A survey of the model theory of tracial von Neumann algebras

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Harvard-MIT Logic Seminar
November 11, 2013

Von Neumann algebras

- Throughout, H is a complex Hilbert space and $\mathcal{B}(H)$ is the $*$-algebra of bounded operators on H.
- For $X \subseteq \mathcal{B}(H)$, we set $X' := \{T \in \mathcal{B}(H) \ : \ TS = ST \text{ for all } S \in X\}$.
- Observe that:
  - X' is a unital subalgebra of $\mathcal{B}(H)$ that is closed under $*$ if X is.
  - $X \subseteq X''$.

Definition

A *von Neumann algebra* is a $*$-subalgebra M of $\mathcal{B}(H)$ such that $M = M''$.

Equivalently, M is a $*$-subalgebra of $\mathcal{B}(H)$ that is closed in either the *weak operator topology* or the *strong operator topology*.

Examples of vNas

Example

$\mathcal{B}(H)$ is a von Neumann algebra.

Example

Suppose that (X, μ) is a finite measure space. Then $L^\infty(X, \mu)$ acts on the Hilbert space $L^2(X, \mu)$ by left multiplication, yielding an embedding

$$L^\infty(X, \mu) \hookrightarrow \mathcal{B}(L^2(X, \mu)),$$

the image of which is a von Neumann algebra. (Actually, all abelian von Neumann algebras are isomorphic to some $L^\infty(X, \mu)$, whence von Neumann algebra theory is sometimes dubbed "noncommutative measure theory.")

Group von Neumann algebras

Example

Suppose that G is a locally compact group and $\alpha : G \to \mathcal{B}(H)$ is a unitary group representation. Then the *group von Neumann algebra of* α is $\alpha(G)''$. (Understanding $\alpha(G)''$ is tantamount to understanding the invariant subspaces of α.)

In the important special case that $\alpha : G \to \mathcal{B}(L^2(G))$ (where G is equipped with its haar measure) is given by left translations

$$\alpha(g)(f)(x) := f(g^{-1}x),$$

we call $\alpha(G)''$ the *group von Neumann algebra of G* and denote it by $L(G)$.

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$\mathcal{R}$

Example

Let M_2 denote the set of 2×2 matrices with entries from $\mathbb{C}$. We consider the canonical embeddings

$$M_2 \hookrightarrow M_2 \otimes M_2 \hookrightarrow M_2 \otimes M_2 \otimes M_2 \hookrightarrow \cdots$$

and set $M := \bigcup_{n=1}^{\infty} \bigotimes_n M_2$.

- The normalized traces on $\bigotimes_n M_2$ form a cohesive family of traces, yielding a trace $\mathrm{tr} : M \to \mathbb{C}$.
- We can define an inner product on M by $\langle A, B \rangle := \mathrm{tr}(B^* A)$. Set H to be the completion of M with respect to this inner product.
- M acts on H by left multiplication, whence we can view M as a $*$-subalgebra of $\mathcal{B}(H)$. We set $\mathcal{R}$ to be the von Neumann algebra generated by M. $\mathcal{R}$ is called *the hyperfinite II_1 factor*.

Tracial von Neumann algebras

Suppose that A is a von Neumann algebra. A *tracial state* (or just *trace*) on A is a linear functional $\tau : A \to \mathbb{C}$ satisfying:

- $\tau(1) = 1$;
- $\tau(x^*x) \geq 0$ for all $x \in A$;
- $\tau(xy) = \tau(yx)$ for all $x, y \in A$.

A *tracial von Neumann algebra* is a pair (A, τ), where A is a von Neumann algebra and τ is a trace on A.

In the case that τ is also *faithful*, meaning that $\tau(x^*x) = 0 \Rightarrow x = 0$, the function $\langle x, y \rangle_\tau := \tau(y^*x)$ is an inner product on A, yielding the so-called *2-norm* $\| \cdot \|_2$ on A. The associated metric is complete on any bounded subset of A.

(A, τ) is called *separable* if the metric associated to the 2-norm is separable.

II_1 Factors

A von Neumann algebra A is said to be a *factor* if $A \cap A' = \mathbb{C} \cdot 1$.

Fact

If A is a von Neumann algebra, then $A \cong \int_X^{\oplus} A_x$ (a *direct integral*) where each A_x is a factor.

A factor is said to be of type II_1 if it is infinite-dimensional and admits a trace.

