The search for mathmematical truth

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The projective sets

Definition

A set $A \subseteq \mathbb{R}^n$ is *projective* if it can be generated from the open subsets of $\mathbb{R}^n$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n.$$

Definition

Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function f *uniformizes* A if for all $x \in \mathbb{R}$:

- if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$.

Two questions of Luzin

Two questions of Luzin

1. *Suppose* $A \subseteq \mathbb{R} \times \mathbb{R}$ *is projective. Can* A *be uniformized by a projective function?*
2. *Suppose* $A \subseteq \mathbb{R}$ *is projective. Is* A *Lebesgue measurable and does* A *have the property of Baire?*

Luzin's questions are questions about $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

Both questions are unsolvable on the basis of the ZFC **axioms**

Determinacy and the answers to Luzin's questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players.

- The players alternate choosing $\epsilon_i \in \{0, 1\}$.
- After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined.
- Player I wins this run of the game if

$$\Sigma_{i=1}^{\infty} \epsilon_i / 2^i \in A$$

  otherwise Player II wins.

Definition

The set A is *determined* if there is a winning strategy for one of the players in the game associated to A.

The Axiom of Determinancy (AD)

Definition (Mycielski-Steinhaus)

Axiom of Determinacy (AD): Every set $A \subseteq \mathbb{R}$ is determined.

Lemma (Axiom of Choice)

There is a set $A \subset \mathbb{R}$ such that A is not determined.

Corollary

AD *is false.*

Projective Determinacy (PD)

Definition

Projective Determinacy (PD): Every projective set $A \subseteq \mathbb{R}$ is determined.

Theorem

Assume every projective set is determined.

1. (Mycielski-Steinhaus) *Every projective set has the property of Baire.*
2. (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*
3. (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

Key questions

Is PD even consistent and if consistent, is PD true?

Basic notions: Logical definability from parameters

- A set N is *transitive* if $A \subset N$ for all $A \in N$.
- A transitive set N is an *ordinal* if $(N, \in)$ is a totally ordered set.
- ω is the least infinite ordinal, $(\omega, \in) \cong (\mathbb{N}, <)$.
- ω_1 is the least uncountable ordinal.

Definition

Suppose N is a transitive set. A subset $X \subseteq N$ is logically definable in $(N, \in)$ from parameters if for some formula $\varphi[x_0, \ldots, x_n]$ and for some parameters $a_1, \ldots, a_n \in N$,

$$X = \{a \in N \mid (N, \in) \models \varphi[a, a_1, \ldots, a_n]\}$$

Basic notions: Elementary embeddings

Definition

Suppose N and M are transitive sets. A function $j : N \to M$ is an *elementary embedding* if for all logical formulas $\varphi[x_0, \ldots, x_n]$ and all $a_0, \ldots, a_n \in N$,

$$(N, \in) \models \varphi[a_0, \ldots, a_n] \text{ if and only if } (M, \in) \models \varphi[j(a_0), \ldots, j(a_n)]$$

Lemma

Suppose that $j : N \to M$ is an elementary embedding and that $N \models \mathrm{ZFC}$. Then the following are equivalent.

(1) *j is not the identity.*

(2) *There is an ordinal $\beta \in N$ such that $j(\beta) \neq \beta$.*

Strong axioms of infinity: large cardinal axioms

Basic template for large cardinal axioms

A cardinal κ is a large cardinal *if there exists an elementary embedding,*

$$j : V \to M$$

such that M is a transitive class and κ is the least ordinal such that $j(\alpha) \neq \alpha$.

- Requiring M be *close* to V yields a hierarchy of large cardinal axioms:
  - simplest case is where κ is a *measurable cardinal*.
- $M = V$ contradicts the Axiom of Choice.

The validation of Projective Determinacy

Theorem (Martin-Steel)

Assume there are infinitely many Woodin cardinals. Then every projective set is determined.

Theorem

The following are equivalent.

(1) *Every projective set is determined.*

(2) *For each $k < \omega$ there is a countable (iterable) transitive set N such that*

$$N \models \mathrm{ZFC} + \text{“There exist } k \text{ Woodin cardinals”},$$

PD **is the missing (and true) axiom for** $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

- *Is there such an axiom for V itself?*

Mathematical truth and two modest claims

- Large cardinal axioms predict facts about our world.

Prediction

There will be no contradiction discovered from PD (by any means) before the year 3010.

