Incompatible Ω-Complete Theories*

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Abstract

In 1985 the second author showed that if there is a proper class of measurable Woodin cardinals and $V^{\mathbb{B}_1}$ and $V^{\mathbb{B}_2}$ are generic extensions of V satisfying CH then $V^{\mathbb{B}_1}$ and $V^{\mathbb{B}_2}$ agree on all Σ^2_1-statements. In terms of the strong logic Ω-logic this can be reformulated by saying that under the above large cardinal assumption ZFC + CH is Ω-complete for Σ^2_1. Moreover, CH is the unique Σ^2_1-statement with this feature in the sense that any other Σ^2_1-statement with this feature is Ω-equivalent to CH over ZFC. It is natural to look for other strengthenings of ZFC that have an even greater degree of Ω-completeness. For example, one can ask for recursively enumerable axioms A such that relative to large cardinal axioms ZFC + A is Ω-complete for all of third-order arithmetic. Going further, for each specifiable segment V_λ of the universe of sets (for example, one might take V_λ to be the least level that satisfies there is a proper class of huge cardinals), one can ask for recursively enumerable axioms A such that relative to large cardinal axioms ZFC + A is Ω-complete for the theory of V_λ. If such theories exist, extend one another, and are unique in the sense that any other such theory B with the same level of Ω-completeness as A is actually Ω-equivalent to A over ZFC, then this would show that there is a unique Ω-complete picture of the successive fragments of the universe of sets and it would make for a very strong case for axioms complementing large cardinal axioms. In this paper we show that uniqueness must fail. In particular, we show that if there is one such theory that Ω-implies CH then there is another that Ω-implies ¬CH.

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In this paper we consider a very optimistic scenario for extending the axioms of ZFC to diminish independence and we show that this scenario must fail. In Section 1 we motivate the scenario by discussing some developments in the search for new axioms. In Section 2 we give a brief overview of Ω-logic and describe the scenario. In Section 3 we prove our main result. The reader who understands the above abstract and is not in need of motivation can turn directly to Section 2.

1 Independence and New Axioms

The independence results in set theory have shown that many basic questions of mathematics cannot be settled on the basis of the standard axioms of mathematics, ZFC. Two classical examples of such statements are PM (the statement that all projective sets are Lebesgue measurable) and CH (Cantor's continuum hypothesis). The first concerns the structure of second-order arithmetic while the second concerns third-order arithmetic. Both of these problems were intensively investigated during the early era of set theory but no progress was made. The explanation for this was ultimately provided by results of Gödel and Cohen. Gödel constructed an *inner model* L of V that satisfies $\neg$PM and CH. Cohen complemented this by constructing an *outer model* (or *forcing extension*) $V^{\mathbb{B}}$ of V that satisfies $\neg$CH. Solovay combined these techniques and, assuming an inaccessible cardinal, constructed a model of ZFC in which PM holds. Together these results show that it is *in principle* impossible to either prove or refute these statements on the basis of the standard axioms of mathematics, ZFC. But this simply raises the question of whether these statements are *absolutely undecidable*, that is, undecidable relative to any collection of justified axioms.

To show that such statements are *not* absolutely undecidable one must find and justify new axioms that settle the undecided statements. This program has both a *mathematical* component and a *philosophical* component. On the mathematical side one must find axioms which are sufficient for the task. On the philosophical side one must show that these axioms are indeed justified. These are not unrelated components. For, on the one hand, philosophical considerations serve as a guide to the initial formulation of the axioms and, on the other hand, the ultimate case for justification will inevitably rest on a network of mathematical results.

In the case of the first problem (and, more generally, a vast array of other problems in second-order arithmetic) there has been remarkable success. We now have axioms that settle PM (affirmatively) and which admit of a strong justification. In fact there are two classes of such axioms—*axioms of definable determinacy* and *large cardinal axioms* (or *axioms of infinity*). These axioms spring from entirely different sources and, in addition to resolving PM and many other questions of second-order arithmetic, these axioms are intimately connected. In the remainder of this introduction we will give a brief sketch of these developments, with the aim of motivating the main results on the prospect of bifurcation at the level of CH.

1.1 Large Cardinal Axioms and Axioms of Definable Determinacy

Large cardinal axioms are (roughly speaking) generalizations of the axioms of extent of ZFC—Infinity and Replacement—in that they assert that there are large levels of the universe of sets. Examples of such axioms are those asserting the existence of strongly inaccessible cardinals, Woodin cardinals, and supercompact cardinals. Axioms of definable determinacy can also be seen as generalizations of a principle inherent in ZFC, namely, Borel determinacy, which was shown to be a theorem of ZFC by Martin. Examples of such axioms are PD (the statement that all projective sets are determined) and $\mathrm{AD}^{L(\mathbb{R})}$ (the statement that all sets of reals in $L(\mathbb{R})$ are determined).

Large cardinal axioms and axioms of definable determinacy are (in many cases) intrinsically plausible but the strongest case for their justification comes through their fruitful consequences, their connections with each other, and an intricate network of theorems relating them to other axioms. We will touch on some of these developments but will necessarily have to be brief.[1]

Let us start with fruitful consequences. In ZFC one can develop a remarkable structure theory for the Borel sets; for example, all such sets are Lebesgue measurable and have the property of Baire. The principle of Borel determinacy lies at the heart of this structure theory and PD lifts this structure theory from the Borel sets to the projective sets, while $\mathrm{AD}^{L(\mathbb{R})}$ lifts it further to the sets of reals in $L(\mathbb{R})$, which lie in a transfinite extension of the projective hierarchy.

Theorem 1.1 (Mycielski-Swierczkowski [17]; Mazur, Banach; Davis [3])**.**

[1] For a more detailed discussion see [9] and the references therein. For further historical and mathematical information see [8].

