Logic and operator algebras

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Complex Hilbert space ℓ_2, C*-algebras

$$\begin{aligned} \ell_2 &= \{a \in \mathbb{C}^{\mathbb{N}} : \textstyle\sum_n |a_n|^2 < \infty\}. \\ \|a\| &= (\textstyle\sum |a_n|^2)^{1/2}. \\ (\mathcal{B}(\ell_2), +, \cdot, ^*, \|\cdot\|): &\quad \text{the algebra of bounded linear operators on } \ell_2. \end{aligned}$$

Definition

C-algebra* is a Banach algebra with involution which is *-isomorphic to a norm-closed self-adjoint subalgebra of $\mathcal{B}(\ell_2)$.

Examples

1. $\mathcal{B}(\ell_2)$.
2. $M_n(\mathbb{C})$, *for* $n \in \mathbb{N}$.
3. $C(X) = \{f : X \to \mathbb{C} | f \text{ is continuous}\}$ *for any compact Hausdorff space* X.

Inductive limits and the CAR algebra

$$M_n(\mathbb{C}) \hookrightarrow M_{2n}(\mathbb{C})$$

via

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}.$$

$$M_{2^\infty}(\mathbb{C}) = \varinjlim M_{2^n}(\mathbb{C}) = \bigotimes_{n \in \mathbb{N}} M_2(\mathbb{C}).$$

(where $\varinjlim$ means 'completion of the direct limit.')

UHF (uniformly hyperfinite) algebras

Lemma

1. *$M_n(\mathbb{C})$ unitally embeds into $M_k(\mathbb{C})$ iff n divides k.*
2. *All unital emeddings of $M_n(\mathbb{C})$ into $M_k(\mathbb{C})$ are unitarily conjugate.*
3. *$M_n(\mathbb{C}) \otimes M_k(\mathbb{C}) \cong M_{nk}(\mathbb{C})$.*

Theorem (Glimm, 1960)

UHF algebras $\bigotimes_i M_{n(i)}(\mathbb{C})$ and $\bigotimes_i M_{m(i)}(\mathbb{C})$ are isomorphic iff there is an 'obvious' isomorphism. In particular,

$$M_{2^\infty} \ncong M_{3^\infty}.$$

Elliott invariant

The Elliott invariant, Ell, is a functor from the category of C*-algebras into a category of K-theoretic invariants.

Lemma

Let A be a UHF algebra, and let

$$\Gamma := \{m/n : m \in \mathbb{Z} : M_n(\mathbb{C}) \text{ embeds unitally into } A\}.$$

Then $\mathrm{Ell}(A) = (\Gamma, 1, \Gamma \cap \mathbb{Q}^+)$.

One of many definitions of nuclearity for C*-algebras

A C*-algebra is *nuclear* if for every C*-algebra B there is a unique C*-algebra norm on $A \otimes B$.

Elliott's program

Conjecture (Elliott, 1990)

Infinite-dimensional, simple, nuclear, unital, separable algebras are classified by Ell. *Classification is strongly functorial:*

$$\begin{array}{ccc} A & \longrightarrow & \mathrm{Ell}(A) \\ \varphi \big\downarrow & & \big\downarrow f = \mathrm{Ell}(\varphi) \\ B & \longrightarrow & \mathrm{Ell}(B) \end{array}$$

For every morphism $f\colon \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ there exist morphism $\varphi\colon A \to B$ such that $\mathrm{Ell}(\varphi) = f$.

Remarkably, this is true for a large class of C*-algebras

(Elliott, Rørdam, Kirchberg–Phillips, Elliott–Gong–Li, Winter,...)

Elliott's program: Counterexamples

Theorem (Jiang–Su, 2000)

There exists an ∞-dimensional simple, nuclear, unital, separable algebra $\mathcal{Z}$ such that $\mathrm{Ell}(\mathcal{Z}) = \mathrm{Ell}(\mathbb{C})$.

Theorem (Toms, 2008)

There are ∞-dimensional simple, nuclear, unital, separable algebras A and B such that $\mathrm{Ell}(A) = \mathrm{Ell}(B)$, moreover $F(A) = F(B)$ for every continuous homotopy-invariant functor F, but $A \ncong B$.

$$A \cong B \otimes \mathcal{Z}.$$

Abstract classification

Almost every classical classification problem (not of 'obviously set-theoretic nature') in mathematics is concerned with definable equivalence relations on a Polish (separable, completely metrizable) space.

If E and F are equivalence relations on Polish spaces, then $E \leq_B F$ if there exists Borel-measurable f such that

$$x\,E\,y \Leftrightarrow f(x)\,F\,f(y).$$

Hjorth developed a tool for proving that an equivalence relation is not classifiable by the isomorphism of countable structures.

