Definability in the Computably Enumerable Sets

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- Gödel [1931] defined his diagonal Klasse K.
  - This is the Π^0_1 complement of our familiar Σ^0_1 class K, which first defined by Kleene in 1936.
  - Kleene showed K is c.e. and noncomputable.
- Kleene-Post [1954] defined the jump: $A' = K^A$

  A' is c.e. in and above A.

All sets and degrees are c.e.

Definition

$L_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}'\}$.

$H_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}''\}$.

Definition

$L_n = \{\mathbf{d} \mid \mathbf{d}^{(\mathbf{n})} = \mathbf{0}^{(\mathbf{n})}\}$

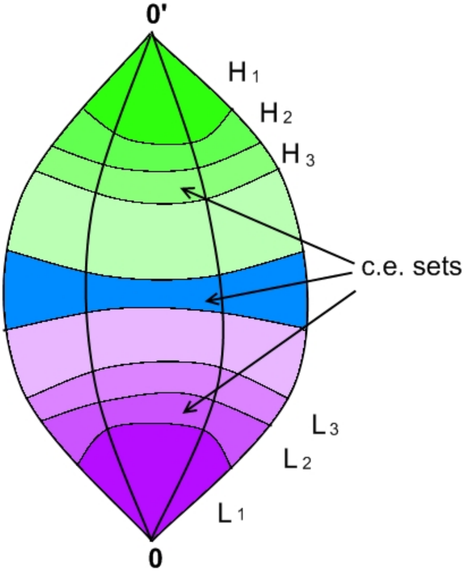
$H_n = \{\mathbf{d} \mid \mathbf{d}^{(\mathbf{n})} = \mathbf{0}^{(\mathbf{n+1})}\}$

- Sacks proved the Jump Inversion Theorem, which led to the following corollary:

Corollary

$\mathbf{0} = L_0 \subsetneq L_1 \subsetneq L_2 \subsetneq L_3 \subsetneq \ldots$, *and*

$\mathbf{0}' = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 \ldots$


0'
H 1
H 2
H 3
c.e. sets
L 3
L 2
L 1
0

- Myhill [1956] studied the lattice $\mathcal{E}$ of c.e. sets under inclusion.
    - $\mathcal{E} = \{\{W_e\}_{e\in\omega}, \cup, \cap, \omega, \emptyset\}$
    - We define $\mathcal{E}^* = \mathcal{E}/F$, where $F = \{W_e \mid W_e \text{ finite}\}$.

Definition

A set A is *maximal* if A^* is a coatom of $\mathcal{E}^*$, i.e. if for all e,

$$A \subset W_e \quad \Longrightarrow \quad W_e =^* A \;\text{ or }\; W_e =^* \omega.$$

- Myhill asked if there exists a maximal set.
- Friedberg [1958]: There exists a maximal set; $\mathcal{E}^*$ is not dense.
- Sacks [1964]: There is an incomplete maximal set.
- Yates [1965]: There is a complete maximal set.

Theorem (Martin, 1966)

H_1 = *the degrees of maximal sets.*

- Definability has played a major role in the field, for the structures $\mathcal{D}$, $\mathcal{R}$, and $\mathcal{E}$.
- Shore and Slaman [1999] wrote: "The overarching goal of these [many years of] investigations has been the definition of the (Turing) jump operator".

Definition

We say a class of degrees $\mathcal{C}$ is *definable* if $\mathcal{C} = \{\deg(W) \mid W \in \mathcal{S}\}$ where $\mathcal{S}$ is a class of sets definable in $\mathcal{E}$.

Question

Which jump classes of degrees are definable in $\mathcal{E}$*?*

Definition

A set A is *atomless* if it is not contained in any maximal set.

- Lachlan [1968]: The atomless sets are contained in the class $\overline{L_2}$.
- Shoenfield [1976]: Every degree in $\overline{L_2}$ contains an atomless set.
- Thus, $\overline{L_2} = \{deg(A) \mid A \text{ atomless}\}$.

