Forcing axioms in $\mathbb{P}_{\max}$ extensions

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For Γ a class of partial orders FA(Γ) is the statement that for all $P \in \Gamma$, and for all collections $\{D_\alpha : \alpha < \omega_1\}$ consisting of dense subsets of P, there is a filter $G \subseteq P$ intersecting each D_α.

Examples.

- FA(c.c.c.) is $\mathrm{MA}_{\aleph_1}$
- FA(proper) is PFA
- FA(preserving stationary subsets of ω_1) is MM
- FA(σ-closed*c.c.c) : stronger than $\mathrm{MA}_{\aleph_1}$, weaker than PFA

Why only $\aleph_1$ many dense sets? Even for Cohen forcing, no filter can meet continuum many dense sets.

Theorem.[Todorcevic, Velickovic] FA(σ-closed * c.c.c.) implies that $2^{\aleph_0} = \aleph_2$.

- Forcing axioms say that the universe is closed under certain forcing operations (i.e., certain objects that can be forced to exist exist already). Models of forcing axioms can be thought of a maximal, or complete, in contrast to fine structural models, which are minimal (with respect to some hypothesis).
- The consistency of forcing axioms can tell you what the absolute objects are in a given class.
    - Destroying stationary subsets of ω_1 is the only impediment to a forcing axiom.
    - (Moore) PFA implies that the uncountable linear orders have a five-element basis.
    - (Velickovic) PFA implies that for all infinite cardinals κ, all automorphisms of $\mathcal{P}(\kappa)/\mathrm{Fin}$ are trivial.

Large cardinal hypotheses statements which assert the existence of infinite cardinals with certain properties.

For example, a strongly inaccessible cardinal is a regular cardinal closed under cardinal exponentiation.

The existence of strongly inaccessible cardinals implies the consistency of ZFC, so cannot be proved in ZFC.

- Empirically, large cardinal axioms are linearly ordered by $\phi < \psi$ iff ZFC + ψ implies ZFC + Con(ϕ).
- Fine structural models have been produced for some initial segment of the hierarchy (roughly a Woodin limit of Woodins).
- Below this, we can show that large cardinals are necessary, and, often, show that statements (often having no obvious relation to large cardinals) are equiconsistent with some large cardinal hypothesis.
- Forcing axioms may be the most important statements beyond this level.

Theorem.[Foreman-Magidor-Shelah] If there exists a supercompact cardinal, then there is a forcing extension in which Martin's Maximum holds.

For a cardinal κ, MM(κ) is the restriction of Martin's Maximum to partial orders of cardinality at most κ.

Martin's Axiom is equivalent to its restriction to partial orders of cardinality $\aleph_1$, but MM is not equivalent to its restriction to any small cardinal.

Theorem.[Woodin] Assuming $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular", there is a forcing extension in which ZFC + MM($\mathfrak{c}$) holds.

MM($\mathfrak{c}$) implies that $\mathfrak{c} = \aleph_2$.

Theorem.[Sargsyan] $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" has consistency strength below a Woodin limit of Woodin cardinals.

What about $\mathrm{MM}(\mathfrak{c}^+)$?

We will show that certain consequences of $\mathrm{MM}(\mathfrak{c}^+)$ can be produced from hypotheses below a Woodin limit of Woodin cardinals.

Traditional consistency proofs for forcing axioms, including the Foreman-Magidor-Shelah proof, are iterated forcing constructions over models of ZFC.

Given $A \subseteq {}^{\omega}\omega$, the game G_A has ω many round, where players I and II alternately choose the members of a sequence $\langle n_i : i \in \omega \rangle$, and player I wins if $\langle n_i : i \in \omega \rangle \in A$.

The set A is determined if either player I or player II has a winning strategy.

- The Axiom of Determinacy (AD) is the statement that every $A \subset {}^\omega\omega$ is determined.
- The Axiom of Real Determinacy ($\mathrm{AD}_\mathbb{R}$) is the corresponding statement for games where the players play elements of ${}^\omega\omega$.
- AD^+ (a statement in between, formulated by Woodin)

Θ is the least ordinal which is not a surjective image of ${}^{\omega}\omega$.

- $\mathbb{P}_{\max}$ is a partial ordered developed by Woodin in the early 1990's.
- Conditions are elements of $H(\aleph_1)$, essentially countable transitive models of ZFC with some additional structure.
- The order is induced by elementary embeddings with critical point ω_1.

- A $\mathbb{P}_{\max}$ extension of a model of AD^+ satisfies MM($\aleph_1$).
- A $\mathbb{P}_{\max}$ extension of a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" satisfies MM($\mathfrak{c}$) + $\mathfrak{c} = \aleph_2$.