Theorem (F.–Toms–Törnquist, 2013)

Isomorphism relation of simple, nuclear, unital, separable algebras is not Borel-reducible to the isomorphism relation of countable structures.

Theorem (F.–Toms–Törnquist, Gao–Kechris, Elliott–F.–Paulsen–Rosendal–Toms–Törnquist, Sabok)

The following isomorphism relations are Borel-equireducible.

1. *Isomorphism relation of arbitrary separable C*-algebras.*
2. *Isomorphism relation of Elliott–classifiable simple, nuclear, unital, separable algebras.*
3. *Isomorphism relation of Elliott invariants.*
4. *The $\leq_B$-maximal orbit equivalence relation of a Polish group action.*

None of these relations is Borel-reducible to the isomorphism relation of countable structures.

Logic of metric structures

Ben Yaacov–Berenstein–Henson–Usvyatsov, 2008.

(Bounded) *metric structure* has a complete metric space (M, d) as its domain.

All functions and predicates are uniformly continuous.

Uniform continuity moduli are a part of the language.

classical logic	logic of metric structures
$\top, \bot$	$[0, \infty)$
$\wedge, \vee, \leftrightarrow$	continuous $f\colon \mathbb{R}^2 \to [0, \infty)$
$\forall, \exists$	$\sup_x$, $\inf_x$.
$\mathrm{Th}(A)$	$\{\varphi \mid \varphi^A = 0\}$.

Lemma

Every formula has a uniform continuity modulus.

Completeness, compactness, ultraproducts, Łos's theorem, Lindström-type theorems, EF-games,...
everything works out as one would expect.

Theorem (Elliott–F.–Paulsen–Rosendal–Toms–Törnquist, 2012)

For any separable metric language L, the isomorphism of separable L-models is Borel-reducible to an orbit equivalence relation of a continuous action of a Polish group $\mathrm{Iso}(\mathbb{U})$ *on a Polish space.*

$$\frac{\text{Classical logic}}{S_\infty} = \frac{\text{Logic of metric structures}}{\mathrm{Iso}(\mathbb{U})}$$

Logic of metric structures was adapted to operator algebras by F.–Hart–Sherman.

Uniform continuity moduli of functions and predicates are attached to bounded balls, and quantification is allowed only over the bounded balls.

In general, *sorts* over which one can quantify correspond to functors from the category of models into metric spaces with uniformly continuous functions that commute with ultraproducts.

Examples

1. $(\sup_{\|x\|\leq 1, \|y\|\leq 1} \|xy - yx\|)^A = 0$ *iff A is abelian.*
2. $(\inf_{\|x\|\leq 1} |1 - \|x\|| + \|x^2\|)^A = 0$ *iff A is non-abelian.*
3. *Being nuclear is not axiomatizable.*
4. *Being simple is not axiomatizable.*

Counterexamples to Elliott's conjecture revisited

Theorem (Toms, 2009)

There are ∞*-dimensional simple, nuclear, unital, separable algebras* A_r *for* $r \in [0,1]$ *such that* $\mathrm{Ell}(A_s) = \mathrm{Ell}(A_r)$*, but* $A_r \not\cong A_s$ *if* $r \neq s$*.*

Theorem (L. Robert)

No two of these algebras are elementarily equivalent.

Question (Strong Conjecture)

For simple, nuclear, unital, separable A *and* B*, do* $\mathrm{Ell}(A) = \mathrm{Ell}(B)$ *and* $\mathrm{Th}(A) = \mathrm{Th}(B)$ *together imply* $A \cong B$*?*

Intertwining

Every known instance of Elliott's conjecture is proved by lifting a morphism between the invariants.

$$
\begin{array}{ccccccccccc}
A_1 & \longrightarrow & A_2 & \longrightarrow & A_3 & \longrightarrow & A_4 & \longrightarrow & \cdots & & A = \lim_n A_n \\
\Phi_1 \downarrow & \Psi_1 \nearrow & \Phi_2 \downarrow & \Psi_2 \nearrow & \Phi_3 \downarrow & \Psi_3 \nearrow & \Phi_4 \downarrow & \nearrow & & & \\
B_1 & \longrightarrow & B_2 & \longrightarrow & B_3 & \longrightarrow & B_4 & \longrightarrow & \cdots & & B = \lim_n B_n
\end{array}
$$

Φ_n, Ψ_n are partial *-homomorphisms. The n-th triangle commutes up to 2^{-n}.

Then $A \cong B$.

Jiang–Su algebra $\mathcal{Z}$ revisited

Revised Elliott's Conjecture (Toms–Winter, 2007)

Infinite-dimensional, simple, nuclear, unital, separable, *$\mathcal{Z}$-stable* (i.e., $A \otimes \mathcal{Z} \cong A$) algebras are classified by Ell.