Fact

A II_1 factor admits a unique weakly continuous trace, which is automatically faithful.

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Examples-revisited

- $\mathcal{B}(H)$ is a factor. If $\dim(H) < \infty$, then $\mathcal{B}(H)$ admits a trace, but is not a II_1 factor. If $\dim(H) = \infty$, then $\mathcal{B}(H)$ admits no trace. Thus, $\mathcal{B}(H)$ is never a II_1 factor.
- $L^\infty(X, \mu)$ admits a trace $f \mapsto \int_X f\, d\mu$ but is not a factor.
- If G is a countable group that is *ICC*, namely all conjugacy classes (other than $\{1\}$) are infinite, then $L(G)$ is a II_1 factor; the trace is given by $T \mapsto \langle T\delta_e, \delta_e \rangle$. In particular, if $n \geq 2$, then $L(\mathbb{F}_n)$ is a II_1 factor.
- $\mathcal{R}$ is a II_1 factor; the trace $\mathrm{tr} : \bigcup_n \bigotimes_n M_2 \to \mathbb{C}$ extends uniquely to the completion. Moreover, $\mathcal{R}$ embeds into any II_1 factor.

Continuous model theory

There is a natural language of continuous logic in which to discuss tracial von Neumann algebras.

Theorem (Farah-Hart-Sherman)

- *The class of tracial von Neumann algebras is universally axiomatizable.*
- *The class of II_1 factors is $\forall\exists$-axiomatizable.*

Ultrapowers of von Neumann algebras

Suppose that (A, τ) is a tracial von Neumann algebra and $\mathcal{U}$ is a nonprincipal ultrafilter on $\mathbb{N}$. We set

$$\ell^\infty(A) := \{(a_n) \in A^{\mathbb{N}} \ : \ \|a_n\| \text{ is bounded}\}.$$

Unfortunately, if we quotient this out by the ideal

$$\{(a_n) \in A^{\mathbb{N}} \ : \ \lim_{\mathcal{U}} \|a_n\| = 0\},$$

the resulting quotient is usually never a von Neumann algebra. Rather, we have to quotient out by the smaller ideal

$$\{(a_n) \in A^{\mathbb{N}} \ : \ \lim_{\mathcal{U}} \|a_n\|_2 = 0\},$$

yielding the *tracial ultrapower* $A^{\mathcal{U}}$ of A. (This is the continuous logic ultrapower, so the result is once again a von Neumann algebra.)

Property (Γ)

- For a while, Murray and von Neumann could not figure out whether $\mathcal{R}$ and $L(\mathbb{F}_2)$ were isomorphic or not.
- They finally figured out a property that distinguished them.
- Say that a II_1 factor M has *property* (Γ) if, for any finite $F \subseteq M$ and any $\epsilon > 0$, there is a trace 0 unitary u such that $\|ux - xu\|_2 < \epsilon$ for all $x \in F$.
- $\mathcal{R}$ has (Γ) (easy) while $L(\mathbb{F}_2)$ does not (M-vN), so $\mathcal{R} \not\cong L(\mathbb{F}_2)$.
- Equivalently, M has property (Γ) if and only if $M' \cap M^{\mathcal{U}} \neq \mathbb{C}$ (M has *nontrivial relative commutant*).
- Note that (Γ) is *axiomatizable* by the sentences

$$\sigma_n := \sup_{\vec{x}} \inf_y \left(\|yy^* - 1\|_2 + |tr(y)| + \sum \|[x_i, y]\|_2 \right).$$

- Therefore, $\mathcal{R}$ and $L(\mathbb{F}_2)$ are not elementarily equivalent!

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McDuff factors

Theorem (McDuff)

For any separable II_1 factor M, $M' \cap M^{\mathcal{U}}$ takes one of the following forms:

1. $\mathbb{C}$;
2. *an abelian von Neumann algebra* $\neq \mathbb{C}$;
3. *a* II_1 *factor.*

We call a II_1 factor *McDuff* if case (3) holds. This is equivalent to $M \otimes \mathcal{R} \cong M$.

- In case (2), $M' \cap M^{\mathcal{U}}$ is independent of the choice of $\mathcal{U}$.
- McDuff asked whether $M' \cap M^{\mathcal{U}}$ is independent of the choice of $\mathcal{U}$ in case M is McDuff.