Which Jump Classes are Definable?

Red = Definable   Blue = Not definable

- $L_0 = \{\mathbf{0}\}$: Definable by $\{\deg(\emptyset)\}$.
- $\overline{L_0} = \{\mathbf{d} \mid \mathbf{d} > \mathbf{0}\}$: Definable by $\{\deg(W) \mid \overline{W} \notin \mathcal{E}\}$.
- $H_0 = \{\mathbf{0}'\}$: Definable because the creative sets are definable [Harrington, 1986].
- $H_1 = \{\mathbf{d} \mid \mathbf{d}' = \mathbf{0}''\}$: Definable by $\{\deg(W) \mid W \text{ maximal}\}$ by Martin.
- $\overline{L_2} = \{\mathbf{d} \mid \mathbf{d}'' > \mathbf{0}''\}$: Definable by $\{\deg(W) \mid W \text{ atomless}\}$ by Lachlan and Shoenfield.

Definition

A class of sets $\mathcal{S} \subseteq \mathcal{E}$ is *invariant* if it is closed under $\mathrm{Aut}(\mathcal{E})$. A class of degrees $\mathcal{C}$ is *invariant* if $\mathcal{C} = \{\deg(W) \mid W \in \mathcal{S}\}$, where $\mathcal{S}$ is invariant.

- Definable classes are invariant.
- To show a class is not definable, we show it is noninvariant.
- For the c.e. degrees $\mathcal{R}$, we don't know any nontrivial automorphisms.

Definition

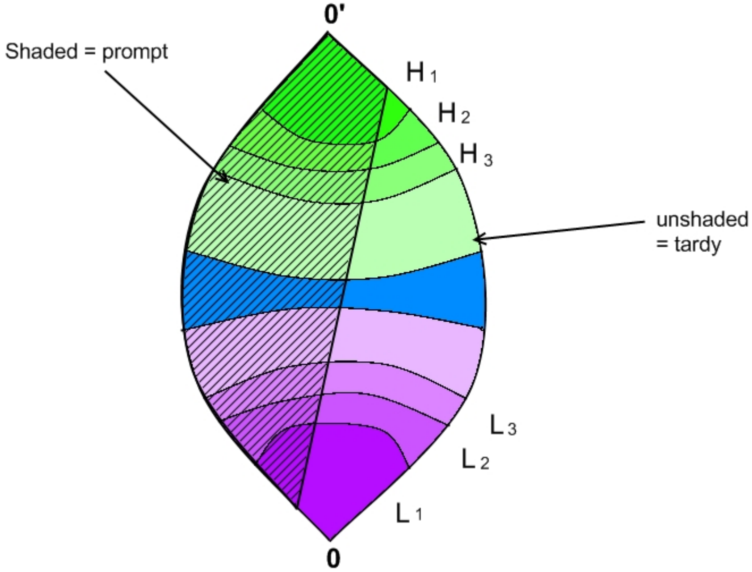
A c.e. set A is *prompt* if there is an enumeration $\{A_s\}$ of A and a computable function p such that for all s, $p(s) \geq s$, and for all e,

$$W_e \text{ infinite} \implies (\exists x)\,(\exists s)\,[x \in W_{e,\text{ at } s}\ \&\ A_s \restriction x \neq A_{p(s)} \restriction x].$$

Theorem (Harrington-Soare, 1996)

For all prompt sets A, there exists $B \equiv_{\mathrm{T}} \mathbf{0}'$ such that $A \simeq B$.

- Cholak, Downey, and Stob [1992] showed this for promptly simple sets.
- There is a low prompt degree. Hence, every set in that degree is automorphic to a complete set.
- Thus, the *downward closed* jump classes $\{L_n\}_{n>0}$ and $\{\overline{H_n}\}_{n\geq 0}$ are *noninvariant*, and thus *not definable*.


