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HARVARD MATH AND PHILOSOPHY LOGIC COLLOQUIUM

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will speak on:

The Canonical Base Property and the Zilber Dichotomy Revisited

Date : Tuesday, October 8, 2013

Time : 4:30 - 5:30 PM

Location: Science Center 507

The truth of the Zilber dichotomy in several first-order theories of fields with additional operators was behind Hrushovski's dramatic application of model theory to diophantine geometry in the nineties. About ten years ago, abstracting from a theorem in complex-analytic geometry, Pillay introduced a new model-theoretic condition now called the canonical base property (CBP). This condition provides a direct proof of the Zilber dichotomy in various contexts, and has other strong geometric consequences. What seems to be required in establishing the CBP in any given situation is an appropriate notion of "jet space". This talk will be a largely expository introduction to, and overview of, the subject.

Three Theories
(each with a distinguished definable field)

1. **DCF_0** Differentially closed fields of char. 0.
Model companion of differential fields in char. 0

$(K, 0, 1, +, -, \times, \delta)$
$$\delta(a+b) = \delta a + \delta b$$
$$\delta(ab) = a\delta b + b\delta a$$

Field of constants $= \{a : \delta a = 0\}$

2. **$SCF_{p,r}$** Separably closed fields of char. p
with $[K : K^p] = p^r$

$(K, 0, 1, +, -, \times)$

$$K^{p^\infty} = \bigcap_n K^{p^n} = \{a : \forall n \exists b \ b^{p^n} = a\}$$

3. **CCM** Compact complex manifolds
Theory of the many-sorted structure:
sorts = compact complex manifolds
language = a predicate for each $\mathbb{C}$-analytic subset of each product of sorts
($\mathbb{C}$-analytic means given locally by zeros of hol. functions.)

$\mathbb{C}$ complex field is dfble on the sort $\mathbb{P}(\mathbb{C})$, projective line.

Each of these theories is tame: all stable.

- DCF_0 and CCM admit QE
- $SCF_{p,\nu}$ admits quantifier simplification up to λ-functions

We work from now on in a large saturated model M of one of these theories, and we let

$$C = \begin{cases} \text{field of constants} & \text{if } M \models DCF_0 \\ K^{p^\infty} & \text{if } M \models SCF_{p,\nu} \\ \mathbb{C}' & \text{if } M \models CCM \end{cases}$$

FACT: C is a dfble algebraically closed field and the structure induced by M on C is that of a pure field.
i.e. $X \subseteq C^\ell$ dfble in $M \Leftrightarrow X$ dfble in $(C, 0, 1, +, -, \times)$

Algebraic geometry lives in M on C.

We view M as an expansion of algebraic geometry to

- differential varieties if $M \models DCF_0$
- λ-varieties if $M \models SCF_{p,\nu}$
- $\mathbb{C}$-analytic varieties if $M \models CCM$

The Dichotomy Theorem

We are interested in the category of dfble sets in M of finite rank.

Finite rank dfble sets are analysable in terms of rank 1 sets.

rank 1 means every dfble subset is finite or cofinite.

C and every algebraic curve in C^n is rank 1.
What others are there?

Dichotomy Theorem: If X is a rank 1 set in M then either X is (essentially) an algebraic curve in C^n or X is modular.

Modular = $X \times X$ has no "rich" dfble family of infinite dfble sets.
There are only 1-parameter families.

Note that C is not modular:
$y = ax + b$ defines a rich family of lines in $C \times C$

Point: Rank 1 sets either come from algebraic geometry or are structurally simple.

The truth of the dichotomy for DCF_0 (Hrush.-Sok.) and for $SCF_{p,r}$ (Hrushovski) is at the heart of Hrushovski's model-theoretic proof of the function field Mordell-Lang Conjecture (JAMS '96)

The proof of the dichotomy in both cases relies on the very difficult theory of Zariski Geometries developed by Hrushovski and Zilber (JAMS '96)

The dichotomy for CCM follows directly from a theorem in complex-analytic geometry due to Campana/Fujiki (observed by Pillay in 2002).