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Assuming (CH)

- If we assume the Continuum Hypothesis, then $M^{\mathcal{U}}$ and $M^{\mathcal{V}}$ are saturated models of the same theory, so they are isomorphic (even over M).
- Consequently, $M' \cap M^{\mathcal{U}} \cong M' \cap M^{\mathcal{V}}$.

Assuming ¬(CH)

- Recall that a (continuous) theory T is said to be *unstable* if there is a formula $\varphi(x; y)$ and a sequence (a_i) from a model M of T such that $\varphi(a_i; a_j) = 0$ if $i < j$ and $\varphi(a_j; a_i) = 1$ if $i \geq j$.
- Using the fact that II_1 factors embed arbitrarily large matrix algebras, one can prove that every II_1 factor is unstable.
- Given an ultrafilter $\mathcal{U}$ on $\mathbb{N}$, let $\kappa(\mathcal{U})$ denote the coinitiality of $\mathbb{N}^{\nearrow\mathbb{N}}/\mathcal{U}$.
- Using the instability of M, one can "encode" $\kappa(\mathcal{U})$ into $M^{\mathcal{U}}$, so $M^{\mathcal{U}} \cong M^{\mathcal{V}} \Rightarrow \kappa(\mathcal{U}) = \kappa(\mathcal{V})$.
- Now use the fact (due to Dow and Shelah independently) that ¬(CH) implies that there exist $\mathcal{U}$ and $\mathcal{V}$ such that $\kappa(\mathcal{U}) = \aleph_1$ and $\kappa(\mathcal{V}) = \aleph_2$.

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Assuming $\neg$(CH)

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Assuming $\neg$(CH)

Theorem (FHS)

Assuming $\neg$(CH), for any II_1 factor M there exist two nonisomorphic ultrapowers of M.

By altering the definition of instability of a relative commutant, the same ideas can be used to prove that if M is McDuff, then M has nonisomorphic relative commutants $M' \cap M^{\mathcal{U}}$.

Also, the same arguments show that there are nonisomorphic matrix ultraproducts $\prod_{\mathcal{U}} M_n(\mathbb{C})$.

Connes' Embedding Problem

- In 1976, Connes proved that $L(\mathbb{F}_2)$ is $\mathcal{R}^\omega$-embeddable.
- He then remarked "Apparently such an embedding ought to exist for all II_1 factors..."
- This remark is now known as the *Connes Embedding Problem* (CEP) and is *the* central question in II_1 factor theory. It has zillions of equivalent reformulations.
- For example, it is known that $L(G)$ is $\mathcal{R}^\omega$-embeddable if and only if G is hyperlinear. So settling the CEP for group von Neumann algebras would settle the question of whether or not all groups are hyperlinear (a serious question in group theory).

CEP (continued)

- Model theory 101: CEP is the statement: for any II_1 factor M, $\mathrm{Th}_\forall(M) = \mathrm{Th}_\forall(\mathcal{R})$.
- Call a separable II_1 factor A *locally universal* if every separable II_1 factor is A^ω-embeddable. (So CEP asks whether or not $\mathcal{R}$ is locally universal.)

Theorem ("Poor Man's CEP"-FHS)

There is a locally universal II_1 factor.

Proof.

Amalgamate to your heart's desire. □

CEP (continued)

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Could $\mathcal{R}$ be rosy?

- OK, so no theory of II_1 factors is stable.
- Nor are they (model-theoretically) simple. (Folklore-Hart)
- Could they admit some “nice” notion of independence (and thus be rosy)?
- A natural candidate exists using conditional expectation in analogy with ordinary probability theory.
- When trying to verify the axioms for being an independence relation, we realized it would be useful to know if $\mathrm{Th}(\mathcal{R})$ had QE.

Bad News

Theorem (G., Hart, Sinclair)

Th($\mathcal{R}$) *does not have QE.*

- The proof uses nontrivial results of Nate Brown concerning embeddings of algebras into $\mathcal{R}^\omega$.
- The same proof actually shows that Th(S) does not have QE if S is locally universal and McDuff.

A reminder on model companions

- Recall that a theory T is *model complete* if any embedding between models of T is elementary.
- If T' is a theory, then a model complete theory T is a *model companion* for T' if any model of T' embeds in a model of T and vic-versa (that is, if $T'_\forall = T_\forall$). A theory can have at most one model companion.
- If T' is universal, then T' has a model companion T if and only if the class of its existentially closed structures is elementary; in this case T is their theory.

