Topological dynamics and ergodic theory of automorphism groups

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Introduction

I will discuss some aspects of the topological dynamics and ergodic theory of automorphism groups of countable first-order structures and their connections with logic, finite combinatorics and probability theory. This is joint work with Omer Angel and Russell Lyons.

Fraïssé theory

I will first review some basic concepts of Fraïssé theory.

Definition

A class $\mathcal{K}$ of finite structures of the same signature is called a Fraïssé class if it satisfies the following properties:

- (HP) Hereditary property.
- (JEP) Joint embedding property.
- (AP) Amalgamation property.
- It is countable (up to $\cong$).
- It is unbounded.

Examples: finite graphs, finite linear orderings, f.d. vector spaces (over a finite field), finite Boolean algebras, finite rational metric spaces, finite posets, ...

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Joint embedding property (JEP)

$$\begin{tikzcd}
& \mathbf{C} & \\
\mathbf{A} \arrow[ur, dashed] & & \mathbf{B} \arrow[ul, dashed]
\end{tikzcd}$$

Amalgamation property (AP)

$$\begin{tikzcd}
& \mathbf{D} & \\
\mathbf{A} \arrow[ur, dashed] & & \mathbf{B} \arrow[ul, dashed] \\
& \mathbf{C} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

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A countable structure $\boldsymbol{K}$ is a Fraïssé structure if it satisfies the following properties:

- It is infinite.
- It is locally finite.
- It is ultrahomogeneous (i.e., an isomorphism between finite substructures can be extended to an automorphism of the whole structure).

Examples: rational order, random graph, (countably) infinite dimensional vector space (over a finite field), countable atomless Boolean algebra, rational Urysohn space.

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For an infinite structure $\boldsymbol{A}$, its age, denoted by $\mathrm{Age}(\boldsymbol{A})$, is the class of finite structures that can be embedded in $\boldsymbol{A}$.

The age of a Fraïssé structure is a Fraïssé class and Fraïssé showed that conversely one can associate to each Fraïssé class $\mathcal{K}$ a canonical Fraïssé structure $\boldsymbol{K} = \mathrm{Frlim}(\mathcal{K})$, called its Fraïssé limit, which is the unique Fraïssé structure whose age is equal to $\mathcal{K}$. Therefore one has a canonical one-to-one correspondence:

$$\mathcal{K} \mapsto \mathrm{Frlim}(\mathcal{K})$$

between Fraïssé classes and Fraïssé structures whose inverse is:

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Fraïssé theory

Examples

- finite graphs $\rightleftarrows$ random graph
- finite linear orderings $\rightleftarrows$ $\langle \mathbb{Q}, < \rangle$
- f.d. vector spaces $\rightleftarrows$ (countable) infinite-dimensional vector space (over a finite field)
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$\mathrm{Aut}(\boldsymbol{A})$ as a topological group

For a countable structure $\boldsymbol{A}$, we view $\mathrm{Aut}(\boldsymbol{A})$ as a topological group with the pointwise convergence topology. It is not hard to check then that it becomes a Polish group. In fact one can characterize these groups as follows:

Theorem

For any Polish group G, the following are equivalent:

- *G is isomorphic to a closed subgroup of S_∞, the permutation group of $\mathbb{N}$ with the pointwise convergence topology.*
- *G is non-Archimedean, i.e., admits a basis at the identity consisting of open subgroups.*
- *$G \cong \mathrm{Aut}(\boldsymbol{A})$, for a countable structure $\boldsymbol{A}$.*
- *$G \cong \mathrm{Aut}(\boldsymbol{K})$, for a Fraïssé structure $\boldsymbol{K}$.*

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Amenability of $\mathrm{Aut}(\boldsymbol{A})$

We will now consider some aspects of the dynamics of automorphism groups, especially the concept of amenability.

Definition

Let G be a topological group. A G-flow is a continuous action of G on a compact Hausdorff space. A group G is called amenable if every G-flow admits an invariant (Borel probability) measure. It is called extremely amenable if every G-flow admits an invariant point.

Remark

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Extreme amenability and Ramsey theory

In a paper of K-Pestov-Todorcevic (2005) a duality theory was developed that relates the Ramsey theory of Fraïssé classes (sometimes called structural Ramsey theory) to the topological dynamics of the automorphism groups of their Fraïssé limits.

Structural Ramsey theory is a vast generalization of the classical Ramsey theorem to classes of finite structures. It was developed primarily in the 1970's by: Graham, Leeb, Rothschild, Nešetřil-Rödl, Prömel, Voigt, Abramson-Harrington, ...