- $\mathbb{P}_{\max}$ is ω-closed, preserves ω_2, and makes Θ into ω_3.
- It forces a wellordering of $\mathbb{R}$ of ordertype ω_2.
- If $G \subseteq \mathbb{P}_{\max}$ is a V-generic filter (for V a model of AD^+) then $\mathcal{P}(\omega_1)^{V[G]} \subseteq L(\mathbb{R})[G]$.
- Forcing with $\mathbb{P}_{\max}$ over a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" does not wellorder $\mathcal{P}(\mathbb{R})$, but $\mathcal{P}(\mathbb{R})$ can be wellordered over the $\mathbb{P}_{\max}$ extension (without adding subsets of ω_2) by forcing with $\mathrm{Add}(1, \omega_3)$.

In $\mathbb{P}_{\max} * \mathrm{Add}(1, \omega_3)$-extensions of suitable models of determinacy (below a Woodin limit of Woodins) one can obtain $\mathrm{MM}(\mathfrak{c}^+)$ for partial orders P for which at least one of the following hold.

- Forcing with P does make make ω_3 have cofinality ω_1.
- P is stationary set preserving in any outer model with the same ω_1-sequences of ordinals.

The following definition is due to Jensen.

For an infinite cardinal κ, $\square_\kappa$ asserts the existence of a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that for all $\alpha < \kappa^+$,

- C_α is a club subset of α
- for all $\beta \in \lim(C_\alpha)$, $C_\beta = C_\alpha \cap \beta$
- $\mathrm{ot}(C_\alpha) \leq \kappa$

- there cannot exist a club $E \subseteq \gamma$ such that for all $\alpha \in \lim(E)$, $C_\alpha = E \cap \alpha$.
- $\square(\kappa^+)$: remove the condition $\mathrm{ot}(C_\alpha) \leq \kappa$ and assert the nonexistence of such an E (so $\square(\kappa^+)$ is weaker)

$\square$ principles illustrate why partial orders preserving stationary subsets of ω_1 don't have to have small subalgebras with the same property.

Much (possibly all) of the known consistency strength of MM comes from the following result.

Theorem.[Todorcevic] If γ is an ordinal of cofinality greater than ω_1, there exists a σ-closed*c.c.c. forcing P of cardinality $|\gamma|^{\aleph_0}$ such that $\mathrm{FA}(\{P\})$ implies $\neg\Box(\gamma)$.

- In the $\mathbb{P}_{\max} * \mathrm{Add}(1, \omega_3)$ extension of a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular", $2^{\aleph_0} = \aleph_2$.
- Given that $2^{\aleph_0} = \aleph_2$, $\mathrm{MM}(\mathfrak{c})$ implies $\neg\square(\omega_2)$ and $\mathrm{MM}(\mathfrak{c}^+)$ implies $\neg\square(\omega_3)$.

Theorem.[CLSSSZ] Assuming a certain determinacy hypothesis below a Woodin limit of Woodin cardinals, the $\mathbb{P}_{\max} * \mathrm{Add}(1, \omega_3)$ extension satisfies $\mathrm{MM}(\mathfrak{c}) + \neg\Box_{\omega_2}$.

- The previous upper bound for $\neg\square(\omega_2) + \neg\square_{\omega_2}$ was a quasicompact cardinal, above the current inner model theory.
- Lower bound : at least $AD^{L(\mathbb{R})}$
- From a stronger hypothesis (beyond a Woodin limit of Woodin cardinals) we get $\neg\square(\omega_3)$.

The Solovay sequence is the unique continuous sequence $\langle \theta_\alpha : \alpha \leq \delta \rangle$ satisfying the following conditions.

- θ_0 is the least ordinal γ for which there does not exist an ordinal definable function from ${}^\omega\omega$ onto γ.
- if $\theta_\alpha < \Theta$, then $\theta_{\alpha+1}$ is the least ordinal γ for which there does not exist an ordinal definable function from $\mathcal{P}(\theta_\alpha)$ onto γ.
- $\theta_\delta = \Theta$.

The following gives some indication of the strength of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular".

Theorem.[Woodin] Assuming AD + $V = L(\mathcal{P}(\mathbb{R}))$, $\mathrm{AD}_{\mathbb{R}}$ holds if and only if the Solovay sequence has limit length.

HOD is the class of hereditarily ordinal definable sets.

Theorem.[Woodin] Assuming AD + DC, all successor elements of the Solovay sequence are Woodin in HOD.

Given $A, B \subseteq {}^\omega\omega$, say that $A \leq_W B$ (A is Wadge below B) if $A = f^{-1}[B]$, for some continuous $f \colon {}^\omega\omega \to {}^\omega\omega$.

Theorem.[Wadge] Under AD, for all $A, B \subseteq {}^\omega\omega$, either $A \leq_W B$ or $B \leq_W {}^\omega\omega \setminus A$.

Theorem.[Martin] Under AD, $\leq_W$ is a wellfounded relation on the $\leq_W$-equivalence classes.

So we can associate to each subset of ${}^\omega\omega$ its Wadge rank. We let $\mathcal{P}_\alpha({}^\omega\omega)$ be the collection of subsets of ${}^\omega\omega$ of Wadge rank less than α.

The hypotheses for our theorems on the failure of square are derived from the following theorem, plus Sargsyan's analysis of HOD.

