The Search for Deep Inconsistency

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March 22, 2014

Introduction

The space of mathematical theories can be ordered under the relation of relative interpretability.

In general, this ordering is chaotic—it is neither linear nor well-founded—but, remarkably, the theories that arise in mathematical practice line up in a well-ordering.

QUESTION: How far does this hierarchy go?

A canonical way to climb the hierarchy is through *principles of pure strength*, most notably, *large cardinal axioms*.

So a related question is:

QUESTION: How far does the hierarchy of large cardinals extend?

Motivations

(1) Intrinsic Interest.

— How high can you count?

(2) Chart out the hierarchy of interpretability.

— At a certain stage we violate the limiting principle $V = L$? Are there stages that violate AC?

(3) The search for deep inconsistency.

— From time to time there are purported proofs that PA is inconsistent, that ZFC is inconsistent, that measurable cardinals are inconsistent, etc. These proofs generally falter. And the one's that haven't—like Kunen's—are relatively transparent. Is there a *deep* inconsistency?

Large Cardinals

Large cardinals divide into the *small* and the *large*. The defining characteristic of *small* large cardinals is that they are compatible with $V = L$.

Small

Some small large cardinals:

- Inaccessible
- Mahlo
- Weakly Compact
- Indescribable
- Subtle
- Ineffable

TEMPLATE #1: Reflection Principles:

$$V \models \varphi(A) \;\rightarrow\; \exists\alpha \; V_\alpha \models \varphi^\alpha(A^\alpha)$$

Theorem (Scott)

Measurable cardinals are not small.

Large

Some large large cardinals:

- Measurable
- Strong
- Supercompact
- Huge
- Rank to Rank
- I_0

TEMPLATE #2: Elementary Embeddings: There exists a non-trivial elementary embedding

$$j : V \to M$$

where M is a transitive class.

Simple Inconsistency

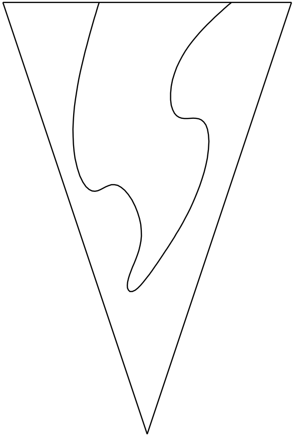
When pushed to the limit both Template #1 and Template #2 lead to inconsistency.

Template #1: Reflection Principles

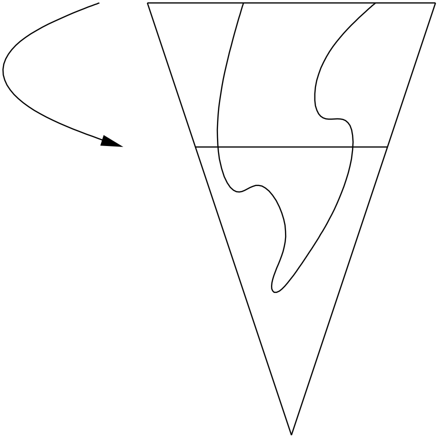
Schematically, a reflection principle has the form

$$V \models \varphi(A) \ \rightarrow \ \exists\alpha \ V_\alpha \models \varphi^\alpha(A^\alpha)$$

where $\varphi^\alpha(\,\cdot\,)$ is the result of relativizing the quantifiers of $\varphi(\,\cdot\,)$ to V_α and A^α is the result of relativizing an arbitrary parameter A to V_α.



$$V \models \varphi(A)$$



$V \models \varphi(A)$

$V_\alpha \models \varphi^\alpha(A^\alpha)$

Use $x, y, z, \ldots$ as variables of the first order and, for $m > 1$, $X^{(m)}$, $Y^{(m)}$, $Z^{(m)}$, $\ldots$ as variables of the m^{th} order.