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A class $\mathcal{K}$ of finite structures (in the same signature) has the Ramsey property (RP) if for any $\boldsymbol{A} \leq \boldsymbol{B}$ in $\mathcal{K}$, and any $n \geq 1$, there is $\boldsymbol{C} \geq \boldsymbol{B}$ in $\mathcal{K}$, such that

$$\boldsymbol{C} \to (\boldsymbol{B})^{\boldsymbol{A}}_n.$$

Examples of classes with Ramsey property:

- finite linear orderings (Ramsey)
- finite Boolean algebras (Graham-Rothschild)
- finite-dimensional vector spaces over a given finite field (Graham-Leeb-Rothschild)
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One of the consequences of the duality theory is the following characterization of extreme amenability of automorphism groups.

Theorem (KPT)

The extremely amenable automorphism groups are exactly the automorphism groups of ordered Fraïssé structures whose age satisfies the Ramsey Property.

Examples

The automorphism groups of the following structures are extremely amenable:

- random ordered graph
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Hrushovski structures

Clearly every extremely amenable group is amenable. There are however many amenable automorphism groups that are not extremely amenable. Many such examples arise in the context of the Hrushovski Property.

Definition

Let $\mathcal{K}$ be a Fraïssé class of finite structures. We say that $\mathcal{K}$ is a Hrushovski class if for any $\boldsymbol{A}$ in $\mathcal{K}$ there is $\boldsymbol{B}$ in $\mathcal{K}$ containing $\boldsymbol{A}$ such that any partial automorphism of $\boldsymbol{A}$ extends to an automorphism of $\boldsymbol{B}$.

Some basic examples of such classes are the pure sets, graphs (Hrushovski), hypergraphs and K_n-free graphs (Herwig), rational valued metric spaces (Solecki), finite dimensional vector spaces over finite fields, etc.

Definition

Let $\mathcal{K}$ be a Fraïssé class of finite structures and $\boldsymbol{K}$ its Fraïssé limit. If $\mathcal{K}$ is a Hrushovski class, then we say that $\boldsymbol{K}$ is a Hrushovski structure.

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This turns out to be a property of automorphism groups:

Proposition

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- $\boldsymbol{K}$ *is a Hrushovski structure.*
- $\mathrm{Aut}(\boldsymbol{K})$ *is compactly approximable, i.e., there is a increasing sequence K_n of compact subgroups whose union is dense in the automorphism group.*

In particular the automorphism group of a Hrushovski structure is amenable. Thus S_∞ and the automorphism groups of the random graph, random n-uniform hypergraph, random K_n-free graph, rational Urysohn space, (countably) infinite-dimensional vector space over a finite field, etc., are amenable (but not extremely amenable).

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Non-amenable groups

At the other end of the spectrum there are also automorphism groups that are not amenable. These include the following:

Theorem (K-Sokić)

The automorphism groups of the random poset and random distributive lattice are not amenable.

Theorem (Malicki)

The automorphism group of the random lattice is not amenable.

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Let G be a topological group and X a G-flow. Consider G-invariant (Borel probability) measures in such a flow.

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Unique ergodicity

Recall that a flow is called minimal if every orbit is dense or equivalently if is has no proper subflows. Every flow contains a minimal subflow.

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Remark: The assumption of minimality is necessary because in general a flow has many minimal subflows which are of course pairwise disjoint. Note also that every uniquely ergodic group is amenable.

Clearly every extremely amenable Polish group is uniquely ergodic and so is every compact Polish group. On the other hand Benjamin Weiss has shown that no infinite countable (discrete) group can be uniquely ergodic and he believes that this extends to Polish locally compact, non-compact groups although this has not been verified in detail.

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Recall that a flow is called minimal if every orbit is dense or equivalently if is has no proper subflows. Every flow contains a minimal subflow.

Definition

Let G be a topological group. We call G uniquely ergodic if every minimal flow admits a unique invariant measure (which must then be ergodic).

Remark: The assumption of minimality is necessary because in general a flow has many minimal subflows which are of course pairwise disjoint. Note also that every uniquely ergodic group is amenable.

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Universal minimal flows

In order to understand better the concept of unique ergodicity we need to discuss first the idea of a universal minimal flow.

A homomorphism between two G-flows X, Y is a continuous G-map $\pi : X \to Y$. If Y is minimal, then π must be onto. An isomorphism is a bijective homomorphism.

Theorem

For any G, there is a minimal G-flow, $M(G)$, called its universal minimal flow with the following property: For any minimal G-flow X, there is a homomorphism $\pi : M(G) \to X$. Moreover $M(G)$ is uniquely determined up to isomorphism by this property.

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The following is a consequence of the Hahn-Banach Theorem.

