I-Indexed Indiscernible Sets and Trees

Lynn Scow

Vassar College

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Outline

1. background
2. the modeling property
3. a dictionary theorem
The modeling property

order indiscernible sets

- Fix a linear order $O$ and an $L$-structure $M$ (we assume $M$ is sufficiently saturated.)
- Let $b_i$ be same-length finite tuples from $M$:

Definition

$B = \{b_i \mid i \in O\}$ is an **order-indiscernible set** if for all $n \geq 1$, for all $i_1, \ldots, i_n, j_1, \ldots, j_n$ from $O$,

$(i_1, \ldots, i_n) \mapsto (j_1, \ldots, j_n)$ is an order-isomorphism $\Rightarrow$

$tp^L(b_{i_1}, \ldots, b_{i_n}; M) = tp^L(b_{j_1}, \ldots, b_{j_n}; M)$
Suppose we have parameters \( A = \{ a_i \mid i < \omega \} \) and
\[
i < j \Rightarrow \models \varphi(a_i, a_j)
\]
(let’s assume \( \varphi(x, x) \) is unsatisfiable)

In a typical application, we use Ramsey’s theorem to find an order-indiscernible set \( B = \{ b_i \mid i < \omega \} \) such that
\[
i < j \Rightarrow \models \varphi(b_i, b_j)
\]

Because \( B \) is indiscernible, for some \( t \in \{0, 1\} \) (\( \varphi^0 = \varphi, \varphi^1 = \neg \varphi \))
\[
i > j \Rightarrow \models \varphi(b_i, b_j)^t
\]

In a well-known characterization: \( \text{Th}(M) \) is stable \( \iff t = 0 \) for all such \( B \)
Consider $O$ as a structure in its own right, $\mathcal{O} = (O, <)$ in the language $L' = \{<\}$, and re-write the definition:

**Definition**

$B = \{b_i : i \in O\}$ is an order-indiscernible set if for all $n \geq 1$, for all $i_1, \ldots, i_n, j_1, \ldots, j_n$ from $O$,

$$(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n) \Rightarrow \text{tp}^L(b_{i_1}, \ldots, b_{i_n}; M) = \text{tp}^L(b_{j_1}, \ldots, b_{j_n}; M)$$

Here $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ means

$$\text{qftp}^{L'}(i_1, \ldots, i_n; O) = \text{qftp}^{L'}(j_1, \ldots, j_n; O)$$
$I$-indexed indiscernible sets

- Now we fix an arbitrary language $L'$, and an $L'$-structure $\mathcal{I}$ in the place of $\mathcal{O}$.

Definition ([She90])

$B = \{b_i : i \in I\}$ is an $\mathcal{I}$-indexed indiscernible set if for all $n \geq 1$, for all $i_1, \ldots, i_n, j_1, \ldots, j_n$ from $I$,

$$(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n) \Rightarrow \text{tp}^L(b_{i_1}, \ldots, b_{i_n}; M) = \text{tp}^L(b_{j_1}, \ldots, b_{j_n}; M)$$

Here $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$ means

$$\text{qftp}^{L'}(i_1, \ldots, i_n; \mathcal{I}) = \text{qftp}^{L'}(j_1, \ldots, j_n; \mathcal{I})$$

- Say that $B$ is $\Delta$-$\mathcal{I}$-indexed indiscernible for $\Delta \subseteq L$ if we replace $L$ above by $\Delta$. 
Suppose $\varphi(x,x)$ is unsatisfiable. Then the “type” of a $\{\varphi\}$-$\mathcal{I}$-indexed indiscernible set $B$ is determined entirely by the data $t = (t_0, t_1, \ldots)$

- If $B$ is an order-indiscernible set:
  \begin{align*}
  i < j & \implies \models \varphi(b_i, b_j)^{t_0} \\
  i > j & \implies \models \varphi(b_i, b_j)^{t_1}
  \end{align*}

- If $B$ is an ordered-graph indexed indiscernible set
  \begin{align*}
  i < j \land i \not< j & \implies \models \varphi(b_i, b_j)^{t_0} \\
  i > j \land i \not< j & \implies \models \varphi(b_i, b_j)^{t_1} \\
  i < j \land \neg i \not< j & \implies \models \varphi(b_i, b_j)^{t_2} \\
  i > j \land \neg i \not< j & \implies \models \varphi(b_i, b_j)^{t_3}
  \end{align*}
ordered graphs