Assume ZFC + PD. *Then all projective sets are Lebesgue measurable and have the property of Baire.*

Theorem 1.2 (Mycielski-Swierczkowski [17]; Mazur, Banach; Martin-Steel [15])**.** *Assume* ZFC + $\mathrm{AD}^{L(\mathbb{R})}$. *Then every set of reals in* $L(\mathbb{R})$ *is Lebesgue measurable and has the property of Baire and* Σ^2_1*-uniformization holds in* $L(\mathbb{R})$*.*

Not only do PD and $\mathrm{AD}^{L(\mathbb{R})}$ lift the structure theory to their respective domains, they appear to be "effectively complete" for their respective realms. A comparison with PA is useful here. In contrast to the case of PA there are many statements of prior mathematical interest concerning second-order arithmetic that are not settled by second-order Peano Arithmetic, PA_2. For example, PM is such a statement. When one adds (schematic) PD to PA_2 this ceases to be the case. In fact, $\mathrm{PA}_2+\mathrm{PD}$ appears to be *more* complete for second-order arithmetic than PA is for first-order arithmetic, in that there is no analogue of the type of result uncovered in [18]. Similar considerations apply to $\mathrm{AD}^{L(\mathbb{R})}$.

Let us turn now to the fruitful consequences of large cardinal axioms. Since these principles assert the existence of very large sets there is perhaps little reason to expect that they will have significant consequences for second- and third-order arithmetic. Of course, we know from the incompleteness phenomenon that they yield new Π^0_1-statements and so have consequences for first-order arithmetic. Gödel had much higher expectations for large cardinal axioms, thinking that they had significant consequences for second- and third-order arithmetic. Indeed he went so far as to entertain a kind of generalized completeness theorem for large cardinal axioms:

> It is not impossible that for such a concept of demonstrability [namely, provability from true large cardinal axioms] some completeness theorem would hold which would say that every proposition expressible in set theory is decidable from the present axioms plus some true assertion about the largeness of the universe of all sets. ([5, p. 151])

It turns out that large cardinal axioms do have significant consequences for second-order arithmetic. For example, they settle PM.

Theorem 1.3 (Shelah-Woodin [19])**.** *Assume* ZFC *and that there are infinitely many Woodin cardinals. Then* PM*.*

To summarize: We have two classes of axioms—large cardinal axioms and axioms of definable determinacy—which settle PM.

Remarkably, although large cardinal axioms and axioms of definable determinacy spring from entirely different sources, there are intimate connections between them. To begin with, large cardinal axioms *imply* axioms of definable determinacy:

Theorem 1.4 (Martin-Steel [16]). *Assume* ZFC *and that there are infinitely many Woodin cardinals. Then* PD.

Theorem 1.5 (Woodin [21]). *Assume* ZFC *and that there are infinitely many Woodin cardinals with a measurable cardinal above them all. Then* $\mathrm{AD}^{L(\mathbb{R})}$.

The connection between Woodin cardinals and axioms of definable determinacy is much deeper—axioms of definable determinacy are in fact equivalent to the existence of certain inner models of Woodin cardinals:

Theorem 1.6 (Woodin). *The following are equivalent*:

(1) PD (*Schematic*).

(2) *For every* $n < \omega$, *there is a fine-structural, countably iterable inner model* M *such that* $M \vDash$ "*There are* n *Woodin cardinals*".

Theorem 1.7 (Woodin). *The following are equivalent*:

(1) $\mathrm{AD}^{L(\mathbb{R})}$.

(2) *In* $L(\mathbb{R})$, *for every set* S *of ordinals, there is an inner model* M *and an* $\alpha < \omega_1^{L(\mathbb{R})}$ *such that* $S \in M$ *and* $M \vDash$ "α *is a Woodin cardinal*".

Let us return to the issues of "fruitful consequences" and "effective completeness". We mentioned earlier that axioms of definable determinacy lift the structure theory of the Borel sets to higher levels. One concern might be that there are other, incompatible axioms that also have this consequence. However, one can show that in certain cases, for example, the case of $L(\mathbb{R})$, one cannot have the structure theory *without* having definable determinacy:

Theorem 1.8 (Woodin). *Assume that every set of reals in* $L(\mathbb{R})$ *is Lebesgue measurable and has the property of Baire and assume* Σ^2_1*-uniformization holds in* $L(\mathbb{R})$. *Then* $\mathrm{AD}^{L(\mathbb{R})}$.

Further, much stronger, evidence that $\mathrm{AD}^{L(\mathbb{R})}$ is giving the correct theory of the sets of reals in $L(\mathbb{R})$ lies in the network of theorems supporting the claim that all sufficiently strong "natural" theories imply $\mathrm{AD}^{L(\mathbb{R})}$. This occurs even in cases where the theories are logically incompatible. So $\mathrm{AD}^{L(\mathbb{R})}$ appears to lie in the overlapping consensus of all sufficiently strong "natural" theories.

The "effective completeness" of large cardinal axioms for the theory of the sets of reals in $L(\mathbb{R})$ and, in fact, all of $L(\mathbb{R})$ can be quantified. Our main technique for establishing independence in set theory is the powerful technique of set-forcing. But in the presence of Woodin cardinals this technique *cannot* be used to establish independence of statements concerning $L(\mathbb{R})$. Woodin cardinals "seal" or "freeze" the theory of $L(\mathbb{R})$. Put otherwise, the theory of $L(\mathbb{R})$ is *generically absolute* in the presence of Woodin cardinals.

Theorem 1.9 (Woodin [12])**.** *Assume there is a proper class of Woodin cardinals. Suppose φ is a sentence, $\mathbb{P}$ is a partial order and $G \subseteq \mathbb{P}$ is V-generic. Then*

$$L(\mathbb{R}) \models \varphi \quad \textit{iff} \quad L(\mathbb{R})^{V[G]} \models \varphi.$$

This situation generalizes beyond $L(\mathbb{R})$ and, in a sense which can be made precise, holds strictly "below" Σ^2_1, the level of CH.[2]

2 An Optimistic Scenario

Our goal now is to investigate how far generic absoluteness extends. To this end it will be useful to reformulate generic absoluteness in terms of a strong logic that is designed to factor out the effects of set-forcing.

2.1 Ω-Logic

We begin by introducing the semantic consequence relation of Ω-logic and then turn to the quasi-syntactic proof relation that aims to capture it.[3]

Definition 2.1. Suppose that T is a countable theory in the language of set theory and φ is a sentence. Then

$$T \models_\Omega \varphi$$

[2]See section 3 of [9] for a precise statement.