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Let G be an amenable group. Then G is uniquely ergodic iff $M(G)$ is uniquely ergodic.

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If G is compact, then $M(G) = G$. If G is non-compact but locally compact, then $M(G)$ is extremely complicated, e.g., it is non-metrizable. However, by definition G is extremely amenable iff $M(G)$ trivializes!

This leads to a general problem in topological dynamics:

For a given G, can one explicitly determine $M(G)$ and show that it is metrizable?

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Universal minimal flows of automorphism groups

The duality theory of K-Pestov-Todorcevic provides tools for computing the universal minimal flows of automorphism groups of Fraïssé structures. We will discuss this next.

Order expansions of Fraïssé classes

Consider a Fraïssé class $\mathcal{K}$. A Fraïssé class $\mathcal{K}^*$ is an order expansion of $\mathcal{K}$ if $\mathcal{K}^*$ consists of structures of the form $\langle \boldsymbol{A}, < \rangle$, where $\boldsymbol{A} \in \mathcal{K}$ and $<$ is a linear ordering on (the universe of) $\boldsymbol{A}$. In this case, if $\langle \boldsymbol{A}, < \rangle \in \mathcal{K}^*$ we call $<$ a $\mathcal{K}^*$-admissible ordering on $\boldsymbol{A}$. The order expansion $\mathcal{K}^*$ of $\mathcal{K}$ is *reasonable* if for every $\boldsymbol{A}, \boldsymbol{B} \in \mathcal{K}$, with $\boldsymbol{A} \subseteq \boldsymbol{B}$ and any $\mathcal{K}^*$-admissible ordering $<$ on $\boldsymbol{A}$, there is a $\mathcal{K}^*$-admissible ordering $<'$ on $\boldsymbol{B}$ such that $< \subseteq <'$.

Order expansions of Fraïssé classes

If $\mathcal{K}$ is a Fraïssé class with $\boldsymbol{K} = \mathrm{Flim}(\mathcal{K})$ and $\mathcal{K}^*$ is a reasonable, order expansion of $\mathcal{K}$, we denote by $X_{\mathcal{K}^*}$ the space of linear orderings $<$ on $\boldsymbol{K}$ such that for any finite substructure $\boldsymbol{A}$ of $\boldsymbol{K}$, $< |A$ is $\mathcal{K}^*$-admissible on $\boldsymbol{A}$. We call these the $\mathcal{K}^*$-admissible orderings on $\boldsymbol{K}$. They form a compact, metrizable, non-empty subspace of 2^{K^2} (with the product topology) on which the group $G = \mathrm{Aut}(\boldsymbol{K})$ acts continuously, thus $X_{\mathcal{K}^*}$ is a G-flow.

Order expansions of Fraïssé classes

Examples

- $\mathcal{K}$ = finite graphs, $\boldsymbol{K} = \boldsymbol{R}$; $\mathcal{K}^*$ = finite ordered graphs. Then $X_{\mathcal{K}*}$ is the space of all linear orderings of the random graph.
- $\mathcal{K}$ = finite sets, $\boldsymbol{K} = \langle \mathbb{N} \rangle$; $\mathcal{K}^*$ = finite orderings. Then $X_{\mathcal{K}*}$ is the space of all linear orderings on $\mathbb{N}$.
- $\mathcal{K}$ = f.d. vector spaces over a fixed finite field, $\boldsymbol{K} = \boldsymbol{V}_\infty$; $\mathcal{K}^*$ = lex. ordered f.d. vector spaces. Then $X_{\mathcal{K}^*}$ is the space of all "lex. orderings" on $\boldsymbol{V}_\infty$.
- $\mathcal{K}$ = finite posets, $\boldsymbol{K} = \boldsymbol{P}$; $\mathcal{K}^*$ = finite posets with linear extensions. Then $X_{\mathcal{K}^*}$ is the space of all linear extensions of the random poset.

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Order expansions of Fraïssé classes

Beyond the Ramsey Property, there is an additional property of classes of finite structures that was introduced by Nešetřil and Rödl in the 1970's and played an important role in the structural Ramsey theory.

Definition

If $\mathcal{K}^*$ is an order expansion of $\mathcal{K}$, we say that $\mathcal{K}^*$ satisfies the ordering property (OP) if for every $\boldsymbol{A} \in \mathcal{K}$, there is $\boldsymbol{B} \in \mathcal{K}$ such that for every $\mathcal{K}^*$-admissible orderings $<$ on $\boldsymbol{A}$ and $<'$ on $\boldsymbol{B}$, $\langle \boldsymbol{A}, < \rangle$ can be embedded in $\langle \boldsymbol{B}, <' \rangle$.