There will be no contradiction discovered from PD (by any means) before all the Clay Millennium problems have been solved.

- A controversial claim.

Claim

Consistency claims for large cardinal axioms **require** *a conception of the Universe of Sets in which large cardinals axioms are* true*.*

- *This ultimately requires that questions such as that of the Continuum Hypothesis also be resolved*
  - *or an explanation of the exact nature of the ambiguity.*

Basic notions: The cumulative hierarchy

- If X is a set then $\mathcal{P}(X)$ denotes the set of all subsets of X:

$$\mathcal{P}(X) = \{Y \mid Y \subseteq X\}.$$

The von Neumann cumulative hierarchy of sets

1. $V_0 = \emptyset$.
2. (Successor case) $V_{\alpha+1} = \mathcal{P}(V_\alpha)$.
3. (Limit case) $V_\alpha = \cup\{V_\beta \mid \beta < \alpha\}$.

The effective cumulative hierarchy: L

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that X is logically definable in the structure $(X, \in)$ from parameters in X.

- (Axiom of Choice) $\mathcal{P}_{\mathrm{Def}}(X) = \mathcal{P}(X)$ if and only if X is finite.
- $\mathcal{P}_{\mathrm{Def}}(V_{\omega+1}) \cap \mathcal{P}(\mathbb{R})$ is exactly the projective sets.

Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$,
2. (Successor case) $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$,
3. (Limit case) $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

L is the class of all sets X such that $X \in L_\alpha$ for some ordinal α.

The axiom $V = L$, the projective sets, and large cardinals

Theorem

Assume $V = L$.

(1) (Gödel) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

(2) (Gödel) *There is a projective set which is not Lebesgue measurable:*

- *there is a projective wellordering of the reals.*

(3) (Scott) *There are no measurable cardinals:*

- *there are no (interesting) large cardinals.*

(meta) Corollary

$V \neq L$.

Basic notions: Enlargements of L, generalizing the projective sets

Definition

1. $L_0(\mathbb{R}) = \mathbb{R}$,
2. (Successor case) $L_{\alpha+1}(\mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(\mathbb{R}))$,
3. (Limit case) $L_\alpha(\mathbb{R}) = \cup\{L_\beta(\mathbb{R}) \mid \beta < \alpha\}$.

$L(\mathbb{R})$ is the class of all sets a such that $a \in L_\alpha(\mathbb{R})$ for some α.

- $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a transfinite extension of the hierarchy of the projective sets.
- Assuming there is a proper class of Woodin cardinals then $L(\mathbb{R}) \models \mathrm{AD}$.

Suppose $\Gamma \subseteq \mathcal{P}(\mathbb{R})$. Then one defines $L(\Gamma, \mathbb{R})$ by setting $L_0(\Gamma, \mathbb{R}) = \mathbb{R} \cup \Gamma \cup \{\Gamma\}$.

Forcing axioms

Definition

Suppose $\mathbb{B}$ is a complete Boolean algebra and κ is a cardinal.

The κ-Baire Category Theorem holds for $\mathbb{B}$ if the following holds in Ω where Ω is the Stone space of $\mathbb{B}$.

- Suppose $\mathcal{A}$ is a family of open dense subsets of Ω and $|\mathcal{A}| \leq \kappa$. Then $\cap\mathcal{A}$ is dense in Ω.

- The ω_1-Baire Category Theorem *cannot* hold for all complete Boolean algebras.

Question

For which complete Boolean algebras $\mathbb{B}$ can the ω_1-Baire Category Theorem hold for $\mathbb{B}$?

Stationary sets in ω_1

Definition

1. A cofinal set $C \subseteq \omega_1$ is *closed and unbounded* if if for all limit ordinals $\alpha < \omega_1$, if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.
2. A set $S \subset \omega_1$ is *stationary* if

$$S \cap C \neq \emptyset$$

for all closed unbounded sets $C \subset \omega_1$.

Assuming the Axiom of Choice, there exist sets $S \subset \omega_1$ such that both S and $\omega_1 \backslash S$ are stationary.

Stationary set preserving

Definition

A complete Boolean algebra $\mathbb{B}$ is *stationary set preserving* if the following holds for all $c \in \mathbb{B}$ with $c > 0$, for all sequences

$$\langle b_\alpha : \alpha < \omega_1 \rangle$$

of elements of $\mathbb{B}$, and for all stationary sets $S \subseteq \omega_1$.

