (Steel). $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ *is a Mitchell-Steel model, where* $\delta = (\underset{\sim}{\delta}^2_1)^{L(\mathbb{R})}$.

For a proof of this result see Steel's chapter in this Handbook. As a corollary one has that $\mathrm{HOD}^{L(\mathbb{R})}$ satisfies GCH along with the combinatorial principles (such as $\diamondsuit$ and $\Box$) that are characteristic of fine-structural models.

The above results suggest that all of $\mathrm{HOD}^{L(\mathbb{R})}$ might be a Mitchell-Steel inner model of the form $L[\vec{E}]$. This is not the case.

Theorem 8.15. $\mathrm{HOD}^{L(\mathbb{R})}$ *is not a Mitchell-Steel inner model.*

Nevertheless, $\mathrm{HOD}^{L(\mathbb{R})}$ is a fine-structural inner model, one that belongs to a new, quite different, hierarchy of models. Let

$$D = \big\{ L[\vec{E}] \bigm| L[\vec{E}] \text{ is an iterate of } M_\omega \text{ by a countable tree which is based on the first Woodin cardinal and has a non-dropping cofinal branch} \big\}.$$

Any two structures in D can be compared and the iteration halts in countably many steps (since we have full iterability) with iterates lying in D. So D is a directed system under the elementary embeddings given by iteration maps. By the Dodd-Jensen lemma the embeddings commute and hence there is a direct limit. Let $L[\vec{E}^\infty]$ be the direct limit of D. Let $\langle \delta_i^\infty \mid i < \omega \rangle$ be the Woodin cardinals of $L[\vec{E}^\infty]$.

Theorem 8.16. *Let $L[\vec{E}^\infty]$ be as above. Then*

(1) $L[\vec{E}^\infty] \subseteq \mathrm{HOD}^{L(\mathbb{R})}$,

(2) $L[\vec{E}^\infty] \cap V_\delta = \mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$, *where* $\delta = \delta_0^\infty$,

(3) $\Theta^{L(\mathbb{R})} = \delta_0^\infty$, *and*

(4) $(\underset{\sim}{\delta}_1^2)^{L(\mathbb{R})}$ *is the least cardinal in* $L[\vec{E}^\infty]$ *which is* λ*-strong for all* $\lambda < \delta_0^\infty$.

To reach $\mathrm{HOD}^{L(\mathbb{R})}$ we need to supplement $L[\vec{E}^\infty]$ with additional inner-model-theoretic information. A natural candidate is the iteration strategy. It turns out that by folding in the right fragment of the iteration strategy one can capture $\mathrm{HOD}^{L(\mathbb{R})}$. Let

$$T^\infty = \big\{ T \bigm| T \text{ is a maximal iteration tree on } L[\vec{E}^\infty] \text{ based on } \delta_0^\infty, \; T \in L[\vec{E}^\infty], \text{ and length}(T) < \sup\{\delta_n^\infty \mid n < \omega\} \big\}$$

and

$$P = \big\{ \langle b, T \rangle \bigm| T \in T^\infty \text{ and } b \text{ is the true branch through } T \big\}.$$

Theorem 8.17. *Let $L[\vec{E}^\infty]$ and P be as above. Then*

$$\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{E}^\infty, P].$$

In fact, there is a single iteration tree $T \in T^\infty$ such that if b is the branch through T chosen by P then

$$\mathrm{HOD}^{L(\mathbb{R})} = L[\vec{E}^\infty, b].$$

This analysis has an interesting consequence. Notice that the model $L[\vec{E}^\infty]$ is of the form $L[A]$ for $A \subseteq \delta_0^\infty$. Thus, although the addition of P does not add any new bounded subsets of $\Theta^{L(\mathbb{R})}$ it does a lot of damage to the model above $\Theta^{L(\mathbb{R})}$, for example, it collapses ω-many Woodin cardinals. One might think that this is an artifact of $L[\vec{E}^\infty]$ but in fact the situation is much more general: Suppose $L[\vec{E}]$ is ω-small, fully iterable, and has ω-many Woodin cardinals. Let P be defined as above except using the Woodin cardinals of $L[\vec{E}]$. Then $L[\vec{E}, P] \cap V_\delta = L[\vec{E}] \cap V_\delta$, where δ is the first Woodin cardinal of $L[\vec{E}]$, and $L[\vec{E}] \subsetneq L[\vec{E}, P] \subsetneq L[\vec{E}^\#]$. For example, applying this result to $L[\vec{E}] = M_\omega$, one obtains a canonical inner-model-theoretic object between M_ω and $M_\omega^\#$. In this way, what appeared to be a coarse approach to inner model theory has actually resulted in a hierarchy that supplements and refines the standard fine-structural hierarchy.

The above results generalize. We need a definition.