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In all the examples of the previous page we have the ordering property.

Calculation of universal minimal flows

Theorem (KPT)

Let $\mathcal{K}$ be a Fraïssé class and $\mathcal{K}^*$ a reasonable order expansion of $\mathcal{K}$. Then if G is the automorphism group of the Fraïssé limit of $\mathcal{K}$ the following are equivalent:

- *$X_{\mathcal{K}^*}$ is the universal minimal flow of the automorphism group of G.*
- *$\mathcal{K}^*$ has the Ramsey Property and the Ordering Property.*

Calculation of universal minimal flows

Examples

- $\mathcal{K}$ = finite graphs, $\boldsymbol{K} = \boldsymbol{R}$; $\mathcal{K}^*$ = finite ordered graphs. Then the space of all linear orderings of the random graph is the UMF of its automorphism group.
- $\mathcal{K}$ = finite sets, $\boldsymbol{K} = \langle \mathbb{N} \rangle$; $\mathcal{K}^*$ = finite orderings. Then the space of all linear orderings on $\mathbb{N}$ is the UMF of S_∞(Glasner-Weiss).
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Unique ergodicity revisited

Let $\mathcal{K}$ be a Fraïssé class and $\mathcal{K}^*$ a reasonable order expansion of $\mathcal{K}$ that has the Ramsey Property and the Ordering Property. We will say then that $\mathcal{K}^*$ is a companion of $\mathcal{K}$. It was shown in the paper of KPT that such a companion, when it exists, is essentially unique.

Thus we have seen that when $\mathcal{K}$ has a companion class $\mathcal{K}^*$, and this happens for many important examples, then the UMF of the automorphism group G of its Fraïssé limit is the compact, metrizable space $X_{\mathcal{K}^*}$. Thus the unique ergodicity of G is equivalent to the unique ergodicity of $X_{\mathcal{K}^*}$. This can then be seen to be equivalent to the following probabilistic notion.

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Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$. A random, consistent $\mathcal{K}^*$-admissible ordering is a map that assigns to each structure $\boldsymbol{A} \in \mathcal{K}$ a probability measure $\mu_{\boldsymbol{A}}$ on the (finite) space of $\mathcal{K}^*$-admissible orderings on $\boldsymbol{A}$, which is isomorphism invariant and has the property that if $\boldsymbol{A} \subseteq \boldsymbol{B}$, then $\mu_{\boldsymbol{B}}$ projects by the restriction map to $\mu_{\boldsymbol{A}}$.

We now have:

Proposition (AKL)

Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$. Then amenability of the automorphism group G of the Fraïssé limit of $\mathcal{K}$ is equivalent to the existence of a random, consistent $\mathcal{K}^*$-admissible ordering and unique ergodicity of G is equivalent to the uniqueness of a random, consistent $\mathcal{K}^*$-admissible ordering.

Example: graphs

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Unique ergodicity as a quantitative version of the Ordering Property

Interestingly it turns out that unique ergodicity fits well in the framework of the duality theory of KPT (which originally was developed in the context of topological dynamics). In many cases it can simply be viewed as a quantitative version of the Ordering Property.

Definition (AKL)

Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$. We say that $\mathcal{K}^*$ satisfies the Quantitative Ordering Property (QOP) if the following holds:

There is an isomorphism invariant map that assigns to each structure $\boldsymbol{A}^* = \langle \boldsymbol{A}, < \rangle \in \mathcal{K}^*$ a real number $\rho(\boldsymbol{A}^*)$ in $(0, 1]$ such that for every $\boldsymbol{A} \in \mathcal{K}$ and each $\epsilon > 0$, there is a $\boldsymbol{B} \in \mathcal{K}$ and a nonempty set of embeddings $E(\boldsymbol{A}, \boldsymbol{B})$ of $\boldsymbol{A}$ into $\boldsymbol{B}$ with the property that for each $\mathcal{K}^*$-admissible ordering $<$ of $\boldsymbol{A}$ and each $\mathcal{K}^*$-admissible ordering $<'$ of $\boldsymbol{B}$ the proportion of embeddings in $E(\boldsymbol{A}, \boldsymbol{B})$ that preserve $<, <'$ is equal to $\rho(\langle \boldsymbol{A}, < \rangle)$, within ϵ.

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Unique ergodicity as a quantitative version of the Ordering Property

Theorem (AKL)

Let $\mathcal{K}^*$ be a companion of $\mathcal{K}$, let G be the automorphism group of the Fraïssé limit of $\mathcal{K}$ and assume that G is amenable. Then QOP implies the unique ergodicity of G. Moreover, if $\mathcal{K}$ is a Hrushovski class, QOP is equivalent to the unique ergodicity of G.