[3]See [2] and the references therein for further details concerning Ω-logic.

if for all complete Boolean algebras $\mathbb{B}$ and for all ordinals α,

$$\text{if } V_\alpha^{\mathbb{B}} \models T \text{ then } V_\alpha^{\mathbb{B}} \models \varphi.$$

This notion of semantic implication is robust in that large cardinal axioms imply that the question of what implies what cannot be altered by forcing:

Theorem 2.2 (Woodin)**.** *Assume* ZFC *and that there is a proper class of Woodin cardinals. Suppose that T is a countable theory in the language of set theory and φ is a sentence. Then for all complete Boolean algebras $\mathbb{B}$,*

$$T \models_\Omega \varphi \quad \textit{iff} \quad V^{\mathbb{B}} \models \text{“}T \models_\Omega \varphi.\text{”}$$

We say that a statement or theory T is Ω-*satisfiable* if there exists an ordinal α and a complete Boolean algebra $\mathbb{B}$ such that $V_\alpha^{\mathbb{B}} \models T$.

It follows immediately from the above that Ω-satisfiability is also generically invariant. To underscore just how remarkable this is we note the following consequence: Suppose that there is a proper class of Woodin cardinals and let φ be a Σ_2-sentence. The statement that φ holds in a generic extension is generically absolute. For example, suppose that φ is the Σ_2-statement asserting that there is a huge cardinal. Let $V^{\mathbb{B}}$ be a generic extension where the huge cardinal is collapsed. It follows from the above that it is possible to further force to “resurrect” the huge cardinal, that is, there is a further forcing extension $V^{\mathbb{B}*\mathbb{C}}$ containing a huge cardinal.

To introduce the “syntactic” proof relation which aims to capture the above semantic notion we first need to introduce the notion of a universally Baire set of reals.

Definition 2.3. Suppose $A \subseteq \omega^\omega$ and δ is a cardinal. The set A is δ-*universally Baire* if for all partial orders $\mathbb{P}$ of cardinality δ there exist trees S and T in $\omega \times \kappa$ for some κ such that

(1) $A = p[T]$ and

(2) if $G \subseteq \mathbb{P}$ is V-generic then in $V[G]$,

$$p[T] = \omega^\omega \setminus p[S].$$

The set A is *universally Baire* if it is δ-universally Baire for all δ.

Universally Baire sets have canonical interpretations in generic extensions $V[G]$: Choose any $T, S \in V$ such that $p[T] = A$ and $p[T]^{V[G]} = (\omega^\omega)^{V[G]} \setminus p[S]^{V[G]}$ and set $A_G = p[T]^{V[G]}$. It is straightforward to see (using the absoluteness of well-foundedness) that A_G is independent of the choice of T and S. See [4] for further details.

Definition 2.4. Suppose that $A \subseteq \omega^\omega$ is universally Baire and that M is a countable transitive model of ZFC. Then M is *strongly A-closed* if for all set generic extensions $M[G]$ of M,

$$A \cap M[G] \in M[G].$$

Definition 2.5. Suppose that there is a proper class of Woodin cardinals, T is a countable theory in the language of set theory and φ is a sentence. Then $T \vdash_\Omega \varphi$ iff there exists a set $A \subseteq \omega^\omega$ such that

(1) A is universally Baire, and

(2) for all countable transitive models M, if M is strongly A-closed and $T \in M$, then
$$M \models \text{“}T \models_\Omega \varphi\text{”}.$$

Like the semantic notion of consequence, this notion of provability is robust under large cardinal assumptions:

Theorem 2.6 (Woodin)**.** *Assume there is a proper class of Woodin cardinals. Suppose T is a countable theory in the language of set theory, φ is a sentence, and $\mathbb{B}$ is a complete Boolean algebra. Then*

$$T \vdash_\Omega \varphi \quad \textit{iff} \quad V^{\mathbb{B}} \models \text{“}T \vdash_\Omega \varphi\text{”}.$$

Thus, we have a semantic consequence relation and a quasi-syntactic proof relation, both of which are robust under the assumption of large cardinal axioms. It is natural to ask whether the soundness and completeness theorems hold. The soundness theorem is known to hold:

Theorem 2.7 (Woodin)**.** *Suppose T is a countable theory in the language of set theory and φ is a sentence. If $T \vdash_\Omega \varphi$ then $T \models_\Omega \varphi$.*

It is open whether the completeness theorem holds for Ω-logic.

Definition 2.8 (Ω Conjecture)**.** Assume ZFC and that there is a proper class of Woodin cardinals. Then for each sentence φ,

$$\varnothing \models_\Omega \varphi \text{ iff } \varnothing \vdash_\Omega \varphi.$$

We shall need to introduce a strengthening of this conjecture.

Definition 2.9 (AD^+ Conjecture)**.** Suppose that A and B are sets of reals such that $L(A,\mathbb{R})$ and $L(B,\mathbb{R})$ satisfy AD^+. Suppose every set

$$X \in \mathscr{P}(\mathbb{R}) \cap \big(L(A,\mathbb{R}) \cup L(B,\mathbb{R})\big)$$

is ω_1-universally Baire. Then either

$$(\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$$

or

$$(\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})}.$$

Definition 2.10 (Strong Ω Conjecture)**.** Assume there is a proper class of Woodin cardinals. Then the Ω Conjecture holds and the AD^+ Conjecture is Ω-valid.

As we shall see this conjecture has profound meta-mathematical consequences.

2.2 Ω-Complete Theories

We are now in a position to reformulate generic absoluteness in terms of Ω-logic.

Definition 2.11. A theory T is Ω-*complete* for a collection of sentences Γ if for each $\varphi \in \Gamma$, $T \models_\Omega \varphi$ or $T \models_\Omega \neg\varphi$.

Remark 2.12. Notice that we are allowing the degenerate case in which T is not Ω-satisfiable, in which case *both* of the above implications hold. In particular, the theory $\mathrm{ZFC} + 0 = 1$ is trivially Ω-complete for any Γ. This choice is merely one of convenience and, of course, in all cases of interest there will be sufficient large cardinals to ensure that the theory will be Ω-satisfiable.