8.18 Definition (Mouse Capturing). MC is the statement: For all $x, y \in \omega^\omega$, $x \in \mathrm{OD}_y$ iff there is an iterable Mitchell-Steel model M of the form $L[\vec{E}, y]$ such that $x \in M$.

The *Mouse Set Conjecture*, MSC, is the conjecture that it is a theorem of AD^+ that MC holds if there is no iterable model with a superstrong cardinal. There should be a more general version of MC, one that holds for extensions of the Mitchell-Steel models that can accommodate long extenders. And this version of MC should follow from AD^+. However, the details are still being worked out. See [13].

Theorem 8.19. *Assume* $\mathrm{AD}^+ + V = L(\mathscr{P}(\mathbb{R})) + \Theta_0 = \Theta + \mathrm{MSC}$. *Then the inner model* $\mathrm{HOD}^{L(\mathscr{P}(\mathbb{R}))}$ *is of the form* $L[\vec{E}^\infty, P]$, *with the key difference being that* $L[\vec{E}^\infty]$ *need not be* ω*-small.*

Theorem 8.20. *Assume* $\mathrm{AD}^+ + V = L(\mathscr{P}(\mathbb{R})) + \Theta_0 < \Theta + \mathrm{MSC}$. *Then*

(1) Θ_0 *is the least Woodin cardinal in* HOD,

(2) $\mathrm{HOD} \cap V_{\Theta_0}$ *is a Mitchell-Steel model,*

(3) $\mathrm{HOD} \cap V_{\Theta_0+1}$ *is not a Mitchell-Steel model, and*

(4) $\mathrm{HOD} \cap V_{\Theta_1}$ *is a model of the form* $L[\vec{E}^\infty, P]$ (*assuming the appropriate form of the Mouse Set Theorem*).

One can move on to stronger hypotheses. For example, assuming AD^+ and $V = L(\mathscr{P}(\mathbb{R}))$, $\mathrm{AD}_{\mathbb{R}}$ is equivalent to the statement that Ω (defined at the beginning of Section 5) is a non-zero limit ordinal. There is a minimal inner model N of $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$ that contains all of the reals. The model HOD^N has ω-many Woodin cardinals and these are exactly the members of the Θ-sequence. This model belongs to the above hierarchy and has been used to calibrate the consistency strength of $\mathrm{AD}_{\mathbb{R}}$ in terms of the large cardinal hierarchy. This hierarchy extends and a good deal is known about it.

We now turn to the case of lightface determinacy and the setting $L[x]$ for a Turing cone of x. Here the situation is less clear. In fact, the basic question is open.

5 Open Question. Assume Δ^1_2-determinacy. For a Turing cone of x, what is $\mathrm{HOD}^{L[x]}$ from a fine-structural point of view?

We close with partial results in this direction and with a conjecture. To simplify the discussion we state these results under a stronger assumption than is necessary: Assume Δ^1_2-determinacy and that for all $x \in \omega^\omega$, $x^\#$ exists.

It follows that M_1 and $M_1^\#$ exist. Let $x_0 \in \omega^\omega$ be such that $M_1^\# \in L[x_0]$. Let κ_{x_0} be the least inaccessible of $L[x_0]$ and let $G \subseteq \mathrm{Col}(\omega, < \kappa_{x_0})$ be $L[x_0]$-generic. The Kechris-Solovay result carries over to show that

$$L[x_0][G] \models \text{OD-determinacy}.$$

Furthermore,

$$\mathrm{HOD}^{L[x_0][G]} = \mathrm{HOD}^{L(\mathbb{R})^{L[x_0][G]}} \quad \text{and} \quad \omega_2^{L[x_0][G]} = \Theta^{L(\mathbb{R})^{L[x_0][G]}}.$$

Thus, the model $L(\mathbb{R})^{L[x_0][G]}$ is a "lightface" analogue of $L(\mathbb{R})$. In fact the conditions of the Generation Theorem hold in $L[x_0][G]$ and as a consequence one has that

$$\mathrm{HOD}^{L[x_0][G]} \models \omega_2^{L[x_0][G]} \text{ is a Woodin cardinal.}$$

For a model $L[\vec{E}]$ containing at least one Woodin cardinal let $\delta_0^{\vec{E}}$ be the least Woodin cardinal. Let

$$D = \left\{ L[\vec{E}] \subseteq L[x_0][G] \;\middle|\; L[\vec{E}] \text{ is an iterate of } M_1 \text{ and } \delta_0^{\vec{E}} < \omega_1^{L[x_0][G]} \right\}.$$

