AN OUTLINE OF RATHJEN'S PROOF THAT CH IS INDEFINITE, GIVEN MY CRITERIA FOR DEFINITENESS

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3d CSLI Workshop on
Logic, Rationality and Intelligent Interaction.
Stanford, 31 May 2014

In memory of my friend and colleague,
Grigori Mints
June 7, 1939-May 29, 2014

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Two Informal Notions of Definiteness

- Notion of a definite totality
- Notion of a definite proposition or property
- Criteria for these can be given in logical terms

The Criteria

- A totality is definite iff quantification over that totality is a definite logical operation.
- A proposition or property is definite iff the Principle of Bivalence holds for it.

The Criteria Interact in Formal Systems

- Internally, quantified variables in definite formulas are restricted to range over definite totalities.
- Externally, classical logic applies only to definite formulas.

A Logical Framework

- A logical framework was introduced in (F 2010) in which different philosophical viewpoints as to which totalities are definite and which not can be represented and investigated by proof-theoretic methods.

Some Philosophical Viewpoints

- According to the finitists, the natural numbers form an “unfinished” or indefinite totality, and quantification over the natural numbers is indefinite, while bounded quantification is definite.
- According to the predicativists, the natural numbers form a definite totality, but not the supposed collection of arbitrary sets of natural numbers.

More Philosophical Viewpoints

- Set theory identifies definite totalities with sets. Then V is not a definite totality by Russell's Paradox. The question then is, which sets exist?
- If the set $\mathbb{N}$ of natural numbers is presumed to exist, but not the power set operation, this leads to predicative set theory.

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One More Viewpoint

- According to the ("classical") Descriptive Set Theorists, the set $\mathbb{R}$ of real numbers is a definite totality but not the supposed totality of arbitrary subsets of $\mathbb{R}$.
- This is equivalent to predicative set theory plus the power set of $\mathbb{N}$.

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Toward Axiomatic Formulations

- Restrict quantifiers in the formulas that are supposed to represent definite properties, e.g. in Comprehension or Separation axioms.
- Quantification over indefinite domains may still be regarded as meaningful, in order to state closure conditions, e.g. under union.

Semi-Intuitionistic Systems

- The underlying logic is intuitionistic.
- This is augmented by classical logic for definite formulas.
- General pattern: start with a system S in classical logic with suitably restricted Comprehension, etc. schemes. Then form associated semi-intuitionistic system SIS.

Semi-Constructive Systems

- Next beef up SIS to a Semi-Constructive System SCS by adjunction of useful principles that can be verified by a constructive functional interpretation.
- Show S, SIS and SCS are equivalent in proof-theoretic strength.

The Basic Semi-Intuitionistic System for Predicative Set Theory

- Start with S = KP, the classical system of predicative (or "admissible") set theory (including the Axiom of Infinity)
- SIS has the same axioms as KP, but is based on intuitionistic logic plus the Law of Excluded Middle for bounded formulas,
- (Δ_0-LEM) $\varphi \vee \neg\varphi$, for all Δ_0 formulas φ.
- SIS = IKP + (Δ_0-LEM)

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Axioms of KP

1. Extensionality
2. Unordered pair
3. Union
4. Infinity
5. Δ_0-Separation
6. Δ_0-Collection
7. The $\in$-Induction Axiom Scheme

A Semi-Constructive System of Predicative Set Theory

- Beef up SIS to a system SCS that includes the Full Axiom of Choice Scheme for sets ,
- (AC_{Set}) $\forall x{\in}a \exists y\ \varphi(x,y) \rightarrow \exists r[Fun(r) \wedge dom(r)=a \wedge \forall x{\in}a\ \varphi(x, r(x))]$

for φ an arbitrary formula,

Notes on SCS-1

- Then SCS proves the Full Collection Axiom Scheme,

$$\forall x \in a \exists y\ \varphi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b\ \varphi(x,y),$$

for φ arbitrary, while only for Σ_1 formulas in SIS.

Even more, SCS proves the Strong Collection Axiom.

Notes on SCS-2

- IKP + AC_{Set} proves (Δ_0-LEM) (by adaptation of the old Diaconescu argument).
- If we add the power set axiom (see next) SCS is a subtheory of Tharp's IZF (1971) based on a system proposed by L. Poszgay in 1967.
- Tharp gave a realizability interpretation of IZF in ZF + V=L.