The result on the generic absoluteness of $L(\mathbb{R})$ (Theorem 1.9) can now be reformulated as follows:

Theorem 2.13 (Woodin)**.** *Assume* ZFC *and that there is a proper class of Woodin cardinals. Then* ZFC *is* Ω*-complete for the collection of sentences of the form "*$L(\mathbb{R}) \models \varphi$*".*

Although we have stated the Ω-completeness with respect to ZFC the large cardinals are really doing the work. For this reason it is perhaps more transparent to formulate the result by saying that "ZFC + there is a proper class of Woodin cardinals" is Ω-complete for the collection of sentences of the form "$L(\mathbb{R}) \models \varphi$", noting that under this formulation the stated Ω-completeness is trivial unless our background assumptions guarantee that "ZFC + there is a proper class of Woodin cardinals" is Ω-satisfiable.

The above result is thus a partial realization of Gödel's conjectured completeness theorem for large cardinal axioms, only now we are invoking a stronger logic and we only have completeness at the level of $L(\mathbb{R})$. Unfortunately, it follows from a series of results originating with Levy and Solovay that the current generation of large cardinal axioms are not Ω-complete at the level of third-order arithmetic, in fact, they are not Ω-complete at the level of Σ^2_1, which is the complexity of CH.

Theorem 2.14. *Assume L is a standard large cardinal axiom. Then* ZFC+L *is not* Ω*-complete for* Σ^2_1*.*

This theorem is stated informally since the notion of a "standard large cardinal axiom" is not precise. However, one can cite examples from across the large cardinal hierarchy. For example, for L one can take "there is a measurable cardinal", "there is a proper class of Woodin cardinals", "there is a non-trivial embedding $j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$ with critical point below λ". See [14], [7] and [13].

Although large cardinal axioms do not provide an Ω-complete picture of Σ^2_1 it turns out that one can attain such a picture provided one supplements large cardinal axioms. Remarkably, one can do this by adding CH.

Theorem 2.15 (Woodin [20])**.** *Assume* ZFC *and that there is a proper class of measurable Woodin cardinals. Then* ZFC + CH *is* Ω*-complete for* Σ^2_1*.*

Moreover, up to Ω-equivalence, CH is the unique Σ^2_1-statement that is Ω-complete for Σ^2_1.

Lemma 2.16. *Suppose A is a Σ^2_1-sentence, $\mathrm{ZFC} + A$ is Ω-satisfiable, and $\mathrm{ZFC} + A$ is Ω-complete for Σ^2_1. Then*

(1) $\mathrm{ZFC} + \mathrm{CH} \models_\Omega A$ *and*

(2) $\mathrm{ZFC} + A \models_\Omega \mathrm{CH}$.

Proof. Since $\mathrm{ZFC} + A$ is Ω-satisfiable, there is an ordinal α, a partial order $\mathbb{P}$, and a V-generic $G \subseteq \mathbb{P}$ such that

$$V[G]_\alpha \models \mathrm{ZFC} + A.$$

Now let $H \subseteq \mathrm{Col}(\omega_1, \mathbb{R})$ be $V[G]$-generic. Thus

$$V[G][H]_\alpha \models \mathrm{ZFC} + \mathrm{CH}.$$

Moreover, since $\mathrm{Col}(\omega_1, \mathbb{R})$ is countably closed it adds no new reals. Thus, since A is Σ^2_1 we have, by upward absoluteness,

$$V[G][H]_\alpha \models \mathrm{ZFC} + A.$$

Thus, ZFC+CH and ZFC+A are Ω-compatible. Since each theory is assumed to be Ω-complete for Σ^2_1 both (1) and (2) follow. □

Thus, up to Ω-equivalence, there is a unique Σ^2_1-sentence which (along with large cardinal axioms) provides an Ω-complete picture of Σ^2_1.

If one shifts perspective from Σ^2_1 to $H(\omega_2)$ there is a companion result for $\neg$CH, *assuming the Strong Ω Conjecture.*

Theorem 2.17 (Woodin [22]). *Assume that there is a proper class of Woodin cardinals and that the Strong Ω-Conjecture holds.*

(1) *There is an axiom A such that*

  (*i*) $\mathrm{ZFC} + A$ *is Ω-satisfiable and*

  (*ii*) $\mathrm{ZFC} + A$ *is Ω-complete for the structure $H(\omega_2)$.*

(2) *Any such axiom A has the feature that*

$$\mathrm{ZFC} + A \models_\Omega \text{“}H(\omega_2) \models \neg\mathrm{CH}\text{”}.$$

Thus, assuming that there is a proper class of Woodin cardinals and that the Strong Ω Conjecture holds, there is an Ω-complete picture of $H(\omega_2)$ and any such picture involves a *failure* of CH.

These two results raise the spectre of bifurcation at the level of CH. There are two key questions. First, are there recursive theories with higher degrees of Ω-completeness? Second, is there a unique such theory (with respect to a given level of complexity)? The answers to these questions turn on the Strong Ω Conjecture.

If there is a proper class of Woodin cardinals and the Strong Ω Conjecture holds then one cannot have an Ω-complete picture of third-order arithmetic.

Theorem 2.18 (Woodin)**.** *Assume that there is a proper class of Woodin cardinals and that the Strong Ω Conjecture holds. Then there is no recursive theory A such that $\mathrm{ZFC} + A$ is Ω-complete for Σ^2_3.*

Even the Ω Conjecture places limitations on the extent of Ω complete theories.

Theorem 2.19 (Woodin)**.** *Assume that there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then there is no recursive theory A such that $\mathrm{ZFC} + A$ is Ω-complete for the theory of $H(\delta_0^+)$, where δ_0 is the least Woodin cardinal.*

It is open whether there is a recursively enumerable theory that is Ω-complete for Σ^2_2. It is known that CH alone will not suffice:

Theorem 2.20 (Jensen, Shelah)**.** $\mathrm{ZFC} + \mathrm{CH}$ *is not Ω-complete for Σ^2_2.*

Jensen obtained a model of $\mathrm{ZFC} + \mathrm{CH} + \mathrm{SH}$ (and hence the failure of $\diamondsuit$) by iterated forcing over L. Later, Shelah obtained such models in the more general setting of proper-forcing iterations that do not add reals. In fact, in [1] it is shown that under $\mathrm{ZFC} + \mathrm{CH}$ there is so much ambiguity at the level of Σ^2_2 that there is a small forcing that adds no new reals but adds a Δ^2_2-well-ordering of the reals. The question of an Ω-complete theory at the level of Σ^2_2 is still open—there is some evidence that (under large cardinal axioms) $\diamondsuit$ is such an axiom, that is, that under large cardinal assumptions $\mathrm{ZFC} + \diamondsuit$ is Ω-complete for Σ^2_2. See [23].