Theorem.[Woodin] It is consistent relative to a Woodin limit of Woodin cardinals that there exist Wadge-incomparable $A, B \subseteq {}^{\omega}\omega$ such that $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ both satisfy AD.

HOD_X is the class of set hereditarily ordinal definable from parameters in X.

Theorem.[Woodin] Suppose that $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds θ_α is a member of the Solovay sequence. Then $\mathrm{HOD}_{\mathcal{P}_{\theta_\alpha}({}^\omega\omega)}$ is a model of AD whose Θ is θ_α and whose $\mathcal{P}({}^\omega\omega)$ is $\mathcal{P}_{\theta_\alpha}({}^\omega\omega)$.

Furthermore, if θ_α is regular in HOD then it is regular in $\mathrm{HOD}_{\mathcal{P}_{\theta_\alpha}({}^\omega\omega)}$, and stationary sets are preserved as well.

Theorem.[CLSSSZ] Assume that $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds, and the cofinalities of the members of the Solovay sequence are unbounded below Θ. Then $\square_{\omega_2}$ fails in the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\theta({}^\omega\omega)}$.

Proof. A name for such a sequence would have to be definable from a subset of ${}^\omega\omega$ of Wadge rank below θ. Fixing a θ on the Solovay sequence above this Wadge rank, the entire $\square_{\omega_2}$-sequence would exist in $\mathrm{HOD}_{\mathcal{P}_\theta({}^\omega\omega)}[G]$, where G is the $\mathbb{P}_{\max}$-generic filter. However, ordinals of cofinality greater than θ must remain so in $\mathrm{HOD}_{\mathcal{P}_\theta({}^\omega\omega)}[G]$, and a $\square_{\omega_2}$-sequence would witness that every ordinal below Θ has cofinality at most ω_2. $\square$

This leaves open the issue of whether a $\square_{\omega_2}$-sequence can be added by $\mathrm{Add}(1, \omega_3)$.

The converse also holds.

Theorem.[CLSSSZ] Assume that $\mathrm{AD}^+ + V = L(\mathcal{P}(\mathbb{R}))$ holds, and the cofinalities of the members of the Solovay sequence are bounded below Θ. Then $\square_{\omega_2}$ holds in the $\mathbb{P}_{\max}$ extension of $\mathrm{HOD}_{\mathcal{P}_\theta({}^\omega\omega)}$.

The hypothesis for the following theorem is below a Woodin limit of Woodin cardinals.

Theorem.[CLSSSZ] Assume that $\mathrm{AD}_{\mathbb{R}} + V = L(\mathcal{P}(\mathbb{R}))$ holds, and that stationarily many elements of the Solovay sequence are regular in HOD. Then in the $\mathbb{P}_{\max} * \mathrm{Add}(1, \omega_3)$-extension there is no partial $\square_{\omega_2}$-sequence defined on all points of cofinality at most ω_1.

The following theorem gives a failure of $\Box(\omega_3)$.

Theorem. Suppose that $M_0 \subseteq M_1$ are models of $\mathrm{ZF} + \mathrm{AD}_{\mathbb{R}}$ with the same reals such that, letting $\Gamma_0 = \mathcal{P}(\mathbb{R}) \cap M_0$, the following hold:

- $M_0 = \mathrm{HOD}^{M_1}_{\Gamma_0}$;
- $M_0 \models$ "Θ is regular";
- $\Theta^{M_0} < \Theta^{M_1}$;
- Θ^{M_0} has cofinality at least ω_2 in M_1.

Let $G \subset \mathbb{P}_{\max}$ be M_1-generic, and let $H \subset \mathrm{Add}(\omega_3, 1)^{M_0[G]}$ be $M_1[G]$-generic. Then $\Box(\omega_3)$ fails in $M_0[G][H]$.

Proof. $M_1[G]$ will satisfy $\mathrm{MM}(\mathfrak{c})$, and this still holds after forcing with $\mathrm{Add}(1, \omega_3)^{M_0[G]}$. So $M_1[G][H]$ will see a thread through any $\Box(\Theta_0)$-sequence in $M_0[G][H]$. The thread is unique, however, so a definable name for it exists in M_0. $\Box$

We have improved the hypothesis for this to just $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" plus the assertion that a certain form of Π^2_1-reflection holds for all subsets of ${}^{\omega}\omega$.

Again, the failure of this hypothesis implies that $\square(\omega_3)$ holds in the $\mathbb{P}_{\max}$ extension.

Woodin's proof of $\mathrm{MM}(\mathfrak{c})$ in the $\mathbb{P}_{\max}$ extension of a model of $\mathrm{AD}_{\mathbb{R}}$ + "Θ is regular" uses tree representations for subsets of ${}^{\omega}\omega$.

One plan for obtaining $\mathrm{MM}(\mathfrak{c}^+)$ in a $\mathbb{P}_{\max}$ extension involves developing a theory of Hom^2_{∞} subsets of $\mathcal{P}({}^{\omega}\omega)$.