- Consider the example $\mathcal{I} = (I, <, E)$ for an order relation $<$ and an edge relation $E$. Suppose we only consider $\mathcal{I}$ that are weakly saturated, i.e., that embed all possible ordered graphs.
- The above kind of $\mathcal{I}$-indexed indiscernible can be applied to characterize NIP theories.
- We call it an ordered graph-indiscernible set.
- Suppose we have an ordered graph-indexed set $B$ such that

  \[ i < j \land iRj \Rightarrow \varphi(b_i, b_j) \]

  \[ i < j \land \neg iRj \Rightarrow \varphi(b_i, b_j)^t \]

  $T$ is NIP $\iff t = 0$ for all such $B$
- In a characterization from [Sco12]: $T$ is NIP iff any ordered graph indiscernible set in a model of $T$ is an order-indiscernible set.
Fix a coloring on $n$-tuples from $I$, where coloring is uniform on pairs:

$$\Rightarrow \exists \text{ large homogeneous } B \subseteq I \text{ s.t. } \forall (i, j) \text{ from } B:$$

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<thead>
<tr>
<th></th>
<th>$iRj$</th>
<th>$\neg iRj$</th>
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</thead>
<tbody>
<tr>
<td>$i &lt; j$</td>
<td>red</td>
<td>blue</td>
</tr>
<tr>
<td>$i &gt; j$</td>
<td>green</td>
<td>purple</td>
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(Ramsey’s theorem)

$$\Rightarrow$$

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<tr>
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<th>$\neg iRj$</th>
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</thead>
<tbody>
<tr>
<td>$i &lt; j$</td>
<td>r (b)</td>
<td>r (b)</td>
</tr>
<tr>
<td>$i &gt; j$</td>
<td>p (g)</td>
<td>p (g)</td>
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(Nešetřil-Rödl theorem)

$$\Rightarrow$$

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\[ \mathcal{I}_s = (\omega^{<\omega}, \sqsubseteq, \land, <_{\text{lex}}, (P_n)_{n<\omega}) \]

where \( \sqsubseteq \) is the partial tree-order, \( \land \) is the meet function in this order, \( <_{\text{lex}} \) is the lexicographical order, and the \( P_n \) are predicates picking out the \( n \)-th level of the tree

\[ \mathcal{I}_1 = (\omega^{<\omega}, \sqsubseteq, \land, <_{\text{lex}}, <_{\text{lev}}) \]

where \( \eta <_{\text{lev}} \nu \Leftrightarrow \ell(\eta) < \ell(\nu) \)

\[ \mathcal{I}_0 = (\omega^{<\omega}, \sqsubseteq, \land, <_{\text{lex}}) \]

\[ \mathcal{I}_t = (\omega^{<\omega}, \sqsubseteq, <_{\text{lex}}) \]
The structure $\mathcal{I}_s$ is ideal to study TP

**Definition**

A theory $T$ has the (2-)tree property (TP) if there is a model $M \models T$, a formula $\varphi(x; y)$ and parameters $a_\eta$ from $M$ with $\ell(a_\eta) = \ell(y)$ such that:

1. $\{\varphi(x; a_\sigma|_n) : \sigma \in \omega^\omega\}$ is consistent
   (nodes on a path “are consistent”),
   and
2. for all $\eta \in \omega^{<\omega}$, pairs from $\{\varphi(x; a_\eta \dashv \langle i \rangle) : i < \omega\}$ are inconsistent
   (siblings “are inconsistent”)

By a well-known result, if a theory has TP, then it has TP as witnessed by $B = \{b_\eta \mid \eta \in \omega^{>\omega}\}$ where $B$ is $\mathcal{I}_s$-indexed indiscernible.

By a series of reductions, one proves the well-known theorem that TP comes in one of two extremal versions...TP1 and TP2.
Fix a class $\mathcal{K}$ of finite $L'$-structures.

**Definition**

For $A, B \in \mathcal{K}$, a *copy of $A$ in $B$* is an embedding $f : A \to B$ modulo the equivalence relation of being the same embedding up to an automorphism of $A$.

1. From now on, assume $L'$ contains a relation $<$ linearly ordering all members of $\mathcal{K}$.
2. Then we may think of a copy of $A$ in $B$ as being the range of an embedding from $A$ into $B$.
3. We denote the copies of $A$ in $B$ as $\left( B_A \right)$.
Given a finite set $X$ of cardinality $k$, we refer to a map $c : \binom{C}{A} \to X$ as a $k$-coloring of the copies of $A$ in $C$.

We say that $B' \subseteq C$ is homogeneous for $c$ if there is an element $x_0 \in X$ such that for all $A' \in \binom{B'}{A}$, $c(A') = x_0$.