- If $c \leq \vee\{b_\alpha \mid \beta < \alpha < \omega_1\}$ for all $\beta < \omega_1$,

then there exists $\eta \in S$ and $0 < d \leq c$ such that

- $d \leq \vee\{b_\alpha \mid \beta < \alpha < \eta\}$ for all $\beta < \eta$.

Martin's Maximum

Theorem (Foreman, Magidor, Shelah)

Suppose that $\mathbb{B}$ is a complete Boolean algebra and that the ω_1-Baire Category Theorem holds for $\mathbb{B}$.

Then $\mathbb{B}$ is stationary set preserving.

Definition (Foreman, Magidor, Shelah)

Martin's Maximum: The ω_1-Baire Category Theorem holds for all stationary set preserving complete Boolean algebras.

Theorem (Foreman, Magidor, Shelah)

Suppose there is a supercompact cardinal. Then there is a stationary set preserving complete Boolean algebra $\mathbb{B}$ such that

$$V^{\mathbb{B}} \models \text{Martin's Maximum}.$$

Consequences of Martin's Maximum

Theorem (Foreman, Magidor, Shelah)

Assume Martin's Maximum. Then $2^{\aleph_0} = \aleph_2$.

Theorem (Martin's Maximum)

Suppose that

$$\pi : C([0,1)] \to \mathcal{A}$$

is an algebra homomorphism of $C([0,1])$ *into a Banach algebra* $\mathcal{A}$.

Then π *is continuous.*

Second generation results

Lemma

There are 2 infinite total orders such that every infinite total order contains an isomorphic copy of them.

- $(\mathbb{Z}^+, <)$ and $(\mathbb{Z}^-, <)$

Theorem (Martin's Maximum: J. Moore)

There are 5 uncountable total orders such that every uncountable total order contains an isomorphic copy of one of them.

The Brown-Douglas-Filmore Problem

Question (Brown-Douglas-Filmore)

Suppose that

$$\pi : \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$$

is an automorphism. Must π be an inner automorphism?

Theorem (CH: Phillips and Weaver)

There is an automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ which is not an inner automorphism.

Theorem (Martin's Maximum: I. Farah)

Every automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ is an inner automorphism.

Is Martin's Maximum true?

Definition

Suppose that κ is a infinite regular cardinal. $H(\kappa)$ is the set of all sets X such that there is a transitive set Y such that

1. $X \in Y$,
2. $|Y| < \kappa$.

Several interesting cases.

1. $H(\omega_1)$. This is logically equivalent to $V_{\omega+1}$.
2. $H(c^+)$. This is logically equivalent to $V_{\omega+2}$.
3. $H(\omega_2)$.
   - Assuming CH, $H(\omega_2)$ is logically equivalent to $V_{\omega+2}$.
   - Assuming Martin's Maximum, $H(\omega_2)$ is *not* logically equivalent to $V_{\omega+2}$.

The structure $(H(\omega_2), \mathcal{I}_{\mathrm{NS}})$

Definition

$\mathcal{I}_{\mathrm{NS}}$ is the ideal of all non-stationary subsets of ω_1.

Lemma (Martin's Maximum)

Suppose that φ is a Π_2 sentence and that there is a stationary set preserving Boolean algebra $\mathbb{B}$ such that

$$V^{\mathbb{B}} \models \text{“}H(\omega_2) \models \varphi\text{”}$$

Then $H(\omega_2) \models \varphi$.

Π_2-maximality

Observation

Martin's Maximum is attempting to maximize *the* Π_2*-theory of* $(H(\omega_2), \mathcal{I}_{\mathrm{NS}})$.

Definition

A Π_2-sentence φ is Ω-*satisfiable* for $(H(\omega_2), \mathcal{I}_{\mathrm{NS}})$ if there is a complete Boolean algebra $\mathbb{B}$ such that

$$V^{\mathbb{B}} \models \text{“}(H(\omega_2), \mathcal{I}_{\mathrm{NS}}) \models \varphi\text{”}.$$

- No requirement that $\mathbb{B}$ be stationary set preserving.

Theorem (Π_2-Maximality)

Assume there is a proper class of Woodin cardinals. There is a partial order $\mathbb{P}_{\max} \in L(\mathbb{R})$ such that the following hold.

(1) $\mathbb{P}_{\max}$ *is homogeneous and* ω*-closed.*

(2) $L(\mathbb{R})^{\mathbb{P}_{\max}} \models \mathrm{ZFC}$.

