I-Indexed Indiscernible Sets and Trees

Lynn Scow

Vassar College

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Outline

order indiscernible sets

- Fix a linear order O and an L-structure M (we assume M is sufficiently saturated.)
- Let b_i be same-length finite tuples from M:

Definition

$B = \{b_i \mid i \in O\}$ is an **order-indiscernible set** if for all $n \geq 1$, for all $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ from O,

$$(i_1, \ldots, i_n) \mapsto (j_1, \ldots, j_n) \text{ is an order-isomorphism} \Rightarrow$$

$$\mathrm{tp}^L(b_{i_1}, \ldots, b_{i_n}; M) = \mathrm{tp}^L(b_{j_1}, \ldots, b_{j_n}; M)$$

typical application

- Suppose we have parameters $A = \{a_i \mid i < \omega\}$ and

$$i < j \Rightarrow \models \varphi(a_i, a_j)$$

  (let's assume $\varphi(x, x)$ is unsatisfiable)

- In a typical application, we use Ramsey's theorem to find an order-indiscernible set $B = \{b_i \mid i < \omega\}$ such that

$$i < j \Rightarrow \models \varphi(b_i, b_j)$$

- Because B is indiscernible, for some $t \in \{0, 1\}$ ($\varphi^0 = \varphi, \varphi^1 = \neg\varphi$)

$$i > j \Rightarrow \models \varphi(b_i, b_j)^t$$

- In a well-known characterization: $\mathrm{Th}(M)$ is stable $\Leftrightarrow t = 0$ for all such B

generalizing order-indiscernible sets

- Consider O as a structure in its own right, $\mathcal{O} = (O, <)$ in the language $L' = \{<\}$, and re-write the definition:

Definition

$B = \{b_i : i \in O\}$ is an order-indiscernible set if for all $n \geq 1$, for all $i_1, \ldots, i_n, j_1, \ldots, j_n$ from O,

$$(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n) \Rightarrow \mathrm{tp}^L(b_{i_1}, \ldots, b_{i_n}; M) = \mathrm{tp}^L(b_{j_1}, \ldots, b_{j_n}; M)$$

Here $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ means $\mathrm{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{O}) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{O})$

I-indexed indiscernible sets

- Now we fix an arbitrary language L', and an L'-structure $\mathcal{I}$ in the place of $\mathcal{O}$.

Definition ([She90])

$B = \{b_i : i \in I\}$ is an $\mathcal{I}$-*indexed indiscernible set* if for all $n \geq 1$, for all $i_1, \ldots, i_n$, $j_1, \ldots, j_n$ from I,

$$(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n) \Rightarrow \mathrm{tp}^L(b_{i_1}, \ldots, b_{i_n}; M) = \mathrm{tp}^L(b_{j_1}, \ldots, b_{j_n}; M)$$

Here $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ means $\mathrm{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{I}) = \mathrm{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{I})$

- Say that B is Δ-$\mathcal{I}$-indexed indiscernible for $\Delta \subseteq L$ if we replace L above by Δ.

overview

Suppose $\varphi(x, x)$ is unsatisfiable. Then the "type" of a $\{\varphi\}$-$\mathcal{I}$-indexed indiscernible set B is determined entirely by the data $t = (t_0, t_1, \ldots)$

- If B is an order-indiscernible set:

$$\begin{aligned} i < j &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_0} \\ i > j &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_1} \end{aligned}$$

- If B is an ordered-graph indexed indiscernible set

$$\begin{aligned} i < j \wedge iRj &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_0} \\ i > j \wedge iRj &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_1} \\ i < j \wedge \neg iRj &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_2} \\ i > j \wedge \neg iRj &\quad\Rightarrow\quad \models \varphi(b_i, b_j)^{t_3} \end{aligned}$$

ordered graphs

- Consider the example $\mathcal{I} = (I, <, E)$ for an order relation $<$ and an edge relation E. Suppose we only consider $\mathcal{I}$ that are **weakly saturated**, i.e., that embed all possible ordered graphs.
- The above kind of $\mathcal{I}$-indexed indiscernible can be applied to characterize NIP theories.
- We call it an **ordered graph-indiscernible** set.
- Suppose we have an ordered graph-indexed set B such that

$$i < j \wedge iRj \Rightarrow \varphi(b_i, b_j)$$

$$i < j \wedge \neg iRj \Rightarrow \varphi(b_i, b_j)^t$$

  T is NIP $\Leftrightarrow t = 0$ for all such B
- In a characterization from [Sco12]: T is NIP iff any ordered graph indiscernible set in a model of T is an order-indiscernible set.