If the Ω Conjecture fails then it is possible that there is a recursively enumerable theory A and a large cardinal axiom L such that $\mathrm{ZFC} + L + A$ is Ω-complete for third-order arithmetic. In fact, it is possible that for each specifiable fragment V_λ of the universe of sets there is a recursively enumerable theory and a large cardinal axiom L such that $\mathrm{ZFC}+L+A$ is Ω-complete for the theory of V_λ. In other words, assuming the failure of the Ω Conjecture it is possible (given our current understanding) that there is an Ω-complete picture of arbitrarily large fragments of the universe of sets. If there were a *unique* such picture then this would make for a compelling case for new axioms that complete the standard axioms of set theory and constitute a realization of a variant of Gödel's conjectured completeness theorem. In the next section we will show that this optimistic scenario must fail; if there is one such Ω-complete picture then there must be another, incompatible Ω-complete picture.

3 Failure of Uniqueness

First, we need a precise specification of a large cardinal property, one that incorporates a key feature shared by customary large cardinal axioms, namely, invariance under small forcing. (Cf. Theorem 2.14.)

Definition 3.1. A *large cardinal property* is a Σ_2-formula $\varphi(x)$ such that (as a theorem of ZFC) if κ is a cardinal and $V \vDash \varphi[\kappa]$ then κ is strongly inaccessible and for all partial orders $\mathbb{P} \in V_\kappa$ and all V-generics $G \subseteq \mathbb{P}$, $V[G] \vDash \varphi[\kappa]$.

This directly captures most of the standard large cardinal properties—for example, "κ is measurable", "κ is a Woodin cardinal", "κ is the critical point of a non-trivial elementary embedding $j : V_\lambda \to V_\lambda$". It does not capture "$\kappa$ is supercompact" but it does capture "$\exists\delta\, V_\delta \vDash \kappa$ is supercompact".

Definition 3.2. Suppose φ is a large cardinal property. Let $\mathrm{PC}(\varphi)$ be the conjunction of the statements "there is a proper class of Woodin cardinals" and "there is a proper class of φ-cardinals".

Above we considered Ω-completeness relative to a fixed pointclass Γ (such as Σ^2_1) but now we shall be dealing with much larger fragments of the universe of sets and so it will be necessary to extend this definition.

Definition 3.3. A sentence Φ is a *specification* if there is a least level V_α that satisfies Φ and $\alpha > \omega$. Suppose Φ is a specification. Let V_Φ denote the level specified by Φ. The sentence Φ is a *robust specification* if, in addition, for all partial orders $\mathbb{P} \in V_\Phi$ and for all V-generic $G \subseteq \mathbb{P}$, in $V[G]$ the ordinal specified by Φ is the same as the ordinal specified by Φ in V.

Remark 3.4. The robustness condition amounts to saying that for all $\mathbb{P} \in V_\Phi$ and for all V-generic $G \subseteq \mathbb{P}$,

$$(V_\Phi)^{V[G]} = (V_\Phi)^V[G].$$

Some such robustness condition is necessary for our purposes. Fortunately, in the cases of interest this condition is met. For example, this is immediately true of large levels (such as the least level V_α satisfying that there is a proper class of measurable cardinals) for any small forcing and we shall see that it is true of the small levels we consider for the particular forcing notions we employ.

Definition 3.5. Let Φ be a robust specification, let φ be a large cardinal property and let A be a recursively enumerable set of axioms. Then $\mathrm{ZFC} + A + \mathrm{PC}(\varphi)$ is Ω-*complete* for $\mathrm{Th}(V_\Phi)$ if for all sentences S of the language of set theory,

$$\mathrm{ZFC} + A + \mathrm{PC}(\varphi) \models_\Omega \text{“}S \in \mathrm{Th}(V_\Phi)\text{”}$$

or

$$\mathrm{ZFC} + A + \mathrm{PC}(\varphi) \models_\Omega \text{“}\neg S \in \mathrm{Th}(V_\Phi)\text{”}.$$

Notice again that we are including the degenerate case in that if $\mathrm{ZFC} + A + \mathrm{PC}(\varphi)$ is Ω-inconsistent then it is Ω-complete for $\mathrm{Th}(V_\Phi)$.

It will be of use to note the following reduction: Conditioning on a recursively enumerable theory A can be subsumed by conditioning on a single Σ_2-sentence. For suppose $\mathrm{ZFC} + A + \mathrm{PC}(\varphi)$ is Ω-complete for $\mathrm{Th}(V_\Phi)$. Let ψ be the Σ_2-sentence which asserts that there exists α such that $V_\alpha \models \mathrm{ZFC} + A + \mathrm{PC}(\varphi)$. Then $\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$ is Ω-complete for $\mathrm{Th}(V_\Phi)$. To see this suppose that β_1 and β_2 are ordinals and $\mathbb{B}_1$ and $\mathbb{B}_2$ are complete Boolean algebras such that $V_{\beta_1}^{\mathbb{B}_1}$ and $V_{\beta_2}^{\mathbb{B}_2}$ satisfy $\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$. Let α_1 and α_2 be the least ordinals witnessing ψ in $V_{\beta_1}^{\mathbb{B}_1}$ and $V_{\beta_2}^{\mathbb{B}_2}$, respectively. By hypothesis,

$$(\mathrm{Th}(V_\Phi))^{V_{\alpha_1}^{\mathbb{B}_1}} = (\mathrm{Th}(V_\Phi))^{V_{\alpha_2}^{\mathbb{B}_2}}.$$

But

$$(\mathrm{Th}(V_\Phi))^{V_{\beta_1}^{\mathbb{B}_1}} = (\mathrm{Th}(V_\Phi))^{V_{\alpha_1}^{\mathbb{B}_1}} \text{ and } (\mathrm{Th}(V_\Phi))^{V_{\beta_2}^{\mathbb{B}_2}} = (\mathrm{Th}(V_\Phi))^{V_{\alpha_2}^{\mathbb{B}_2}}.$$

Thus,

$$(\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_1}_{\beta_1}} = (\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_1}_{\beta_1}},$$

which completes the proof.