Relativization: If $A^{(2)}$ is a second-order parameter over V_α, then the relativization of $A^{(2)}$ to V_β, written $A^{(2),\beta}$, is $A \cap V_\beta$. For $m > 1$, $A^{(m+1),\beta} = \{B^{(m),\beta} \mid B^{(m)} \in A^{(m+1)}\}$.

Strength is obtained by moving to higher-order languages and higher-order parameters.

Some Basic Facts:

(1) First-order formulas with first-order parameters:
- Infinity
- Replacement

(2) Higher-order formulas with second-order parameters:
- Inaccessibles
- Mahlos
- Weakly Compact
- Indescribables

One immediately faces an obstacle when moving to *third*-order parameters: One has to restrict the language to "positive" formulas:

Definition

A formula in the language of finite orders is *positive* iff it is built up by means of the operations $\vee$, $\wedge$, $\forall$ and $\exists$ from atoms of the form $x = y$, $x \neq y$, $x \in y$, $x \notin y$, $x \in Y^{(2)}$, $x \notin Y^{(2)}$ and $X^{(m)} = X'^{(m)}$ and $X^{(m)} \in Y^{(m+1)}$, where $m \geq 2$.

Theorem (Tait)

Suppose $n < \omega$ and $V_\kappa \models \Gamma_n^{(2)}$-reflection. Then κ is n-ineffable.

Two questions:

(1) How strong is $\Gamma_n^{(2)}$-reflection?

(2) How strong are the stronger generalized reflection principles?

Theorem

Assume $\kappa = \kappa(\omega)$ exists. Then there is a $\delta < \kappa$ such that V_δ satisfies $\Gamma_n^{(2)}$-reflection for all $n < \omega$.

Corollary

$\Gamma^{(2)}$-reflection is compatible with $V = L$.

What about the other principles in the hierarchy?

$\Gamma_1^{(2)}$*-reflection, . . . ,* $\Gamma_n^{(2)}$*-reflection, . . .*
$\Gamma_1^{(3)}$*-reflection, . . . ,* $\Gamma_n^{(3)}$*-reflection, . . .*
. . .

Theorem
$\Gamma_1^{(3)}$*-reflection is inconsistent.*

Template #2: Elementary Embeddings

The template has the form: There is a non-trivial elementary embedding

$$j : V \to M$$

where M is a transitive class.

The least ordinal moved—$\mathrm{CRT}(j)$, the critical point of j—is the large cardinal.

Strength is obtained by demanding the M resemble V more and more.

Some Milestones:

(1) Measurable
- $(V_{\kappa+1})^M = V_{\kappa+1}$

(2) λ-Strong
- $(V_\lambda)^M = V_\lambda$

(2) λ-Supercompact
- ${}^\lambda M \subseteq M$

In the limit, the ultimate large cardinal axiom would involve the ultimate degree of resemblance, where $M = V$. Reinhardt proposed this axiom in his dissertation.

Theorem (Kunen)

Assume AC. *Then there is no non-trivial elementary embedding*

$$j : V \to V.$$

Summary

Each template, when pursued far enough, leads to inconsistency. However, in each case the inconsistency is simple.

In the search for a deep inconsistency we shall turn to very large large cardinals.

In moving from small large cardinals we broke the $V = L$ barrier, and now, in moving from large large cardinals to *very* large large cardinals, we shall break the AC barrier.

The Choiceless Hierarchy—Reinhardt Cardinals

Joint work with Bagaria and Woodin.

Our hierarchy *starts* with the large cardinal that Kunen showed to be inconsistent with AC. Work in ZF.

Definition

A cardinal κ is *Reinhardt* if there exists a non-trivial elementary embedding $j : V \to V$ such that $\mathrm{CRT}(j) = \kappa$.

Definition

A cardinal κ is *super Reinhardt* if for all ordinals λ there exists a non-trivial elementary embedding $j : V \to V$ such that $\mathrm{CRT}(j) = \kappa$ and $j(\kappa) > \lambda$.