different partition properties

Fix a coloring on n-tuples from I, where coloring is uniform on pairs:

$\Rightarrow \exists$ large homogeneous $B \subseteq I$ s.t. $\forall (i,j)$ from B:

	iRj	$\neg iRj$
$i < j$	red	blue
$i > j$	green	purple

$\longrightarrow$

	iRj	$\neg iRj$
$i < j$	r (b)	r (b)
$i > j$	p (g)	p (g)

(Ramsey's theorem)

$\longrightarrow$

	iRj	$\neg iRj$
$i < j$	red	blue
$i > j$	green	purple

(Nešetřil-Rödl theorem)

trees

- $\mathcal{I}_s = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}}, (P_n)_{n<\omega})$

  where $\trianglelefteq$ is the partial tree-order, $\wedge$ is the meet function in this order, $<_{\text{lex}}$ is the lexicographical order, and the P_n are predicates picking out the n-th level of the tree
- $\mathcal{I}_1 = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}}, <_{\text{lev}})$

  where $\eta <_{\text{lev}} \nu \Leftrightarrow \ell(\eta) < \ell(\nu)$
- $\mathcal{I}_0 = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}})$
- $\mathcal{I}_t = (\omega^{<\omega}, \trianglelefteq, <_{\text{lex}})$

a typical dichotomy result

- The structure $\mathcal{I}_s$ is ideal to study TP

Definition

A theory T has the *(2-)tree property (TP)* if there is a model $M \vDash T$, a formula $\varphi(x; y)$ and parameters a_η from M with $\ell(a_\eta) = \ell(y)$ such that:

1. $\{\varphi(x; a_{\sigma\restriction n}) : \sigma \in \omega^\omega\}$ is consistent
   (nodes on a path "are consistent"),
   and
2. for all $\eta \in \omega^{<\omega}$, pairs from $\{\varphi(x; a_{\eta^\frown\langle i\rangle}) : i < \omega\}$ are inconsistent
   (siblings "are inconsistent")

- By a well-known result, if a theory has TP, then it has TP as witnessed by $B = \{b_\eta \mid \eta \in {}^{\omega>}\omega\}$ where B is $\mathcal{I}_s$-indexed indiscernible.
- By a series of reductions, one proves the well-known theorem that TP comes in one of two extremal versions...TP1 and TP2.

ramsey classes: I

- Fix a class $\mathcal{K}$ of finite L'-structures.

Definition

For $A, B \in \mathcal{K}$, a *copy of A in B* is an embedding $f : A \to B$ modulo the equivalence relation of being the same embedding up to an automorphism of A

- From now on, assume L' contains a relation $<$ linearly ordering all members of $\mathcal{K}$.
- Then we may think of a copy of A in B as being the range of an embedding from A into B.
- We denote the copies of A in B as $\binom{B}{A}$.

ramsey classes: II

- Given a finite set X of cardinality k, We refer to a map $c : \binom{C}{A} \to X$ as a *k-coloring of the copies of A in C*.
- We say that $B' \subseteq C$ is *homogeneous for c* if there is an element $x_0 \in X$ such that for all $A' \in \binom{B'}{A}$, $c(A') = x_0$.

Definition

A class $\mathcal{K}$ of finite L'-structures is a **Ramsey class** if for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ such that for any 2-coloring of $\binom{C}{A}$, there is a $B' \subseteq C$, isomorphic to B that is homogeneous for this coloring.