0'
Shaded = prompt
H 1
H 2
H 3
unshaded
= tardy
L 3
L 2
L 1
0

Upward Closed Jump Classes

Theorem (Cholak-Harrington, 2002)

For $n \geq 2$, H_n and $\overline{L_n}$ are definable.

Corollary (Lachlan-Shoenfield)

$\overline{L_2}$ is definable.

- Nies, Shore, and Slaman showed that in the c.e. degrees $(\mathcal{R}, <_T)$, H_n and L_n are all definable except possibly L_1.

Red = Definable　　Blue = Not definable

Upward Closed		Downward Closed	
nonlow_n	high_n	low_n	nonhigh_n
$\overline{L_1}$	H_1	L_1	$\overline{H_1}$
$\overline{L_2}$	H_2	L_2	$\overline{H_2}$
$\overline{L_3}$	H_3	L_3	$\overline{H_3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

- The only remaining class is $\overline{L_1}$.
- For the c.e. degrees $\mathcal{R}$, the definability of L_1 is unknown.

Conjecture (Harrington-Soare, 1996)

$\overline{L_1}$ *is noninvariant.*

Theorem (Epstein)

$\overline{L_1}$ is noninvariant, and thus not definable.

Red = Definable   Blue = Not definable

Upward Closed

nonlow$_n$	high$_n$
$\overline{L_1}$	H_1
$\overline{L_2}$	H_2
$\overline{L_3}$	H_3
⋮	⋮

Theorem (Epstein)

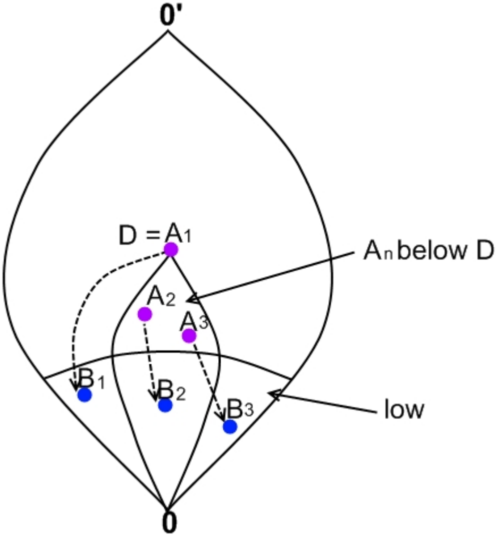
There exists a nonlow D such that for all $A \leq_{\mathrm{T}} D$, there exists a low set B such that $A \simeq B$.

Corollary (Epstein)

The nonlow degrees are noninvariant, and thus not definable.

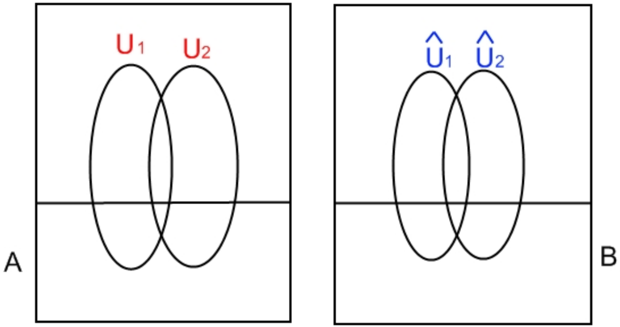
Proof: Let $\mathbf{d} = \deg(D)$. Then $\mathbf{d}$ is an $\overline{L_1}$ degree such that all sets in $\mathbf{d}$ are automorphic to low sets.

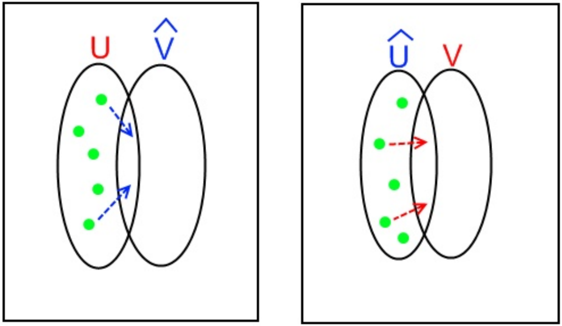
- D must be L_2.
- We will focus on a single set $A = \Psi^D$.