(3) *Suppose that* φ *is a* Π_2*-sentence which is* Ω*-satisfiable for* $(H(\omega_2), \mathcal{I}_{\mathrm{NS}})$. *Then*

$$L(\mathbb{R})^{\mathbb{P}_{\max}} \models \text{“}(H(\omega_2), \mathcal{I}_{\mathrm{NS}}) \models \varphi\text{”}.$$

The Axiom $(*)$

1. $L(\mathbb{R}) \models \mathrm{AD}$.
2. *There is an* $L(\mathbb{R})$*-generic filter* $G \subset \mathbb{P}_{\max}$ *such that*

$$H(\omega_2) \subset L(\mathbb{R})[G].$$

Extending the Axiom $(*)$ to $H(c^+)$: Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is *universally* Baire if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}^n$, the preimage of A by π has the property of Baire in the space Ω.

Theorem

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.

Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.

An abstract generalization of the projective sets

Theorem (Martin-Steel, Woodin)

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.

Then A is determined.

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is universally Baire.

Then A can be uniformized by a universally Baire function.

The ultimate forcing axiom?

The Axiom $(*)^+$

There is a proper class of measurable Woodin cardinals. Let Γ be the collection of all universally Baire sets.

1. $\Gamma = L(\Gamma) \cap \mathcal{P}(\mathbb{R})$.
2. *There is an $L(\Gamma)$-generic filter $G \subset \mathbb{P}_{\max}$ such that*

$$H(c^+) \subset L(\Gamma)[G].$$

- The Axiom $(*)^+$ implies the Axiom $(*)$.

Measuring the complexity of universally Baire sets: calibrating Axiom $(*)^+$

Definition

Suppose A and B are subsets of $\mathbb{R}$.

1. A is *borel reducible* to B, $A \leq_{\mathrm{borel}} B$, if there is a borel function $\pi : \mathbb{R} \to \mathbb{R}$ such that
   - either $A = \pi^{-1}[B]$ or $A = \mathbb{R} \backslash \pi^{-1}[B]$.
2. A and B are *borel bi-reducible* if $A \leq_{\mathrm{borel}} B$ and $B \leq_{\mathrm{borel}} A$.
3. The *borel degree* of A is the equivalence class of all sets which are borel bi-reducible with A.

Theorem (Martin-Steel, Martin, Wadge)

Assume there is a proper class of Woodin cardinals.

Then the borel degrees of the universally Baire sets are linearly ordered by borel reducibility and moreover this is a wellorder.

Claim

The Axiom $(*)$ is the ultimate forcing axiom as far as the structure of $H(\omega_2)$ is concerned. This is certified by:

- *The Π_2-Maximality Theorem.*
- *The extension of $(*)$ to $(*)^+$.*

Theorem

Assume there is a proper class of Woodin cardinals and that the Axiom $(*)$ holds.

Then $H(\omega_2)$ is logically bi-interpretable with $H(\omega_1)$.

Conclusion

The ultimate forcing axiom logically reduces $H(\omega_2)$ to $H(\omega_1)$.

In search of V ... a generic-multiverse of sets?

Suppose that M is a countable transitive set and that

$$M \models \mathrm{ZFC}.$$

Let $\mathbb{V}_M$ be the smallest set of countable transitive sets such that $M \in \mathbb{V}_M$ and such that for all pairs, (M_1, M_2), of countable transitive sets, if

1. $M_1 \models \mathrm{ZFC}$,
2. M_2 is a generic extension of M_1,
3. $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$,

then both M_1 and M_2 are in $\mathbb{V}_M$.

Definition

$\mathbb{V}_M$ is the *generic-multiverse* generated from M.

Evaluating truth in the generic-multiverse...

Theorem

For each sentence φ there is a sentence φ^* such that for all countable transitive sets M if

$$M \models \mathrm{ZFC}$$

then the following are equivalent.

1. $M \models \varphi^*$,
2. *For all $N \in \mathbb{V}_M$, $N \models \varphi$.*

The generic-multiverse view of truth

A Π_2-sentence, φ, is a generic-multiverse truth *if φ holds in each universe of the generic-multiverse generated by V.*

Theorem

Suppose there is a proper class of strongly inaccessible cardinals. Then the following are equivalent in the generic-multiverse view of truth (each if true implies the truth of the other).