Lemma

Being $\mathcal{Z}$-stable is $\forall\exists$-axiomatizable, for separable algebras.

Therefore a positive answer to 'Strong Conjecture' implies a positive answer to the Revised Elliott's Conjecture.

Omitting Types

Definition

Type $\mathbf{p}(\bar{x})$ is a set of *conditions* $\varphi_\gamma(\bar{x}) = r_\gamma$, for $\gamma \in I$.

It is *realized by* $\bar{a}$ *in* A if $\varphi_\gamma(\bar{a})^A = r_\gamma$ for all γ.

Theorem (F.–Hart–Tikuisis–Robert–Lupini–Winter, 2014)

Each of the following classes of algebras: UHF, AF, AT, AI, nuclear, simple, nuclear dimension $< n$, decomposition rank $< n$ ($n \leq \aleph_0$), ... is characterized as the set of all algebras that omit a sequence of types.

Given a complete theory $\mathbf{T}$, one defines 'Henkin forcing' $\mathbb{P}_{\mathbf{T}}$ whose conditions are of the form $\varphi(\bar{d}) < \varepsilon$, consistent with $\mathbf{T}$. The generic model is denoted M_G.

Topologies on the space of complete types in a complete theory $\mathbf{T}$.

Logic topology

is defined as in the discrete case: basic open sets are conditions in $\mathbb{P}_{\mathbf{T}}$.

Metric topology

$\mathbf{d}(\mathbf{t},\mathbf{s}) = \inf\{d(a,b) : (\exists A \models \mathbf{T})\mathbf{t}(a)^A, \mathbf{s}(b)^A\}$.

A type is *isolated* if none of its metric open neighbourhoods is nowhere dense in the logic topology.

Theorem (BYBHU, 2008)

Given a separable language L, complete L-theory $\mathbf{T}$, a complete type $\mathbf{p}$ is omissible in a model of $\mathbf{T}$ if and only if it is not isolated.

Non-complete types over a complete theory

Lemma (Ben Yaacov, 2010)

There are types that are neither isolated nor omissible.

Theorem (F.–Magidor, 2014)

(1) There is a theory $\mathbf{T}$ *in a separable language such that*

$$\{\mathbf{t} : \mathbf{t} \textit{ is omissible in a model of } \mathbf{T}\}$$

is a complete Σ^1_2 *set.*
(2) There is a complete theory $\mathbf{T}$ *in a separable language such that*

$$\{\mathbf{t} : \mathbf{t} \textit{ is omissible in a model of } \mathbf{T}\}$$

is Π^1_1 *hard.*

Lemma (F–Magidor, 2014)

The set of (ground-model) types forced by $\mathbb{P}_{\mathbf{T}}$ to be omitted in M_G is $\Pi^1_1(\mathbf{T})$.

Theorem (F.–Magidor, 2014)

There is a separable complete theory $\mathbf{T}$ and an omissible type $\mathbf{t}(\bar{x})$ which is realized in $\mathbb{P}_{\mathbf{T}}$-generic model.

Uniform sequences of types

A sequence of types $\mathbf{t}_n(\bar{x})$ for $n \in \mathbb{N}$ is *uniform* if there are formulas $\varphi_j(\bar{x})$, with the same modulus of uniform continuity, such that

$$\mathbf{t}_n(\bar{x}) = \{\varphi_j(\bar{x}) \geq 1/n : j \in \mathbb{N}\}, \qquad \text{for all } n.$$

Equivalently, the interpretation of the $L_{\omega_1,\omega}$ formula $\inf_j \varphi_j(\bar{x})$ is a uniformly continuous function in every model of the theory.

Theorem (F.–Magidor, 2014)

A uniform sequence of types $\{\mathbf{t}_n\}$ is omissible in a model of a complete theory $\mathbf{T}$ if and only if for every n type $\mathbf{t}_n$ is not isolated.

Theorem (F.–Hart–Tikuisis–Robert–Lupini–Winter, 2014)

Each of the following classes of algebras: UHF, AF, AT, AI, nuclear, simple, nuclear dimension $< n$, decomposition rank $< n$ ($n \leq \aleph_0$), . . . is characterized as the set of all algebras that omit a uniform sequence of types.

Corollary

Sets of theories of UHF, AF, AT, AI, nuclear,. . . algebras are Borel.