Theorem 3.6. *Assume* ZFC *and that there is a proper class of Woodin cardinals. Suppose* Φ *is a robust specification,* φ *is a large cardinal property, and* ψ *is a* Σ_2*-sentence such that*

$$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \text{ is } \Omega\text{-complete for } \mathrm{Th}(V_\Phi)$$

and $\mathrm{ZFC} + \mathrm{PC}(\varphi)$ *proves that there is a level satisfying* Φ*. Let* $\mathbb{P} \in V_\Phi$ *be a homogeneous partial order that is definable (without parameters) in* V_Φ *and let* $\psi_{\mathbb{P}}$ *be the* Σ_2*-sentence:*

> *There exists* (κ, N, G) *such that* κ *is strongly inaccessible,* $N \models \mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$*,* G *is* N*-generic for* $\mathbb{P}^{(V_\Phi)^N}$*, and* $V_\kappa = N[G]$*.*

Then

$$\mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi) \text{ is } \Omega\text{-complete for } \mathrm{Th}(V_\Phi).$$

Proof. In the statement of the theorem and in the proof we view $\mathbb{P}$ as presented by its definition. Thus, in a given a model M of ZFC, $\mathbb{P}^M$ denotes $\mathbb{P}$ as calculated in M.

Without loss of generality we may assume that $\mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$ is Ω-satisfiable, otherwise there is nothing to prove. (In the cases of interest this condition will be met.)

Lemma. *Suppose* $V^{\mathbb{B}}$ *is a generic extension of* V *such that for some ordinal* α*,* $V^{\mathbb{B}}_\alpha \models \mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$*. Let* (κ, N, G) *be as in the definition of* $\psi_{\mathbb{P}}$*. Then, in* $V^{\mathbb{B}}$*, for each sentence* S *the following are equivalent (where, for notational convenience, we have written* V *for* $V^{\mathbb{B}}$*) :*

(1) $S \in (\mathrm{Th}(V_\Phi))^V$

(2) $S \in (\mathrm{Th}(V_\Phi))^{V_\kappa}$

(3) "$1_{\mathbb{P}^{(V_\Phi)^N}} \Vdash S \in \mathrm{Th}(V_\Phi)$" $\in (\mathrm{Th}(V_\Phi))^N$

(4) $N \models$ "$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega$ " "$1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)$" $\in \mathrm{Th}(V_\Phi)$" " "

(5) $V_\kappa \models$ "$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega$ " "$1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)$" $\in \mathrm{Th}(V_\Phi)$" " "

(6) $V \models$ "$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega$ " "$1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)$" $\in \mathrm{Th}(V_\Phi)$" " ".

Proof. Some remarks on the notation are in order. First, in statements such as (4) the partial order $\mathbb{P}$ is computed via its definition in V_Φ, while the latter is itself computed in various locations (in this case, rank initial segments of generic extensions of N). Second, we are using "$S \in \mathrm{Th}(V_\Phi)$" as shorthand for "either there is an infinite ordinal β such that $V_\beta \models \Phi$ and, letting β be the least such ordinal, $V_\beta \models S$, or there is no such ordinal and both Φ and S hold". This conditional formulation is needed to handle the case of local settings where V is V_Φ.

Let $V^{\mathbb{B}}$ be a generic extension of V such that for some ordinal α, $V_\alpha^{\mathbb{B}} \models \mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$. Let S be a sentence. For notation convenience we shall write V for $V^{\mathbb{B}}$. Let (κ, N, G) be as in the definition of $\psi_{\mathbb{P}}$.

(1) $\leftrightarrow$ (2): Since $N \models \mathrm{ZFC} + \mathrm{PC}(\varphi)$ and V_κ is a small generic extension of N it follows (by the invariance of large cardinal axioms under small forcing) that

$$V_\kappa \models \mathrm{ZFC} + \mathrm{PC}(\varphi).$$

But ZFC+PC(φ) proves that there is a level satisfying Φ. Since specifications are absolute across rank initial segments

$$(V_\Phi)^{V_\kappa} = (V_\Phi)^V$$

and so

$$(\mathrm{Th}(V_\Phi))^{V_\kappa} = (\mathrm{Th}(V_\Phi))^V.$$

(2) $\leftrightarrow$ (3): Since $\mathbb{P}^{(V_\Phi)^N}$ is homogeneous, for each S',

$$V_\kappa \models S' \text{ iff } N \models \text{"}1_{\mathbb{P}^{(V_\Phi)^N}} \Vdash S'\text{"}.$$

In particular, taking S' to be "$S \in \mathrm{Th}(V_\Phi)$",

$$V_\kappa \models \text{"}S \in \mathrm{Th}(V_\Phi)\text{" iff } N \models \text{"}1_{\mathbb{P}^{(V_\Phi)^N}} \Vdash \text{"}S \in \mathrm{Th}(V_\Phi)\text{" "}.$$

But since Φ is a robust specification,

$$(V_\Phi)^{N[G]} = (V_\Phi)^N[G]$$

and so it follows that

$$N \models \text{"}1_{\mathbb{P}^{(V_\Phi)^N}} \Vdash \text{"}S \in \mathrm{Th}(V_\Phi)\text{" " iff } (V_\Phi)^N \models \text{"}1_{\mathbb{P}^{(V_\Phi)^N}} \Vdash \text{"}S \in \mathrm{Th}(V_\Phi)\text{" "}.$$

Thus,

$$V_\kappa \models \text{“}S \in \mathrm{Th}(V_\Phi)\text{”} \;\text{ iff }\; (V_\Phi)^N \models \text{“}1_{\mathbb{P}(V_\Phi)^N} \Vdash \text{“}S \in \mathrm{Th}(V_\Phi)\text{” ”},$$

which completes the proof.