1. $L(\mathbb{R}) \models \mathrm{AD}$.
2. $L(\mathbb{R}) \not\models \text{Axiom of Choice}$.

Ω-logic
(The logic of the generic-multiverse)

Definition

Suppose φ is a Π_2-sentence. Then

$$\models_\Omega \varphi$$

if φ holds in all generic extensions of V.

Theorem

Suppose there is a proper class of Woodin cardinals and that φ is a Π_2-sentence.

Then φ is a generic-multiverse truth if and only if $\models_\Omega \varphi$.

Universally Baire sets and strong closure

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that M is a countable transitive set such that $M \models \mathrm{ZFC}$.

Then M is *strongly A-closed* if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

The definition of $\vdash_\Omega \varphi$

Definition

Suppose there is a proper class of Woodin cardinals. Suppose that φ is a Π_2-sentence.

Then $\vdash_\Omega \varphi$ if there exists a set $A \subset \mathbb{R}$ such that:

1. A is universally Baire,
2. for all countable transitive sets, $M \models \mathrm{ZFC}$, if M is strongly A-closed then
$$M \models \text{“}\models_\Omega \varphi\text{”}.$$

The Ω Conjecture

Theorem (Ω Soundness)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is Π_2-sentence.

If $\vdash_\Omega \varphi$ then $\models_\Omega \varphi$

Definition (Ω Conjecture)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is a Π_2-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

Theorem

The Ω Conjecture is invariant across the generic-multiverse.

The Ω Conjecture and generic-multiverse view of truth

Definition

1. T_0 is the set of sentences φ such that $\models_\Omega$ "$H(\omega_2) \models \varphi$".
2. T is the set of Π_2-sentences φ such that $\models \varphi$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that the Ω Conjecture holds.

Then T is (recursively) reducible to T_0.

Claim

If there is a proper class of Woodin cardinals and the Ω Conjecture holds then the generic-multiverse view of truth is **not** *viable.*

The generic-multiverse view of truth is simply a form of formalism; it reduces truth to Third Order Number Theory.

Claim

If there is a proper class of Woodin cardinals and the Ω Conjecture holds then **there is no (mathematical) evidence that the Continuum Hypothesis has no answer**.

Gödel's transitive class HOD

Definition

HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $Y \subseteq \alpha$ such that

1. Y is definable in V_α without parameters,
2. $X \in L[Y]$.

- (ZF) The Axiom of Choice holds in HOD.

HOD and the Ω Conjecture

Definition

A set $A \subset \mathbb{R}$ is *ordinal definable* if there exists an ordinal α such that A is definable in V_α without parameters.

Theorem

Suppose that there is a proper class of Woodin cardinals and that for all sets $A \subset \mathbb{R}$, if A is ordinal definable then A is universally Baire.

Then $\mathrm{HOD} \models$ " The Ω Conjecture ".

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and large cardinal axioms

Definition

Suppose that $A \subseteq \mathbb{R}$. Then $\mathrm{HOD}^{L(A,\mathbb{R})}$ is the class HOD as defined within $L(A, \mathbb{R})$.

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and enlargements of L

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ *is a Mitchell-Steel inner model.*

Theorem

Suppose that there is a proper class of Woodin cardinals.

Then $\mathrm{HOD}^{L(\mathbb{R})}$ *is* **not** *a Mitchell-Steel inner model.*

Ultimate L

(Conjecture) The axiom scheme for $V =$ ultimate L

There is a proper class of Woodin cardinals. Further for each sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

Two possible futures

Future 1: The axiom "$V =$ ultimate L" is false

Assume there is a proper class of supercompact cardinals. The following must hold.

- *There is a ordinal definable set $A \subset \mathbb{R}$ which is not universally Baire.*
- *Let Γ be the set of all universally Baire sets. Then $\Gamma = L(\Gamma) \cap \mathcal{P}(\mathbb{R})$.*

Future 2: The axiom "$V =$ ultimate L" is possibly true

Assume there is a proper class of supercompact cardinals. The following must hold.

- *There exists an infinite cardinal κ such that*

$$\kappa^+ = (\kappa^+)^{\mathrm{HOD}}$$

(meta) Conjecture

This axiom "V = ultimate L" will be validated on the basis of compelling and accepted principles of infinity just as the axiom PD *has been.*

- *This axiom will reduce all questions of Set Theory to axioms of infinity,*
  - *ending the age of (forcing) independence.*