Proposition

The ultraproduct $\prod_{\mathcal{U}} M_n(\mathbb{C})$ is not elementarily equivalent to a nuclear C-algebra.*
$C^*_r(F_\infty)$ is not elementarily equivalent to a nuclear C-algebra.*

Strongly self-absorbing (s.s.a.) C*-algebras

Definition (Toms–Winter, after McDuff/Connes)

A separable algebra A is *s.s.a.* if

1. $A \cong A \otimes A$,
2. The inner automorphism group is dense in $\mathrm{Aut}(A)$.

Lemma

If A is s.s.a., then

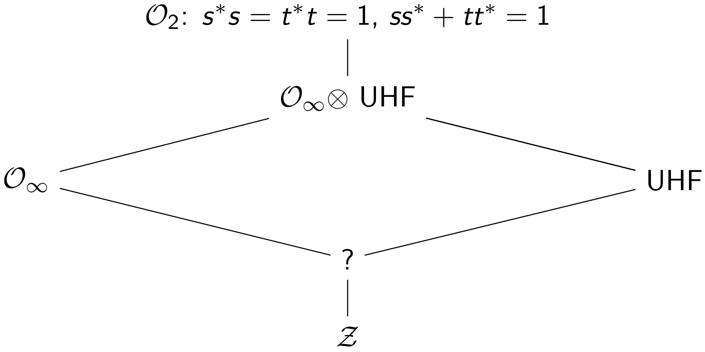
1. $A \cong \bigotimes_{\aleph_0} A$.
2. *(Effros–Rosenberg, 1978) A is simple and nuclear.*
3. *A is a prime model of its theory.*
4. *If $B \equiv A$ then every endomorphism $f : A \to B$ is elementary.*

All s.s.a. algebras

Proposition

If D and E are s.s.a. algebras then TFAE.

1. $E \otimes D \cong E$.
2. *D is isomorphic to a subalgebra of E.*
3. $\mathrm{Th}_\exists(D) \subseteq \mathrm{Th}_\exists(E)$.



Relative commutants

If A is a C*-algebra, identify A with its diagonal image in $A^{\mathcal{U}}$, let

$$A' \cap A^{\mathcal{U}} = \{b \in A^{\mathcal{U}} : (\forall a \in A) ab = ba\}.$$

Theorem (McDuff for II_1 factors, Toms–Winter 2007)

If D is s.s.a. and A is separable then $A \otimes D \cong A$ iff D embeds into $A' \cap A^{\mathcal{U}}$.

All ultrafilters are nonprincipal ultrafilters on $\mathbb{N}$

Question (McDuff 1970, Kirchberg, 2004)

Assume A is separable. Does $A' \cap A^{\mathcal{U}}$ depend on $\mathcal{U}$?

Theorem (Ge–Hadwin, F., F.–Hart–Sherman, F.–Shelah)

If CH fails and A is infinite-dimensional, then there are $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers of A and nonisomorphic relative commutants of A.

Theorem (F.–Hart–Sherman, F.–Shelah 2011)

For a separable model A the following are equivalent.

1. $\mathrm{Th}(A)$ *is not stable.*
2. $\neg CH$ *implies A has nonisomorphic ultrapowers.*
3. $\neg CH$ *implies A has $2^{2^{\aleph_0}}$ nonisomorphic ultrapowers.*

Corollary

For A as above

$CH \Leftrightarrow$ all ultrapowers of A are isomorphic.

Theorem (F.–Hart–Robert–Tikuisis, 2014)

Assume D is s.s.a.. Then $D' \cap D^{\mathcal{U}} \prec D^{\mathcal{U}}$.

Corollary

1. *Every embedding of D into $D' \cap D^{\mathcal{U}}$ is elementary.*
2. *All embeddings of D into $D' \cap D^{\mathcal{U}}$ are unitarily conjugate.*
3. *CH implies $D' \cap D^{\mathcal{U}} \cong D^{\mathcal{U}}$.*
4. *CH implies $\bigotimes_{\aleph_1} D$ embeds into $D^{\mathcal{U}}$ so that its relative commutant in $D^{\mathcal{U}}$ is trivial.*

Problems

1. Construct interesting C*-algebras by using omitting types theorem.
2. Develop theory of Borel-reductions between 'Polish categories.' (Some preliminary results by Lupini.)
3. Further develop model theory of II_1 factors (only three different theories of II_1 factors are known!).
4. If $\mathbf{T}$ is a complete metric theory and types $\mathbf{s}$ and $\mathbf{t}$ are separately omissible in models of $\mathbf{T}$, are they jointly omissible in a model of $\mathbf{T}$?
   (F.–Magidor: There are a complete separable theory $\mathbf{T}$ and types $\mathbf{t}_n$ for $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$ types $\mathbf{t}_n$, for $n \leq m$, are simultaneously omissible but $\mathbf{t}_n$, for $n \in \mathbb{N}$ are not simultaneously omissible.)