(3) $\leftrightarrow$ (4): We first claim that

$$N \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \text{ is } \Omega\text{-complete for } \mathrm{Th}(V_\Phi)\text{”}.$$

Suppose not. Since N satisfies that “$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$” is Ω-satisfiable it follows that there is a sentence S' such that

$$N \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \not\models_\Omega S' \in \mathrm{Th}(V_\Phi)\text{”}$$

and

$$N \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \not\models_\Omega \neg S' \in \mathrm{Th}(V_\Phi)\text{”}.$$

However, by Theorem 2.2, N and V_κ agree on Ω-logic (since $V_\kappa = N[G]$ and N satisfies ZFC and that there is a proper class of Woodin cardinals). So V_κ agrees with N on the above two statements. So V must also satisfy these statements. But this is a contradiction since V satisfies that “$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$” is Ω-complete for $\mathrm{Th}(V_\Phi)$.

Now N also has levels satisfying “$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi)$” (since κ is strongly inaccessible). So the frozen theory must be $(\mathrm{Th}(V_\Phi))^N$. Thus, generally we have

$$S' \in (\mathrm{Th}(V_\Phi))^N \;\text{ iff }\; N \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“}S' \in \mathrm{Th}(V_\Phi)\text{” ”}$$

and the equivalence of (3) and (4) is a special case.

(4) $\leftrightarrow$ (5): This follows from Theorem 2.2.

(5) $\leftrightarrow$ (6): The right-to-left direction is immediate. For the other direction suppose for contradiction that

$$V_\kappa \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“ “}1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)\text{”} \in \mathrm{Th}(V_\Phi)\text{” ” ”}$$

and

$$V \not\models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“ “}1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)\text{”} \in \mathrm{Th}(V_\Phi)\text{” ” ”}.$$

Since V satisfies that "ZFC+ψ+PC(φ)" is Ω-complete for $\mathrm{Th}(V_\Phi)$, it follows that

$$V \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“ “}1_{\mathbb{P}} \nVdash S \in \mathrm{Th}(V_\Phi)\text{” } \in \mathrm{Th}(V_\Phi)\text{” ” ”}.$$

However, V_κ satisfies that "ZFC + ψ + PC(φ)" is Ω-satisfiable. Let $\mathbb{Q} \in V_\kappa$ and $\alpha < \kappa$ be such that

$$V_\alpha^{\mathbb{Q}} \models \mathrm{ZFC} + \psi + \mathrm{PC}(\varphi).$$

By the previous displayed statement concerning V, it follows that

$$V_\alpha^{\mathbb{Q}} \models \text{“ “}1_{\mathbb{P}} \nVdash S \in \mathrm{Th}(V_\Phi)\text{” } \in \mathrm{Th}(V_\Phi)\text{”},$$

which is a contradiction.

□

The lemma ties the theory of V_Φ as computed in $V^{\mathbb{B}}$ to the Ω-consequence relation and so the generic invariance of the former is inherited from that of the latter. More precisely: Let $\mathbb{B}_1$ and $\mathbb{B}_2$ be complete Boolean algebras such that for some α_1 and α_2, $V_{\alpha_1}^{\mathbb{B}_1}$ and $V_{\alpha_2}^{\mathbb{B}_2}$ satisfy $\mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$. Then

$$\begin{aligned}
&S \in (\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_1}} \\
&\leftrightarrow V^{\mathbb{B}_1} \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“ “}1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)\text{” } \in \mathrm{Th}(V_\Phi)\text{” ”} \\
&\leftrightarrow V^{\mathbb{B}_2} \models \text{“ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \text{“ “}1_{\mathbb{P}} \Vdash S \in \mathrm{Th}(V_\Phi)\text{” } \in \mathrm{Th}(V_\Phi)\text{” ”} \\
&\leftrightarrow S \in (\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_2}}.
\end{aligned}$$

The first and third equivalence hold by the Lemma and the second equivalence holds by the generic invariance of Ω-logic.

But PC(φ) proves that there is a Φ-cardinal and clearly

$$(\mathrm{Th}(V_\Phi))^{V_{\alpha_1}^{\mathbb{B}_1}} = (\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_1}} \quad \text{and} \quad (\mathrm{Th}(V_\Phi))^{V_{\alpha_2}^{\mathbb{B}_2}} = (\mathrm{Th}(V_\Phi))^{V^{\mathbb{B}_2}}.$$

Thus,

$$(\mathrm{Th}(V_\Phi))^{V_{\alpha_1}^{\mathbb{B}_1}} = (\mathrm{Th}(V_\Phi))^{V_{\alpha_2}^{\mathbb{B}_2}}.$$

In other words, $\mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$ is Ω-complete for $\mathrm{Th}(V_\Phi)$. □

Remark 3.7. In the statement of the above theorem we have not assumed that $\mathrm{ZFC} + \psi_{\mathbb{P}} + \mathrm{PC}(\varphi)$ is Ω-satisfiable. Under appropriate large cardinal assumptions this theory is Ω-satisfiable and the theorem applies in a substantive way.

Theorem 3.8. *Assume* ZFC *and that there is a proper class of Woodin cardinals. Suppose* Φ *is a robust specification such that the partial orders* $\mathrm{Add}(\omega_2, \omega)$ *and* $\mathrm{Col}(\omega_1, \mathbb{R})$ *are definable in* V_Φ*,* φ *is a large cardinal property, and* ψ *is a* Σ_2*-sentence such that*

$$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \text{ is } \Omega\text{-complete for } \mathrm{Th}(V_\Phi)$$

and $\mathrm{ZFC} + \mathrm{PC}(\varphi)$ *proves that there is a level satisfying* Φ*. Then there is a* Σ_2*-sentence* ψ' *such that*

$$\mathrm{ZFC} + \psi' + \mathrm{PC}(\varphi) \text{ is } \Omega\text{-complete for } \mathrm{Th}(V_\Phi)$$

and the first theory Ω*-implies* CH *if and only the second theory* Ω*-implies* $\neg$CH*.*

Proof. If $\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \mathrm{CH}$ then let $\mathbb{P} = \mathrm{Add}(\omega_2, \omega)$ and if $\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \neg\mathrm{CH}$ then let $\mathbb{P} = \mathrm{Col}(\omega_1, \mathbb{R})$. Letting ψ' be the Σ_2-sentence $\psi_{\mathbb{P}}$ from Theorem 3.6, we have

$$\mathrm{ZFC} + \psi + \mathrm{PC}(\varphi) \models_\Omega \mathrm{CH} \quad \text{iff} \quad \mathrm{ZFC} + \psi' + \mathrm{PC}(\varphi) \models_\Omega \neg\mathrm{CH},$$

which completes the proof. □

Remark 3.9. In the previous theorem one can work with $H(\omega_2)$ instead of V_Φ. The reason is that $\mathbb{P}$ (in either case) is a homogeneous partial order that is definable over $H(\omega_2)$ and it has the feature that if $G \subseteq \mathbb{P}$ is V-generic then truth in $H(\omega_2)^{V[G]}$ is reducible to truth in $H(\omega_2)^V$, which suffices for the proof of Theorem 3.6.