Theorem

Suppose that κ is a super Reinhardt cardinal. Then there exists $\gamma < \kappa$ such that

$$(V_\gamma, V_{\gamma+1}) \models \mathsf{ZF}_2 + \text{“\textit{There is a Reinhardt cardinal.}”}$$

Question

Assume κ is a super Reinhardt cardinal. Must there exist a Reinhardt cardinal below κ?

Berkeley Cardinals —Proto-Berkeley Cardinals

For a transitive set M, let $\mathscr{E}(M)$ be the collection of all non-trivial elementary embeddings $j : M \to M$.

Definition

An ordinal δ is a *proto-Berkeley cardinal* if for all transitive sets M such that $\delta \in M$ there exists $j \in \mathscr{E}(M)$ with $\mathrm{CRT}(j) < \delta$.

Notice that if δ_0 is the least proto-Berkeley cardinal then every ordinal greater than δ_0 is also a proto-Berkeley cardinal. We wish to isolate the *genuine* Berkeley cardinals, those which are like δ_0.

Lemma

For any set A there exists a transitive set M such that $A \in M$ and A is definable (without parameters) in M.

Theorem

Let δ_0 be the least proto-Berkeley cardinal. For all transitive sets M such that $\delta_0 \in M$ and for all $\eta < \delta_0$ there exists an elementary embedding $j : M \to M$ such that

$$\eta < \mathrm{CRT}(j) < \delta_0.$$

Berkeley Cardinals

Definition

A cardinal δ is a Berkeley cardinal if for every transitive set M such that $\delta \in M$, and for every ordinal $\eta < \delta$ there exists $j \in \mathscr{E}(M)$ with $\eta < \mathrm{CRT}(j) < \delta$.

Remark

Notice that:

(1) *The least proto-Berkeley cardinal is a Berkeley cardinal.*

(2) *If δ is a limit of Berkeley cardinals, then δ is a Berkeley cardinal. In other words, the class of Berkeley cardinals is "closed".*

Lemma

Let δ_0 be the least Berkeley cardinal. Then, for a tail of $\beta \in \mathsf{Lim}$, if $j : V_\beta \to V_\beta$ is an elementary embedding with $\mathrm{CRT}(j) < \delta_0$, then

(1) *$j(\delta_0) = \delta_0$, and*

(2) *the set $\{\eta < \delta_0 \mid j(\eta) = j(\eta)\}$ is cofinal in δ_0.*

Theorem

Suppose that δ_0 is the least Berkeley cardinal. Then there exists $\gamma < \delta_0$ such that

$$V_{\gamma+1} \models \mathsf{ZF}_2 + \text{"}\textit{there exists an extendible cardinal and there exists a Reinhardt cardinal.}\text{"}$$

HOD Conjecture

There is a proper class of regular uncountable cardinals λ such that for all $\kappa < \lambda$, if $(2^{\kappa})^{\mathrm{HOD}} < \lambda$ then there is a partition

$$\langle S_\alpha \mid \alpha < \kappa \rangle \in \mathrm{HOD}$$

of S^λ_ω into stationary sets.

It follows from the above theorem and deep results of Woodin that if the HOD Conjecture is true then there are no Berkeley cardinals. Thus, if one could prove the HOD Conjecture one would almost certainly have a deep inconsistency.

But let us proceed upward...

Club Berkeley Cardinals

It is unclear whether the least Berkeley cardinal rank-reflects a super Reinhardt cardinal.

Definition

A cardinal δ is a *club Berkeley cardinal* if δ is regular and for all clubs $C \subseteq \delta$ and for all transitive M with $V_{\delta+1} \in M$ there exists $j \in \mathscr{E}(M)$ with $\mathrm{CRT}(j) \in C$.

Theorem

Suppose δ is a club Berkeley cardinal. Then

$$V_{\delta+1} \models \mathsf{ZF}_2 + \text{“}\textit{there is a super Reinhardt cardinal.}\text{”}$$

The next questions that arise are (1) whether it is possible to rank-reflect a Berkeley cardinal and (2) whether it is possible to have a Berkeley cardinal which is also extendible (or, even better, super Reinhardt).