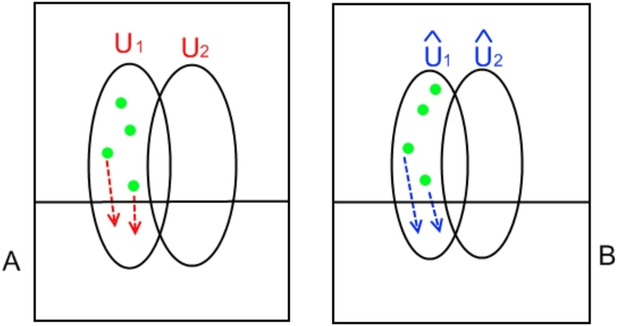

0'
D = A1
An below D
A2
A3
B1
B2
B3
low
0

Building an automorphism

- Given an enumeration $\{U_n\}_{n\in\omega}$ of the c.e. sets, where $U_0 = A$.
- Build an enumeration $\{\widehat{U_n}\}_{n\in\omega}$ of the c.e. sets. Let $B = \widehat{U_0}$.
- We build $\widehat{U_n}$ so that $\Theta : U_n \mapsto \widehat{U_n}$ is an automorphism.


U1
U2
A
Û1
Û2
B


U
V̂
Û
V

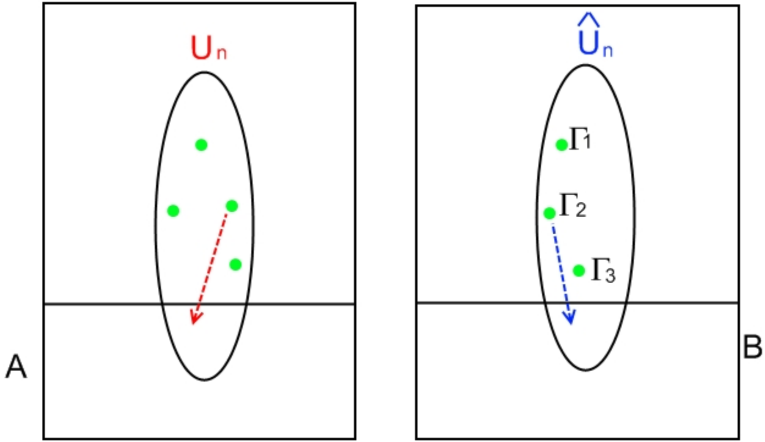

U1
U2
A
Û1
Û2
B

Recall:

Theorem (Harrington-Soare, 1996)

For all prompt sets A, there exists $B \equiv_T \mathbf{0}'$ such that $A \simeq B$

To achieve $K \leq_T B$, as n enters K, enumerate Γ_n into B.