The above theorem is, of course, just a sample. One can replace CH by anything that can be forced with a definable, homogeneous partial order that satisfies the robustness condition. Thus, if there is one theory with the above degree of Ω-completeness then there is a "bifurcation" into a host of incompatible Ω-complete theories with the same degree of Ω-completeness.

The question of whether such a "bifurcation" can be obtained is sensitive to the Strong Ω Conjecture. On the one hand, if there is a proper class of Woodin cardinals and the Strong Ω Conjecture holds then by Theorem 2.18 there can be no Ω-complete theory for even third-order arithmetic (or even the Σ^3_2-fragment). On the other hand, it is currently an open possibility that the Ω Conjecture fail in such a way that for each robustly specifiable λ there is a recursively enumerable theory that is Ω-complete for the theory of V_λ. Should all such theories to agree on their common domain then this would make a strong case for new axioms completing the axioms of set theory. However, the above result shows that this will not happen. Instead there would be a radical "bifurcation" into a multitude of incompatible Ω-complete theories.[4]

4 Conclusion

There is evidence that the Ω Conjecture holds. There are two key points. First, many of the meta-mathematical consequences of the Ω Conjecture follow from the non-trivial Ω-satisfiability of the Ω Conjecture. This latter statement is a Σ_2-statement and there are no known examples of Σ_2-statements that are provably absolute and not settled by large cardinals. So it is reasonable to expect this statement to be settled by large cardinal axioms. Moreover, it seems unlikely that the Ω conjecture be false while its non-trivial Ω-satisfiability be true. Second, recent results have shown that if inner model theory can reach one supercompact cardinal then it can reach all of the traditional large cardinal axioms and, moreover, the Ω Conjecture holds in all of these models. This provides evidence that no traditional large cardinal can refute the Ω-satisfiability of the Ω Conjecture and (by the first point) this is evidence that the Ω Conjecture is true. Thus there is evidence that the above form of bifurcation will not occur. In fact, there is evidence that the Strong Ω Conjecture holds and thus there is evidence that bifurcation cannot even occur at the level of third-order arithmetic.[5]

Nevertheless, even in the presence of the Ω Conjecture there are "local" bifurcations that one can consider. We close with a brief discussion. There are two settings in which one can consider local bifurcation.

The first setting is that of Theorem 2.15 which shows that (granting large cardinals) CH is a Σ^2_1-sentence such that ZFC+CH is Ω-complete for Σ^2_1 and, moreover, that CH is the unique such sentence (up to Ω equivalence). We mentioned above that if the Strong Ω Conjecture holds then (granting large cardinals) there can be no recursively enumerable theory that is Ω-complete for Σ^2_3. Two questions remain. First, is there an axiom A such that (granting large cardinals) ZFC $+ A$ is Ω-complete for Σ^2_2. Second, assuming that there

[4]For a discussion of the potential philosophical significance of such a scenario see [10].
[5]See [24] for further discussion.

is such a theory, do all such theories agree (in Ω-logic) on their computation of Σ^2_2?

The question of existence is open. Let us assume that it is answered positively and consider the question of uniqueness. For each A such that (granting large cardinals) $\mathrm{ZFC} + A$ is Ω-complete for Σ^2_2 let T_A be the Σ^2_2 theory computed by $\mathrm{ZFC} + A$ in Ω-logic. The question of uniqueness simply asks whether T_A is unique. A refinement of the results in this paper can be used to answer this question negatively. This lead to the natural question of how much variability there is among the T_A. It is not known whether CH must belong to them all and a natural conjecture is that it must. Do some contain $\Diamond$ while others contain $\neg\Diamond$? Do some contain SH (Suslin's hypothesis) while others contain $\neg$SH?

We shall address these questions in a sequel to this paper. But let us note the following: It is known (by a result of Woodin in 1985) that if there is a proper class of measurable Woodin cardinals then there is a forcing extension satisfying all Σ^2_2 sentences φ such that $\mathrm{ZFC} + \mathrm{CH} + \varphi$ is Ω-satisfiable. (See [11].) It follows that if the question of existence is answered positively with an A that is Σ^2_2 then T_A must be this maximum Σ^2_2 theory and, consequently, all T_A agree when A is Σ^2_2. (A natural conjecture is that $\Diamond$ is such an A. But even if $\Diamond$ is not such an axiom A it will be in T_A.) So, assuming that all such T_A contain CH and that there is a T_A where A is Σ^2_2, then, although not all T_A agree (when A is arbitrary) there is one that stands out, namely, the one that is maximum for Σ^2_2 sentences.

The second setting is that of Theorem 2.17 which shows that (granting large cardinals and the Strong Ω Conjecture) there is an axiom A such that $\mathrm{ZFC} + A$ is Ω-complete for $H(\omega_2)$ and, moreover, any such axiom has the feature that $\mathrm{ZFC} + A \models_\Omega$ "$H(\omega_2) \models \neg\mathrm{CH}$". For each such axiom A let T_A be the theory of $H(\omega_2)$ as computed by $\mathrm{ZFC} + A$ in Ω-logic. Thus, the theorem shows that all such T_A agree in containing $\neg$CH. The question then naturally arises whether T_A is unique. A refinement of the techniques of this paper can be used to answer this question negatively. But again, there is a T_A that stands out, namely, the maximum theory given by the axiom $(*)$. (See [22].)

We shall prove the above localizations and explore the above questions in a sequel to this paper.

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