EM-types

- For $A = \{a_i \mid i \in I\}$ we can formally define a type in variables $\{x_i \mid i \in I\}$ called the **Ehrenfeucht-Mostowski type of** A,

$$\mathrm{EM}(A)$$

- If $\models \varphi(a_{i_1}, \ldots, a_{i_n})$ for all $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$, then

$$\varphi(x_{j_1}, \ldots, x_{j_n}) \in \mathrm{EM}(A)$$

- If $B \models \mathrm{EM}(A)$, and q is a complete quantifier free type in the language of I, then if

$$\forall \bar{\imath} \ \left(q(\bar{\imath}) \Rightarrow \models \varphi(\overline{a}_{\bar{i}})\right)$$

then

$$\forall \bar{\imath} \ \left(q(\bar{\imath}) \Rightarrow \models \varphi(\overline{b}_{\bar{i}})\right)$$

In fact B will have a rule such as the above for all quantifier-free types q ; whereas A could have rules for none.

the modeling property

Definition

$\mathcal{I}$-indexed indiscernibles have the **modeling property** if for all I-indexed parameters $A = (a_i : i \in I)$ **in any structure** M, there exists $\mathcal{I}$-indexed indiscernible parameters

$$B \models \mathrm{EM}(A)$$

- For which $\mathcal{I}$ do $\mathcal{I}$-indexed indiscernibles have the modeling property?

translation

Theorem (dictionary theorem)

Suppose that $\mathcal{I}$ is a **qfi**, *locally finite structure in a language L' with a relation $<$ linearly ordering I. Then $\mathcal{I}$-indexed indiscernible sets have the modeling property just in case age($\mathcal{I}$) is a Ramsey class.*

- Recall $\mathcal{I}_0 = (\omega^{<\omega}, \trianglelefteq, \wedge, <_{\text{lex}})$

Theorem (Takeuchi-Tsuboi)

I_0-indexed indiscernibles have the modeling property.

Corollary

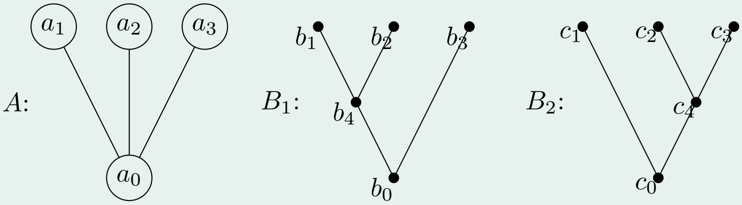
age(I_0) is a Ramsey class.

- Removing $\wedge$ destroys the Ramsey property.

$\mathcal{K} = \text{age}(\mathcal{I}_t)$ not a Ramsey class

Proof.

By [Neš05], if $\mathcal{K}$ is a Ramsey class, then $\mathcal{K}$ has the amalgamation property. However, an example analyzed in Takeuchi-Tsuboi provides a counterexample to amalgamation. A L_t-embeds into B_1, B_2 by $a_i \mapsto b_i, c_i$.



Suppose there exists some amalgam C for (A, B_1, B_2). Observe that b_4, c_4 in C must be $\trianglelefteq$-comparable in C, as both points are $\trianglelefteq$-predecessors of the same point, $b_2 (= c_2)$. If $b_4 \trianglelefteq c_4$, then $b_4 \trianglelefteq c_4 \trianglelefteq c_3 = b_3$, contradicting the data in B_1. If $c_4 \trianglelefteq b_4$, then $c_4 \trianglelefteq b_4 \trianglelefteq b_1 = c_1$, contradicting the data in B_2. □

finitary infinitary

Theorem ([She90])

For every $n, m < \omega$ there is some $k = k(n,m) < \omega$ such that for any infinite cardinal χ, the following is true of $\lambda := \beth_k(\chi)^+$: for every $f : ({}^{n\geq}\lambda)^m \to \chi$ there is a level-preserving, orientation-preserving subtree $I \subseteq {}^{n\geq}\lambda$ such that

(i) $\langle\rangle \in I$ *and whenever* $\eta \in I \cap {}^{n>}\lambda$, $||\{\alpha < \lambda : \eta^\frown \langle\alpha\rangle \in I\}|| \geq \chi^+$.

(ii)$_f$ *If* $\bar{\eta}, \bar{\nu} \in I$ *are such that* $\bar{\eta} \sim_{\mathcal{I}_s} \bar{\nu}$ *then* $f(\eta_0, \ldots, \eta_{m-1}) = f(\nu_0, \ldots, \nu_{m-1})$.

Theorem ([Fou99])

age($\mathcal{I}_s$) is a Ramsey class

Both yield that $\mathcal{I}_s$-indexed indiscernibles have the modeling property, the second by way of the dictionary theorem. The first result yields a height-n indiscernible subtree with m-types from the original tree.

Thanks

Thanks for your attention!

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L. Scow.
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