**Definition**

A class $\mathcal{K}$ of finite $L'$-structures is a **Ramsey class** if for all $A, B \in \mathcal{K}$ there is a $C \in \mathcal{K}$ such that for any 2-coloring of $\binom{C}{A}$, there is a $B' \subseteq C$, isomorphic to $B$ that is homogeneous for this coloring.
EM-types

- For $A = \{a_i \mid i \in I\}$ we can formally define a type in variables $
\{x_i \mid i \in I\}$ called the **Ehrenfeucht-Mostowski type of $A$**, $EM(A)$

- If $\models \varphi(a_{i_1}, \ldots, a_{i_n})$ for all $(i_1, \ldots, i_n) \sim (j_1, \ldots, j_n)$, then
  $$\varphi(x_{j_1}, \ldots, x_{j_n}) \in EM(A)$$

- If $B \models EM(A)$, and $q$ is a complete quantifier free type in the language of $I$, then if
  $$\forall \bar{i} \ (q(\bar{i}) \Rightarrow \models \varphi(\bar{a}_i))$$
  then
  $$\forall \bar{i} \ (q(\bar{i}) \Rightarrow \models \varphi(\bar{b}_i))$$

In fact $B$ will have a rule such as the above for all quantifier-free types $q$; whereas $A$ could have rules for none.
the modeling property

\[ \mathcal{I}\text{-indexed indiscernibles have the } \textbf{modeling property} \text{ if for all } \mathcal{I}\text{-indexed parameters } A = (a_i : i \in I) \text{ in any structure } \mathcal{M}, \text{ there exists } \mathcal{I}\text{-indexed indiscernible parameters } B \models \text{EM}(A) \]

For which \( \mathcal{I} \) do \( \mathcal{I} \)-indexed indiscernibles have the modeling property?
Theorem (dictionary theorem)

Suppose that $\mathcal{I}$ is a qfi, locally finite structure in a language $L'$ with a relation $<$ linearly ordering $I$. Then $\mathcal{I}$-indexed indiscernible sets have the modeling property just in case $\text{age}(\mathcal{I})$ is a Ramsey class.

- Recall $\mathcal{I}_0 = (\omega^{<\omega}, \sqsubseteq, \land, <_{\text{lex}})$

Theorem (Takeuchi-Tsuboi)

$I_0$-indexed indiscernibles have the modeling property.

Corollary

$\text{age}(I_0)$ is a Ramsey class.

- Removing $\land$ destroys the Ramsey property.
Proof.

By [Neš05], if $\mathcal{K}$ is a Ramsey class, then $\mathcal{K}$ has the amalgamation property. However, an example analyzed in Takeuchi-Tsuboi provides a counterexample to amalgamation. A $L_t$-embeds into $B_1, B_2$ by $a_i \mapsto b_i, c_i$.

Suppose there exists some amalgam $C$ for $(A, B_1, B_2)$. Observe that $b_4, c_4$ in $C$ must be $\leq$-comparable in $C$, as both points are $\leq$-predecessors of the same point, $b_2 (= c_2)$. If $b_4 \leq c_4$, then $b_4 \leq c_4 \leq c_3 = b_3$, contradicting the data in $B_1$. If $c_4 \leq b_4$, then $c_4 \leq b_4 \leq b_1 = c_1$, contradicting the data in $B_2$. 

\[ \mathcal{K} = \text{age}(\mathcal{I}_t) \text{ not a Ramsey class} \]
Theorem ([She90])

For every $n, m < \omega$ there is some $k = k(n, m) < \omega$ such that for any infinite cardinal $\chi$, the following is true of $\lambda := \bigcup_k (\chi)^+$: for every $f : (n \geq \lambda)^m \to \chi$ there is a level-preserving, orientation-preserving subtree $I \subseteq n \geq \lambda$ such that

(i) $\langle \rangle \in I$ and whenever $\eta \in I \cap n > \lambda$, $\|\{\alpha < \lambda : \eta \cap \langle \alpha \rangle \in I\}\| \geq \chi^+$.

(ii) If $\bar{\eta}, \bar{\nu} \in I$ are such that $\bar{\eta} \sim_{\mathcal{I}_s} \bar{\nu}$ then $f(\eta_0, \ldots, \eta_{m-1}) = f(\nu_0, \ldots, \nu_{m-1})$.

Theorem ([Fou99])

$\text{age}(\mathcal{I}_s)$ is a Ramsey class

Both yield that $\mathcal{I}_s$-indexed indiscernibles have the modeling property, the second by way of the dictionary theorem. The first result yields a height-$n$ indiscernible subtree with $m$-types from the original tree.
Thanks for your attention!
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(revised edition).  