Relevant to this discussion: any ec vNa is a McDuff II_1 factor. Why?

- Any vNa M embeds into a II_1 factor: $M \subseteq M * L(\mathbb{Z})$.
- Any II_1 factor N embeds into a McDuff II_1 factor: $N \subseteq N \otimes \mathcal{R}$.
- Being a McDuff II_1 factor is $\forall\exists$-axiomatizable (FHS).

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Really bad news

Theorem (G., Hart, Sinclair-2012)

T_{vNa} *does not have a model companion.*

Theorem (Farah, G., Hart-2013)

$\mathrm{Th}_{\forall}(\mathcal{R})$ *does not have a model companion.*

T_{vNa} does not have a model companion

- Suppose that that the theory of vNas did have a model companion, say T. Let $\mathcal{S}$ be a separable model of T. We already know that $\mathcal{S}$ is McDuff.
- It can be shown that S is also locally universal.
- It follows that T does not admit quantifier-elimination.
- But a basic model theoretic fact says that a model-companion of a theory with the amalgamation property has quantifier-elimination, a contradiction.

$\mathrm{Th}_\forall(\mathcal{R})$ does not have a model companion

- The only possible model of $\mathrm{Th}_\forall(\mathcal{R})$ that could be model-complete (even $\forall\exists$-axiomatizable) is $\mathcal{R}$. (Draw crude diagram on board!)
- This already tells us that CEP implies that there is no model-complete theory of II_1 factors.
- In fact, a weaker version of CEP, namely that $\mathrm{Th}_\forall(\mathcal{R})$ having the amalgamation property, tells us that $\mathrm{Th}(\mathcal{R})$ is not model complete.
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- The proof uses an analysis of automorphisms of e.c. models of $\mathrm{Th}_\forall(\mathcal{R})$ together with a nontrivial theorem of Kenley Jung characterizing $\mathcal{R}$ as the only $\mathcal{R}^\omega$-embeddable II_1 factor for which all embeddings into $\mathcal{R}^\omega$ are unitarily conjugate. (We showed that any nonstandard model of $\mathrm{Th}(\mathcal{R})$ also had this property.)

$\mathrm{Th}_\forall(\mathcal{R})$ does not have a model companion

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A computability-theoretic consequence of CEP

- Suppose that M is a vNa. We say that $\mathrm{Th}_\forall(M)$ is *computable* if there is an algorithm such that, upon inputs universal sentence σ and (dyadic rational) $\epsilon > 0$, returns an interval $I \subseteq \mathbb{R}$ of length $\leq \epsilon$ such that $\sigma^M \in I$.

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A converse

Theorem (G., Hart)

CEP is equivalent to: for every vNa M that contains $\mathcal{R}$, we have $\mathrm{Th}_\forall(M)$ is computable.

Proof.

If CEP fails, then there is a universal sentence σ and a II_1 factor M such that $\sigma^{\mathcal{R}} = 0$ and $\sigma^M > 0$. For $t \in [0,1]$, let $M_t := t\mathcal{R} + (1-t)M$. The map $t \mapsto \sigma^{M_t} : [0,1] \to [0,1]$ is continuous and has range bigger than a point, whence has size continuum. But there can only be countably many algorithms for computing universal theories. □

More computability-theoretic consequences of CEP

- Suppose that (M, X) is a separable II_1 factor with countable dense set $X = (x_n)$.
- Certainly, for any universal sentence $\sigma = \sup_x \varphi(x)$ and any $\epsilon > 0$, there is $n \in \omega$ such that $\varphi^M(x_n) > \sigma^M - \epsilon$; call such n good for (M, X, σ, ϵ).

Theorem (G., Hart)

Assume CEP. Then there is a computable partial function $f : \mathbb{N} \times \mathbb{N} \times \mathbb{D}^{>0} \rightharpoonup \mathbb{N}$ such that, if m is the Gödel code for a universal sentence σ and k is the Gödel code for a recursively presented II_1 factor (M, X), then there is $n \leq f(m, k, \epsilon)$ good for (M, X, σ, ϵ).

Is this evidence that CEP is false?

Things we're working on now

- Finding many theories of II_1 factors (we know 3 right now)
- Other possible quantifier-simplifications (via augmenting the language)
- Complexity of axiomatizations
- The relation between $\equiv$ and $\otimes, *, \rtimes_\alpha$
- Other properties of ec models
- Theories of pairs of algebras
- Changing the category/language

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