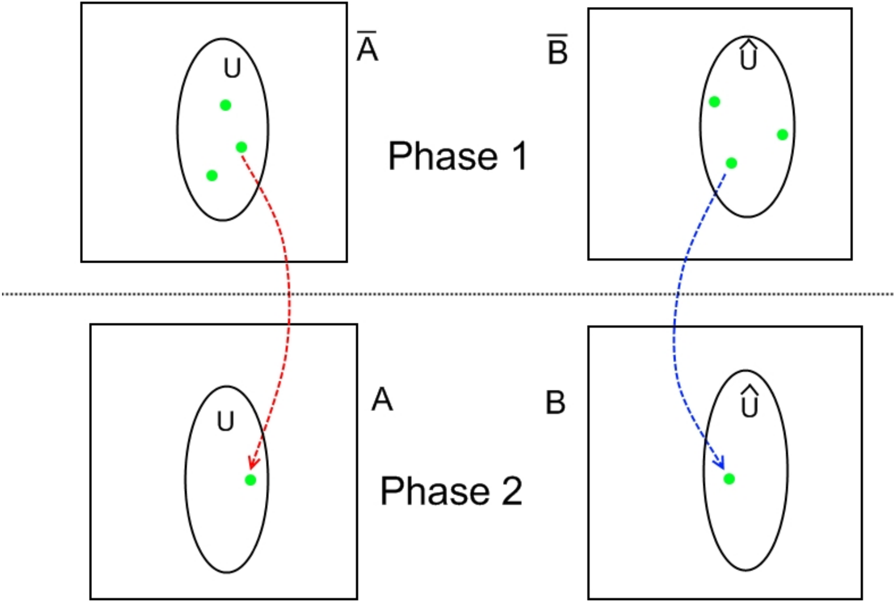
Theorem (Cholak 1995, Harrington-Soare 1996)

Every noncomputable c.e. set is automorphic to a high set.

These theorems move sets *up* in degree. We move sets *down*.

- The Harrington-Soare machinery is inflexible.
- It does not allow us to restrain elements from falling into A.

- We must build an automorphism taking $A \leq_{\mathrm{T}} D$ down to a low set B.
- We restrain B to make it low, so we must also restrain A.
- This is the first automorphism theorem that uses restraint.
- We divide the theorem into two phases because we need two sets of machinery.



$\bar{A}$
U
$\bar{B}$
$\hat{U}$
Phase 1
A
U
B
$\hat{U}$
Phase 2

Phase 1

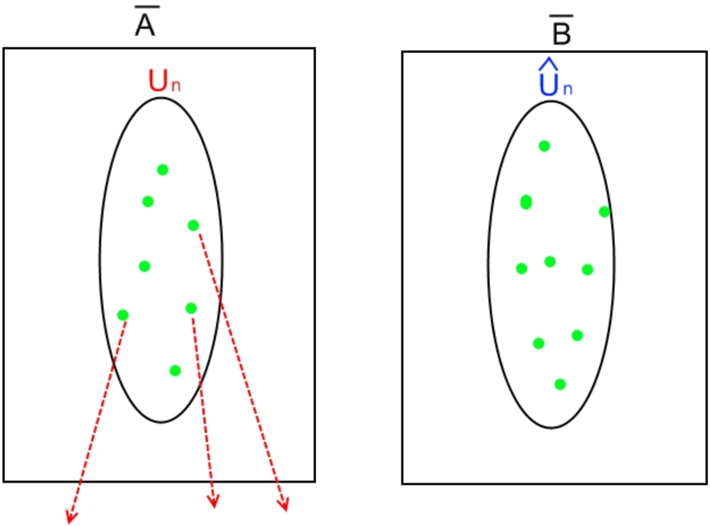
Definition

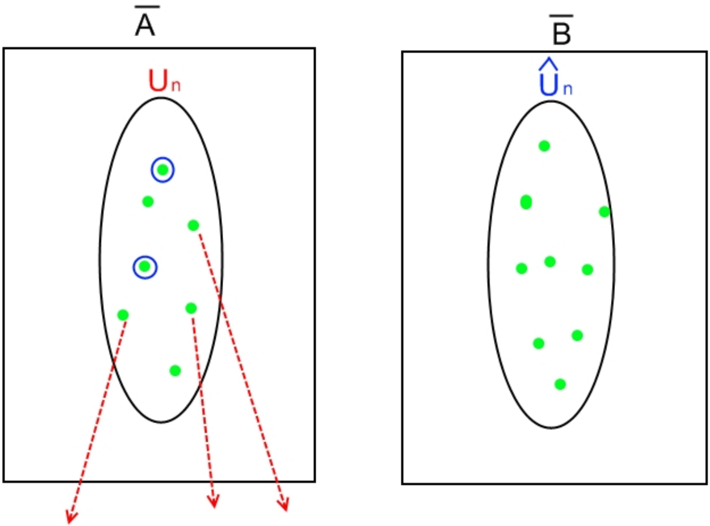
Let $\mathcal{E}(X) = \{W_e \cap X |\ W_e \in \mathcal{E}\}$