Lemma

Suppose δ_0 is the least Berkeley cardinal. Then δ_0 is not extendible.

The proof motivates the notion implicit in the following theorem—a *limit club Berkeley cardinal*.

Theorem

Suppose δ is a club Berkeley cardinal which is a limit of Berkeley cardinals. Then

$V_{\delta+1} \models \mathsf{ZF}_2 +$ "*there is a Berkeley cardinal that is super Reinhardt.*"

Cofinality

A key question that emerges is the cofinality of δ_0. Must it be regular? Can it have cofinality ω?

This question is connected with choice:

Theorem

Suppose that δ_0 is the least Berkeley cardinal $\mathrm{cof}(\delta_0) = \gamma$. *Then* γ-DC *fails.*

Theorem

Suppose that δ is a club Berkeley cardinal which is a limit of Berkeley cardinals. Then there is a forcing extension $V[G]_{\delta+1}$ such that

$$V[G]_{\delta+1} \models \mathrm{cof}(\delta_0) = \omega$$

where δ_0 is the least Berkeley cardinal (as computed in $V[G]_{\delta+1}$).

Theorem

Suppose that δ is a club Berkeley cardinal which is a limit of Berkeley cardinals. Then there is a forcing extension $V[G]_{\delta+1}$ such that

$$V[G]_{\delta+1} \models \mathrm{cof}(\delta_0) > \omega$$

where δ_0 is the least Berkeley cardinal (as computed in $V[G]_{\delta+1}$).

Thus, assuming that there is a club Berkeley cardinal which is a limit of Berkeley cardinals, the question of the cofinality of the least Berkeley cardinal is independent.

This might strike one as suspicious.

Perhaps there is an inconsistency lurking below the surface...

Toward Deep Inconsistency

There are two plausible hypotheses concerning the cofinality of δ_0, each of which, if true, would lead to a new inconsistency result.

We saw that (assuming very strong hypotheses) the cofinality of δ_0 is independent. We also saw that the cofinality of δ_0 is connected with the failure of choice—the lower the cofinality, the greater the failure of choice.

Now, since these principles violate AC, it would seem that we can obtain stronger and stronger principles by folding in more and more AC. This leads to a hierarchy:

(1) $\mathsf{ZF} + \mathsf{BC} + \mathrm{cof}(\delta_0) = \omega$
(2) $\mathsf{ZF} + \mathsf{BC} + \mathsf{DC} + \mathrm{cof}(\delta_0) = \omega_1$
(3) $\mathsf{ZF} + \mathsf{BC} + \omega_1\text{-}\mathsf{DC} + \mathrm{cof}(\delta_0) = \omega_2$
(4) Etc.

If this is indeed a hierarchy of stronger and stronger principles, then one would expect:

PLAUSIBLE HYPOTHESIS #1: $\mathsf{ZF} + \mathsf{BC} + \mathrm{cof}(\delta_0) > \omega$ proves that there exists $\gamma < \delta_0$ such that

$$V_\gamma \models \text{“}\mathsf{ZF} + \mathsf{BC} + \mathrm{cof}(\delta_0) = \omega\text{”}.$$

Theorem

Plausible Hypothesis #1 implies the inconsistency of

$$\mathrm{ZF} + \mathrm{BC} + \mathrm{DC}.$$

This would constitute a *moderately* deep inconsistency.

But Plausible Hypothesis #1 might fail. One plausible reason for its failure is:

PLAUSIBLE HYPOTHESIS #2: ZF + BC proves $\mathrm{cof}(\delta_0) = \delta_0$.

Theorem

Plausible Hypothesis #2 implies the inconsistency of

$$\mathsf{ZF} + \text{“}\textit{There is a limit club Berkeley cardinal}\text{”}.$$

This would constitute a *genuinely* deep inconsistency.

Summary:

(1) PH #1 points toward a moderately deep inconsistency.
(2) PH #2 points toward a genuine deep inconsistency.
(3) The HOD Conjecture points toward a very deep inconsistency.

Thank you.