Let $L[\vec{E}^\infty]$ be the direct limit of D. Let δ^∞ be the least Woodin of $L[\vec{E}^\infty]$ and let κ^∞ be the least inaccessible above δ^∞. Let

$$T^\infty = \big\{T \bigm| T \text{ is a maximal iteration tree on } L[\vec{E}^\infty], \\ T \in L[\vec{E}^\infty], \text{ and length}(T) < \kappa^\infty\big\}$$

and

$$P = \big\{\langle b, T\rangle \bigm| T \in T^\infty \text{ and } b \text{ is the true branch through } T\big\}.$$

Theorem 8.21. *Let $L[\vec{E}^\infty, P]$ be as above. Then*

(1) $\mathrm{HOD}^{L[x_0][G]} \cap V_{\delta^\infty} = L[\vec{E}^\infty] \cap V_{\delta^\infty}$,

(2) $\mathrm{HOD}^{L[x_0][G]} = L[\vec{E}^\infty, P]$, *and*

(3) $\omega_2^{L[x_0][G]} = \delta^\infty$.

A similar analysis can be carried out for other hypotheses that place one in an "$L(\mathbb{R})$-like" setting. For example, suppose again that x_0 is such that $M_1^\# \in L[x_0]$. One can "generically force" MA as follows: In $L[x_0]$ let $\mathbb{P}$ be the partial order where the conditions $\langle \mathbb{B}_\alpha \mid \alpha < \gamma\rangle$ are such that (i) for each $\alpha < \gamma$, $\mathbb{B}_\alpha$ is c.c.c., (ii) $|\mathbb{B}_\alpha| = \omega_1$, (iii) if $\alpha \leqslant \beta < \gamma$ then $\mathbb{B}_\alpha$ is a complete subalgebra of $\mathbb{B}_\beta$, and (iv) $\gamma < \omega_2$, and the ordering is by extension. The forcing is $<\omega_2$-closed. Let $G \subseteq \mathbb{P}$ be $L[x_0]$-generic and let $\mathbb{B}_G$ be the union of the algebras $\mathbb{B}_\alpha$ appearing in the conditions in G. It follows that $\mathbb{B}_G$ is c.c.c in $L[x_0][G]$. Now, letting $H \subseteq \mathbb{B}_G$ be $L[x_0][G]$-generic, we have that $L[x_0][G][H]$ satisfies MA. The result is that

$$\mathrm{HOD}^{L[x_0][G][H]} = L[\vec{E}^\infty, P]$$

for the appropriate $\vec{E}^\infty$ and P. However, in this context

$$\mathrm{HOD}^{L[x_0][G][H]} \models \omega_3^{L[x_0][G][H]} \text{ is a Woodin cardinal.}$$

In the case of $L(\mathbb{R})$ the non-fine-structural analysis showed that $(\underset{\sim}{\delta}{}_1^2)^{L(\mathbb{R})}$ is λ-strong in $\mathrm{HOD}^{L(\mathbb{R})}$ for all $\lambda < \Theta^{L(\mathbb{R})}$ and the HOD-analysis showed that in fact $(\underset{\sim}{\delta}{}_1^2)^{L(\mathbb{R})}$ is the least ordinal with this feature. In the case of $L[x_0][G]$ the non-fine-structural analysis shows that some ordinal δ is λ-strong in $\mathrm{HOD}^{L[x_0][G]} = \mathrm{HOD}^{L(\mathbb{R})^{L[x_0][G]}}$ for all $\lambda < \omega_2^{L[x_0][G]} = \Theta^{L(\mathbb{R})^{L[x_0][G]}}$. Numerology would suggest that δ is $\underset{\sim}{\delta}^1_2$ as computed in $L[x_0][G]$. It turns out this analogy fails: the least cardinal δ that is λ-strong in $\mathrm{HOD}^{L[x_0][G]}$ for all $\lambda < \omega_2^{L[x_0][G]}$ is in fact strictly less δ^1_2 as computed in $L[x_0][G]$.

But there is another analogy that does hold. First we need some definitions. A set $A \subseteq \omega^\omega$ is *γ-Suslin* if there is an ordinal γ and a tree T on $\omega \times \gamma$ such that $A = p[T] = \{x \in \omega^\omega \mid \exists y \in \gamma^\omega\, \forall n\, (x{\restriction}n, y{\restriction}n) \in T\}$. A cardinal κ is a *Suslin cardinal* if there exists a set $A \subseteq \omega^\omega$ such that A is κ-Suslin but not γ-Suslin for any $\gamma < \kappa$. A set $A \subseteq \omega^\omega$ is *effectively γ-Suslin* if there is an ordinal γ and an OD tree $T \subseteq \omega \times \gamma$ such that $A = p[T]$. A cardinal κ is an *effective Suslin cardinal* if there exists a set $A \subseteq \omega^\omega$ such that A is effectively κ-Suslin but not effectively γ-Suslin for any $\gamma < \kappa$.