Theorem (Soare, 1982)

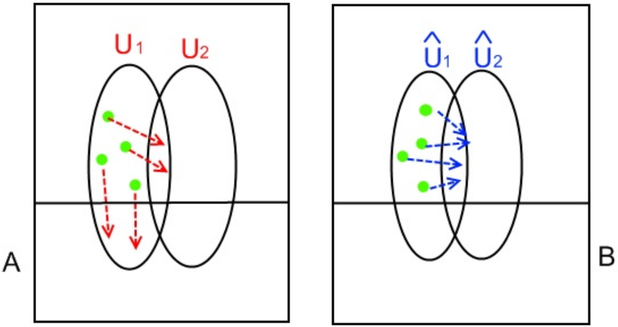
For all coinfinite low B, $\mathcal{E}(\overline{B}) \cong \mathcal{E}$.

Thus, to make A automorphic to B low, we must have $\mathcal{E}(\overline{A}) \cong \mathcal{E}$.


$\bar{A}$
U_n
$\bar{B}$
$\hat{U}_n$



$\bar{A}$
U_n
$\bar{B}$
$\hat{U}_n$

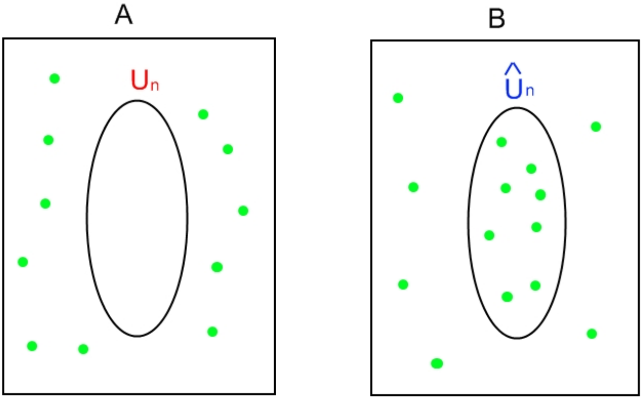


U1
U2
A
Û1
Û2
B

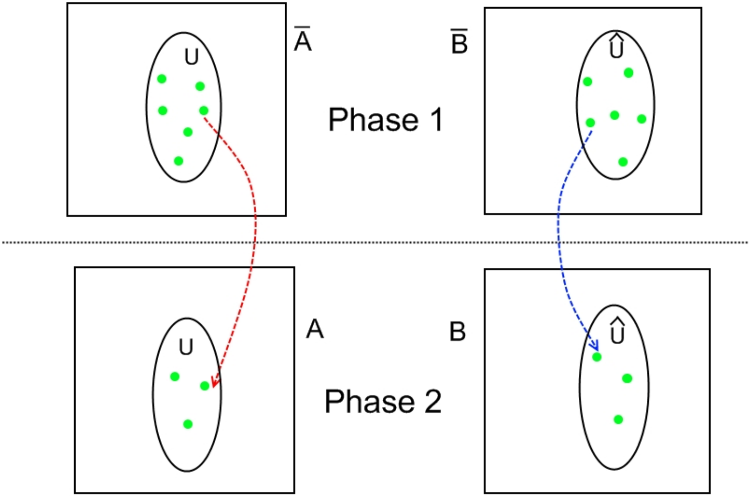
Phase 2: $\mathcal{E}(A) \cong \mathcal{E}(B)$

- Elements enter the Phase 2 when they enter A or B.
- They may already be enumerated into U_n or $\widehat{U_n}$.
- Difficulty: $U_n \cap A$ finite, $\widehat{U_n} \cap B$ infinite.
- We can't let Phase 1 do anything it wants.