Unique ergodicity as a quantitative version of the Ordering Property

Theorem (AKL)

The QOP holds for the following Fraïssé classes :

- *ordered graphs*
- *ordered K_n-free graphs*
- *ordered n-uniform hypergraphs*
- *rational ordered metric spaces*

In particular, in all these cases there is a unique random, consistent ordering, namely the uniform one.

The proofs use probabilistic arguments (deviation or concentration inequalities).

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For example, if $\mathcal{K}$ is the class of finite graphs, we establish QOP by showing that for any finite graph $\boldsymbol{A}$ with n vertices and $\epsilon > 0$, there is a graph $\boldsymbol{B}$, containing a copy of $\boldsymbol{A}$, such that given any orderings $<$ on $\boldsymbol{A}$ and $<'$ on $\boldsymbol{B}$, the proportion of all embeddings of $\boldsymbol{A}$ into $\boldsymbol{B}$ that preserve the orderings $<, <'$ is, up to ϵ, equal to $1/n!$.

Proving unique ergodicity

Theorem (AKL, except for S_∞)

The following automorphism groups are uniquely ergodic:

- S_∞ *(Glasner-Weiss)*
- *The isometry group of the Baire space*
- *The general linear group of the (countably) infinite-dimensional vector space over a finite field*
- *The automorphism group of the random graph*
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Unique Ergodicity Problem

In fact I do not know any counterexample to the following problem:

Problem (Unique Ergodicity Problem)

Let G be an amenable automorphism group of a countable structure with a metrizable universal minimal flow. Is G uniquely ergodic?

Next I will consider the problem of determining the support of the unique measure (in the uniquely ergodic case).

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Generic Orbit Problem

Definition

Let X be a G-flow. A comeager orbit of this action is called a generic orbit. (It is of course unique if it exists.) We say that G has the generic orbit property if every minimal G-flow has a generic orbit.

It turns out that G has the generic orbit property iff its universal minimal flow has a generic orbit. Using this one can show:

Theorem (AKL)

Let $\mathcal{K}$ be a Fraïssé class that admits a companion $\mathcal{K}^*$. Then the automorphism group of the Fraïssé limit of $\mathcal{K}$ has the generic orbit property.

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Again it can be shown that no non-compact locally compact Polish group can satisfy the generic orbit property.

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It turns out that G has the generic orbit property iff its universal minimal flow has a generic orbit. Using this one can show:

Theorem (AKL)

Let $\mathcal{K}$ be a Fraïssé class that admits a companion $\mathcal{K}^*$. Then the automorphism group of the Fraïssé limit of $\mathcal{K}$ has the generic orbit property.

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Again it can be shown that no non-compact locally compact Polish group can satisfy the generic orbit property.

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Unique Ergodicity - Generic Orbit Problem

Merging the two previous problems, we can ask whether an even stronger property is true, which specifies the support of the unique invariant measure.

Problem (Unique Ergodicity - Generic Orbit Problem)

Let G be an amenable automorphism group of a countable structure with a metrizable universal minimal flow. Is G uniquely ergodic, has the generic orbit property and moreover in every minimal G-flow the unique invariant measure is supported by the generic orbit orbit?

A positive answer has been obtained in many cases, e.g., S_∞ (Glasner-Weiss); the automorphism group of the random graph, random n-uniform hypergraph, random K_n-free graph, rational Urysohn space, etc. (AKL). Note that when Unique Ergodicity – Generic Orbit holds one has the interesting phenomenon that measure and category agree instead of being, as usual, orthogonal.

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Unique Ergodicity - Generic Point Problem

I will finish with two very recent interesting (and contrasting) answers to the Unique Ergodicity-Generic Orbit Problem.

Andrew Zucker (Caltech undergraduate) has surprisingly shown that for the general linear group of the (countably) infinite-dimensional vector space over a finite field, which is uniquely ergodic and has the generic orbit property, the unique invariant measure of its universal minimal flow does not live on the generic orbit! Thus the Unique Ergodicity - Generic Orbit Problem has a negative answer in general.

On the other hand, András Pongrácz subsequently showed that when a Fraïssé class $\mathcal{K}$ admits an order-forgetful companion $\mathcal{K}^*$ and the automorphism group of its Fraïssé limit is amenable, then it is uniquely ergodic, has the generic orbit property and the unique invariant measure lives on the generic orbit, provided that the language of $\mathcal{K}$ is relational. Thus the Unique Ergodicity - Generic Orbit Problem has a positive answer in this situation. Vector spaces satisfy all these properties but the language is not relational!

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