Canonical Base Property (CBP)

Abstract model-theoretic analogue of Campana/Fujiki

CBP $\Rightarrow$ Dichotomy (Pillay 2002)

DCF_0 has CBP (Pillay, Ziegler 2003)

The Canonical Base Property

M has the CBP if the following holds:

Given
- X a finite rank A-dfble set in M
- $c \in X$
- an A-dfble family of subsets of X

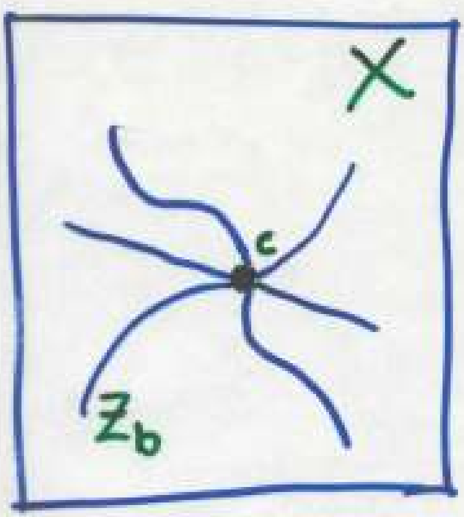
  $\mathcal{Z} = \langle Z_b \subseteq X : b \in B \rangle$ with

  - $tp(b/A) = tp(b'/A)$ all $b, b' \in B$
  - $c \in Z_b$ generic for all $b \in B$

Then there exists a dfble map $f : B \to \mathbb{C}^n$ s.t.

$f(b) = f(b') \iff Z_b = Z_{b'}$ up to smaller rank

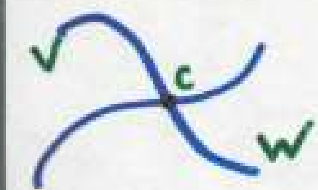
(write this as $Z_b \sim Z_{b'}$)



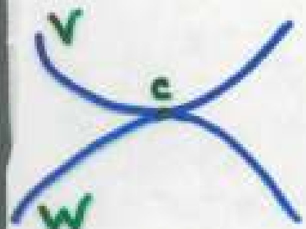
So $\mathcal{Z}/\sim$ is an algebraic – i.e. dfble in $\mathbb{C}$ – family

CCM has CBP : Jet spaces

X a compact $\mathbb{C}$-manifold, $c \in X$, V, W $\mathbb{C}$-analytic subsets



$V \neq W$ $\left(\text{Example: } \begin{matrix} V = \Gamma(g) \\ W = \Gamma(h) \end{matrix} \right)$



$V \neq W$ but $T_c V = T_c W$ in $T_c X$

$g'(c) = h'(c)$

$V \neq W$ but $T_c^{(2)} V = T_c^{(2)} W$ in $T_c^{(2)} X$

$g''(c) = h''(c)$

$\vdots$

If $T_c^{(n)} V = T_c^{(n)} W$ all $n \in \mathbb{N}$, then $V = W$

nth Jet space of V at c $= T_c^{(n)} V$

$$= \mathrm{Hom}_{\mathbb{C}} \left(m_{V,c} / m_{V,c}^{n+1}, \mathbb{C} \right)$$

These are finite dimensional vector spaces over $\mathbb{C}$

– subspaces of $T_c^{(n)} X$.

Given a dfble family $Z = \langle Z_b \subseteq X : b \in B \rangle$ of $\mathbb{C}$-analytic subsets of X through c, there exists N such that

$$T_c^{(N)} Z_b = T_c^{(N)} Z_{b'} \Rightarrow Z_b = Z_{b'}$$

So, $f(b) := T_c^{(N)} Z_b$ is the dfble map $f : B \to \mathrm{Grass}(T_c^{(N)} X) \subseteq \mathbb{C}^n$ witnessing the CBP. $\square$

Cheating : standard model is not saturated, need to do this uniformly.

DCF_0 has CBP : Differential Jets

Pillay and Ziegler develop jet spaces for finite rank differential varieties.
They are finite dimensional vector spaces over the field of constants and they determine differential subvarieties.

Differential jet spaces are used, as in the $\mathbb{C}$-analytic case, to prove CBP.

Question: Does $SCF_{p,r}$ have CBP?

Counterexamples to CBP in general
- Hrushovski and Hrushovski - Palacín - Pillay

Other uses of the CBP

- Techniques used to prove CBP are of independent interest in model theory and geometry: Jets
  - M., Scanlon 2010, 2011, 2013
- Leads to infinite rank analogues of the dichotomy theorem for regular types.
  - M., Pillay, Scanlon $DCF_{0,m}$
- CBP has strong structural consequences for finite rank >1 definable sets beyond what is given by the dichotomy
  - Chatzidakis 2012
  - M., Pillay 2008