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Adding Power Set Axioms

- The Power Set axiom Pow is given via a new constant symbol $\mathcal{P}$, and written as $x \in \mathcal{P}(a) \leftrightarrow x \subseteq a$.
- Pow(ω) is the special case of Pow: $x \in \mathcal{P}(\omega) \leftrightarrow x \subseteq \omega$. We also write $\mathbb{R}$ for $\mathcal{P}(\omega)$.
- The semi-constructive system for classical DST is the system SCS + Pow(ω).

What Properties are Definite?

- From the overall logical point of view taken here, $\varphi(x)$ is formally definite relative to a given system if $\forall x[\varphi(x) \vee \neg\varphi(x)]$ is provable there.

- Conjecture: If $\varphi(x)$ is formally definite relative to SCS (SCS + Pow(ω)) then it is equivalent to a formula that is provably Δ_1 (Δ_1 in $\mathcal{P}(\omega)$).

- That would tell us that definite formulas are model-theoretically absolute.

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Doing Mathematics Semi-Constructively

- Let T = SCS + Pow(ω).
- Conjecture: All of "classical" DST can be carried out in T.
- NB. The Descriptive Set Theorists of the 20s and 30s were called "semi-intuitionists".

What Statements are Definite?

- A sentence φ is formally definite in one of our systems if $\varphi \vee \neg\varphi$ is provable there.
- Conjecture (F 2011). The Continuum Hypothesis (CH) is not definite in T.
- Note that CH is meaningful in SCS + Pow(ω) and *is* formally definite in SCS + Pow(Pow(ω)).
- Theorem (Rathjen 2014). CH is not definite in T.

Rathjen's Proof-1

- Work informally in set theory, using definable classes as usual.
- If A is a set, distinguish two notions of the sets constructible from A: L(A) and L[A].
- The set A belongs to L(A) but is treated as a predicate in the inductive definition of L[A]; in general A does not belong to L[A].

Rathjen's Proof-2

- L[A] shares a number of properties with L, including that it has a Σ_1 well-ordering (relative to L[A]).
- Recursion theory can be generalized to L[A] using Σ_1 definable partial functions with parameters, relative to L[A].
- These are given by indices e, $[e]^{L[A]}(x,\ldots)\simeq y$.

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Rathjen's Proof-3

- A notion of realizability over L[A] is defined using indices e of partial $\Sigma_1^{L[A]}$ functions: $e \Vdash_A \varphi$ (by adaptation of Tharp's realizability notion for IZF).
- Theorem. Associate with each proof D of a $\varphi(\underline{x})$ in T an e_D in HF such that for any A and $\underline{a}$ in A, $[e_D]^{L[A]}(\underline{a}, \mathbb{R}^{L[A]}) \Vdash_A \varphi(\underline{a})$.

Rathjen's Proof-4

- To show CH not definite in T, suppose to the contrary that D is a proof in T of CH ∨ ¬CH.

- Then can produce a hered. finite e_D such that for any A and for $[e_D]^{L[A]}(\mathbb{R}^{L[A]}) \simeq b$ we have $b \Vdash_A CH \vee \neg CH$; b is independent of A for suitable A, and $(b)_0 = 0$ or 1.

Rathjen's Proof-5

- Using forcing, first construct A such that in L[A], the cardinality of $\mathbb{R}$ is ω_2; then CH is false in L[A] and there is *no* d in L[A] with d $\Vdash_A$ CH.
- Also need to make sure that A is chosen so that the value b of $[e_D]^{L[A]}(\mathbb{R}^{L[A]})$ will be independent of A under suitable conditions.

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Rathjen's Proof-6

- So $(b)_0 = 1$.
- Now form a suitable forcing extension L[A∪B] of L[A] in which there are no new real numbers but CH is true.
- $\mathbb{R}^{L[A]} = \mathbb{R}^{L[A \cup B]}$.
- Can actually *realize* CH in L[A∪B]. So $(b)_0 = 0$ and we have a contradiction!

Rathjen's Proof--PS

- The following was communicated to me by Michael Rathjen following the lecture:
- His methods also show that if φ is any analytic sentence consistent with ZFC then CH is indefinite w.r.t. T + φ.
- In particular, this holds for φ expressing Borel Determinacy and any φ in the scheme for Projective Determinacy.

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