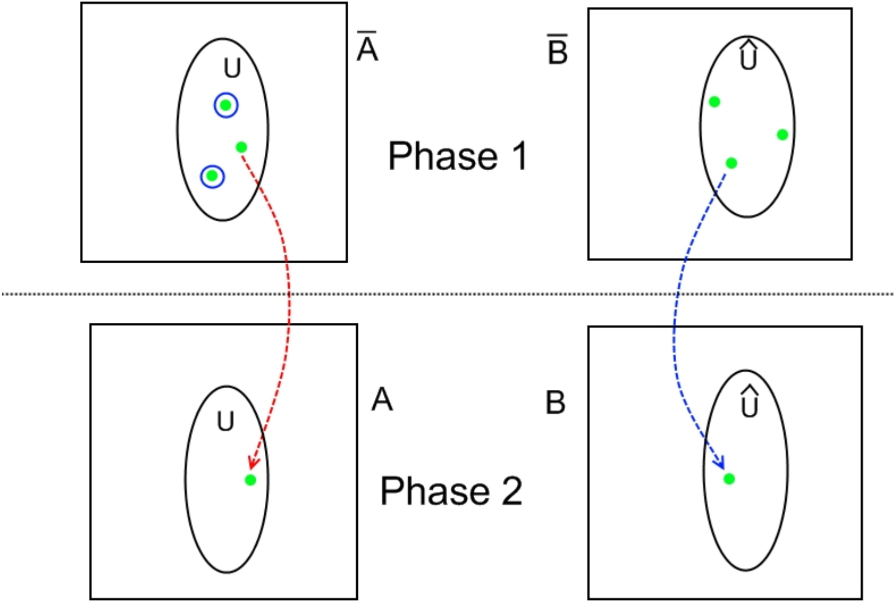
Bad situation:



- The states that have infinitely many elements enter into them as they enter *A* or *B* are the gateway states.
- We make the gateway states equal



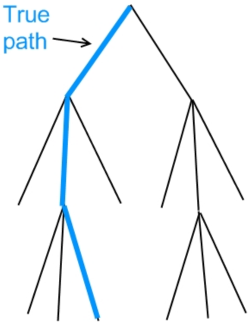
- Summary of automorphism construction:
  - Phase 1: Ensure $\mathcal{E}(\overline{A}) \cong \mathcal{E}(\overline{B})$, while matching gateway states, and
  - Phase 2: Ensure $\mathcal{E}(A) \cong \mathcal{E}(B)$.
- We extend the map $\Theta' : \mathcal{E}(\overline{A}) \to \mathcal{E}(\overline{B})$ to an automorphism Θ of $\mathcal{E}$.