In $L(\mathbb{R})$, $\underset{\sim}{\delta}^2_1$ is the largest Suslin cardinal. Since $L[x_0][G]$ is a lightface analogue of $L(\mathbb{R})$ one might expect that in $L[x_0][G]$, δ^1_2 is the largest *effective* Suslin cardinal in $L[x_0][G]$. This is indeed the case.

There is one more advance on the HOD-analysis for $L[x]$ that is worth mentioning.

Theorem 8.22. *Assume $\underset{\sim}{\Delta}^1_2$-determinacy. For a Turing cone of x there is a predicate A such that*

(1) $\mathrm{HOD}_A^{L[x]}$ *has the form $L[\vec{E}, P]$ where P is a fragment of the iteration strategy,*

(2) $\mathrm{HOD}_A^{L[x]} \models \omega_2^{L[x]}$ *is a Woodin cardinal,*

(3) $\mathrm{HOD}_A^{L[x]}$ *is of the form $L[\vec{E}]$ below $\omega_2^{L[x]}$,*

(4) $L[x] \models \mathrm{ST}^A$*-determinacy, and*

(5) $\mathrm{HOD}^{L[x]} \cap V_\delta = \mathrm{HOD}_A^{L[x]} \cap V_\delta$ *where δ is the least cardinal of* $\mathrm{HOD}_A^{L[x]}$ *that is λ-strong for all $\lambda < \omega_2^{L[x]}$.*

Moreover, there exists a definable collection of such A and the collection has size $\omega_1^{L[x]}$.

This provides some evidence that $\mathrm{HOD}^{L[x]}$ is of the form $L[\vec{E}]$ below $\omega_2^{L[x]}$ and that $\mathrm{HOD}^{L[x]}$ is not equal to a model of the form $L[\vec{E}]$.

8.23 Conjecture. $\mathrm{HOD}^{L[x]}$ is of the form $L[\vec{E}, P]$ where P selects branches through all trees in $L[\vec{E}]$ based on the Woodin cardinal and with length less than the successor of the Woodin cardinal.

Bibliography

[1] Thomas J. Jech. *Set Theory, Third Millenium Edition, Revised and Expanded*. Springer-Verlag, Berlin, 2003.

[2] Akihiro Kanamori. *The Higher Infinite: Large Cardinals in Set Theory from their Beginnings*. Springer Monographs in Mathematics. Springer, Berlin, second edition, 2003.

[3] Alexander S. Kechris. The axiom of determinacy implies dependent choices in $L(\mathbb{R})$. *The Journal of Symbolic Logic*, 49(1):161–173, 1984.

[4] Alexander S. Kechris, Donald A. Martin, and Robert M. Solovay. Introduction to Q-theory. In *Cabal Seminar 79–81*, volume 1019 of *Lecture Notes in Mathematics*, pages 199–282. Springer, 1983.

[5] Alexander S. Kechris, Yiannis N. Moschovakis, W. Hugh Woodin, and E. M. Kleinberg. The axiom of determinacy, partition properties, and non-singular measures. In *Cabal Seminar 77–79*, volume 839 of *Lecture Notes in Mathematics*, pages 75–99. Springer, 1981.

[6] Alexander S. Kechris and Robert M. Solovay. On the relative strength of determinacy hypotheses. *Transactions of the American Mathematical Society*, 290(1):179–211, 1985.

[7] Alexander S. Kechris and W. Hugh Woodin. Equivalence of partition properties and determinacy. *Proceedings of the National Academy of Science U.S.A*, 80:1783–1786, 1983.

[8] Peter Koellner. On the question of absolute undecidability. *Philosophia Mathematica*, 14(2):153–188, 2006. Revised and reprinted in *Kurt Gödel: Essays for his Centennial*, edited by Solomon Feferman, Charles Parsons, and Stephen G. Simpson. Lecture Notes in Logic, 33. Association of Symbolic Logic, 2009.

[9] Paul B. Larson. *The Stationary Tower: Notes on a Course by W. Hugh Woodin*, volume 32 of *University Lecture Series*. American Mathematical Society, 2004.

[10] Yiannis N. Moschovakis. *Descriptive Set Theory*. North-Holland, Amsterdam, 1980.

[11] Itay Neeman. *The Determinacy of Long Games*, volume 7 of *de Gruyter Series in Logic and its Applications*. de Gruyter, Berlin, 2004.

[12] John Steel. Mathematics needs new axioms. *The Bulletin of Symbolic Logic*, 6(4):422–433, December 2000.

[13] W. Hugh Woodin. Suitable extender sequences.

[14] W. Hugh Woodin. The continuum hypothesis. In Rene Cori, Alexander Razborov, Stevo Todorcevic, and Carol Wood, editors, *Logic Colloquium 2000*, volume 19 of *Lecture Notes in Logic*, pages 143–197. Association of Symbolic Logic, 2005.

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