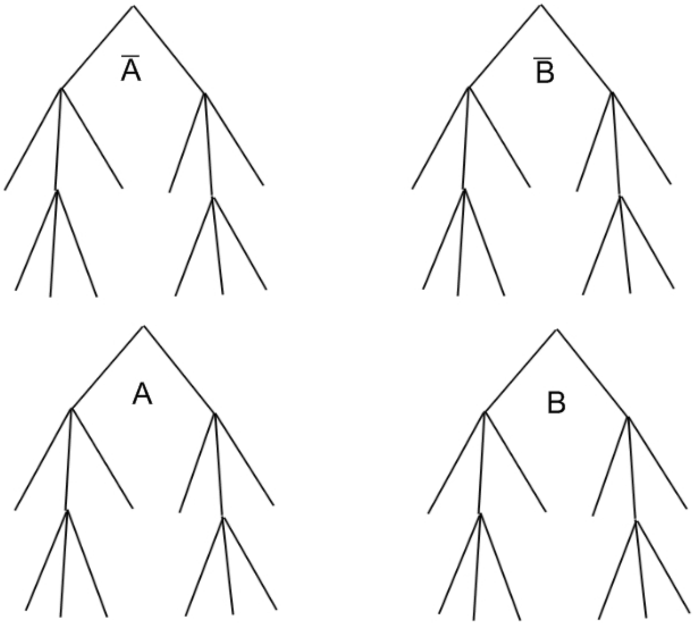

$\bar{A}$
U
$\bar{B}$
$\hat{U}$
Phase 1
A
U
B
$\hat{U}$
Phase 2

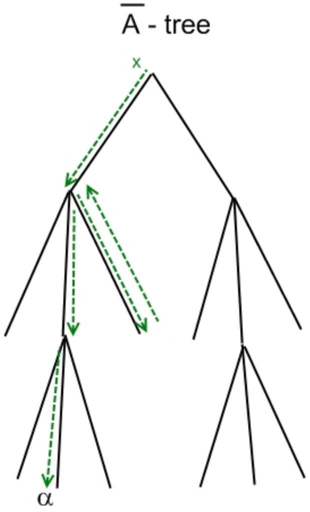
Building D

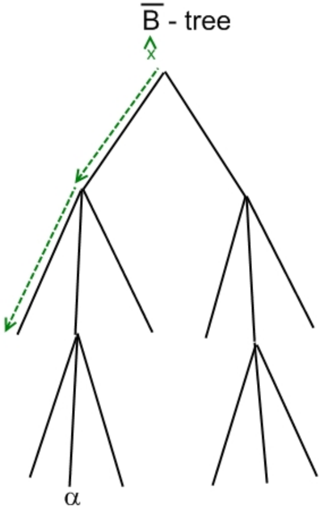
- There are infinitary positive requirements to make D nonlow.
- We also restrain D to keep elements in $\overline{A}$.
- This causes conflict between the positive and negative requirements on D.

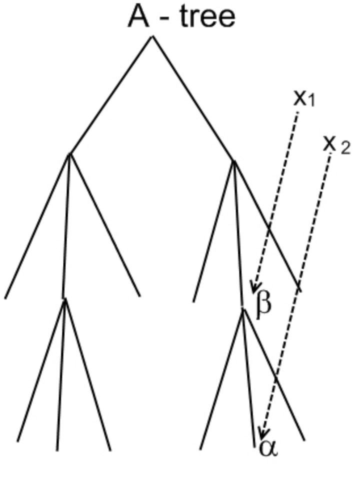
- We can build automorphisms on a tree of strategies.
- We use one tree to keep track of positive and automorphism requirements.
- Negative requirements are built in to automorphism nodes.




$\bar{A}$
$\bar{B}$
A
B


$\overline{A}$ - tree
x
α


$\overline{B}$ - tree
x
α


A - tree
x1
x 2
β
α
B - tree
x̂1
x̂2
β
α

Conclusion

$\overline{L_1}$ is the only upward closed jump class that is not definable.

Upward Closed		Downward Closed	
nonlow_n	high_n	low_n	nonhigh_n
$\overline{L_1}$	H_1	L_1	$\overline{H_1}$
$\overline{L_2}$	H_2	L_2	$\overline{H_2}$
$\overline{L_3}$	H_3	L_3	$\overline{H_3}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

Open Questions

- For all L_2 degrees $\mathbf{a}$, does there exists $A \in \mathbf{a}$ such that A is automorphic to a low set?

- For all c.e. sets $A < \mathbf{0}'$ and noncomputable c.e. sets C, is A automorphic to a c.e. set B, $C \not\leq_{\mathrm{T}} B$? What if A is L_2?

For Further Reading

R. Epstein,
The Structure and Applications of the Computably Enumerable Sets, (2010), PhD thesis.

R. Epstein,
Definability and Automorphisms of the Computably Enumerable Sets, ip.

L. Harrington and R. I. Soare,
The Δ^0_3 automorphism method and noninvariant classes of degrees,
Jour. Amer. Math. Soc., **9** (1996), 617–666.

www.math.harvard.edu/$\sim$ *repstein*

Thank you for listening.