The search for mathematical truth

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The projective sets

**Definition**

A set $A \subseteq \mathbb{R}^n$ is *projective* if it can be generated from the open subsets of $\mathbb{R}^n$ in finitely many steps of taking complements and images by continuous functions, $f : \mathbb{R}^n \to \mathbb{R}^n$.

**Definition**

Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function $f$ *uniformizes* $A$ if for all $x \in \mathbb{R}$:

- if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$. 
Two questions of Luzin

| 1. Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is projective. Can $A$ be uniformized by a projective function? |
| 2. Suppose $A \subseteq \mathbb{R}$ is projective. Is $A$ Lebesgue measurable and does $A$ have the property of Baire? |

Luzin’s questions are questions about $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

Both questions are unsolvable on the basis of the ZFC axioms
Determinacy and the answers to Luzin’s questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players.

- The players alternate choosing $\epsilon_i \in \{0, 1\}$.
- After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined.
- Player I wins this run of the game if
  \[ \sum_{i=1}^{\infty} \epsilon_i / 2^i \in A \]
  otherwise Player II wins.

**Definition**

The set $A$ is *determined* if there is a winning strategy for one of the players in the game associated to $A$. 
The Axiom of Determinacy (AD)

<table>
<thead>
<tr>
<th>Definition (Mycielski-Steinhaus)</th>
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<tr>
<td>Axiom of Determinacy (AD): Every set $A \subseteq \mathbb{R}$ is determined.</td>
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<tr>
<th>Lemma (Axiom of Choice)</th>
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<tr>
<td>There is a set $A \subseteq \mathbb{R}$ such that $A$ is not determined.</td>
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<th>Corollary</th>
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<td>AD is false.</td>
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### Projective Determinacy (PD)

**Definition**

*Projective Determinacy (PD)*: Every projective set $A \subseteq \mathbb{R}$ is determined.

**Theorem**

Assume every projective set is determined.

1. (Mycielski-Steinhaus) *Every projective set has the property of Baire.*

2. (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*

3. (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

**Key questions**

*Is PD even consistent and if consistent, is PD true?*
Basic notions: Logical definability from parameters

- A set \( N \) is *transitive* if \( A \subset N \) for all \( A \in N \).
- A transitive set \( N \) is an *ordinal* if \((N, \in)\) is a totally ordered set.
- \( \omega \) is the least infinite ordinal, \((\omega, \in) \cong (\mathbb{N}, <)\).
- \( \omega_1 \) is the least uncountable ordinal.

**Definition**

Suppose \( N \) is a transitive set. A subset \( X \subseteq N \) is logically definable in \((N, \in)\) from parameters if for some formula \( \varphi[x_0, \ldots, x_n] \) and for some parameters \( a_1, \ldots, a_n \in N \),

\[
X = \{ a \in N | (N, \in) \models \varphi[a, a_1, \ldots, a_n] \}
\]
Basic notions: Elementary embeddings

**Definition**

Suppose $N$ and $M$ are transitive sets. A function $j : N \rightarrow M$ is an *elementary embedding* if for all logical formulas $\varphi[x_0, \ldots, x_n]$ and all $a_0, \ldots, a_n \in N$,

$$( N, \in ) \models \varphi[a_0, \ldots, a_n] \text{ if and only if } (M, \in ) \models \varphi[j(a_0), \ldots, j(a_n)]$$

**Lemma**

Suppose that $j : N \rightarrow M$ is an elementary embedding and that $N \models \text{ZFC}$. Then the following are equivalent.

1. $j$ is not the identity.
2. There is an ordinal $\beta \in N$ such that $j(\beta) \neq \beta$. 

A cardinal $\kappa$ is a large cardinal if there exists an elementary embedding,

$$j : V \rightarrow M$$

such that $M$ is a transitive class and $\kappa$ is the least ordinal such that $j(\alpha) \neq \alpha$.

- Requiring $M$ be close to $V$ yields a hierarchy of large cardinal axioms:
  - simplest case is where $\kappa$ is a measurable cardinal.
  - $M = V$ contradicts the Axiom of Choice.
The validation of Projective Determinacy

**Theorem (Martin-Steel)**

Assume there are infinitely many Woodin cardinals. Then every projective set is determined.

**Theorem**

The following are equivalent.

1. Every projective set is determined.
2. For each $k < \omega$ there is a countable (iterable) transitive set $N$ such that

   $N \models \text{ZFC} + \text{“There exist } k \text{ Woodin cardinals”},$

**PD** is the missing (and true) axiom for $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

- Is there such an axiom for $V$ itself?
Mathematical truth and two modest claims

- Large cardinal axioms predict facts about our world.

**Prediction**

There will be no contradiction discovered from PD (by any means) before the year 3010.

There will be no contradiction discovered from PD (by any means) before all the Clay Millennium problems have been solved.

- A controversial claim.

**Claim**

Consistency claims for large cardinal axioms require a conception of the Universe of Sets in which large cardinals axioms are true.

- This ultimately requires that questions such as that of the Continuum Hypothesis also be resolved
  - or an explanation of the exact nature of the ambiguity.
If $X$ is a set then $\mathcal{P}(X)$ denotes the set of all subsets of $X$:

$$\mathcal{P}(X) = \{ Y \mid Y \subseteq X \}.$$
The effective cumulative hierarchy: \( L \)

The definable power set

For each set \( X \), \( P_{\text{Def}}(X) \) denotes the set of all \( Y \subseteq X \) such that \( X \) is logically definable in the structure \( (X, \in) \) from parameters in \( X \).

- (Axiom of Choice) \( P_{\text{Def}}(X) = P(X) \) if and only if \( X \) is finite.
- \( P_{\text{Def}}(V_{\omega+1}) \cap P(\mathbb{R}) \) is exactly the projective sets.

Gödel’s constructible universe, \( L \)

Define \( L_\alpha \) by induction on \( \alpha \) as follows.

1. \( L_0 = \emptyset \),
2. (Successor case) \( L_{\alpha+1} = P_{\text{Def}}(L_\alpha) \),
3. (Limit case) \( L_\alpha = \bigcup \{ L_\beta \mid \beta < \alpha \} \).

\( L \) is the class of all sets \( X \) such that \( X \in L_\alpha \) for some ordinal \( \alpha \).
**The axiom $V = L$, the projective sets, and large cardinals**

<table>
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<tr>
<th>Theorem</th>
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<tr>
<td><strong>Assume $V = L$.</strong></td>
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<tr>
<td><strong>(1)</strong> (Gödel) <em>Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.</em></td>
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<td><strong>(2)</strong> (Gödel) <em>There is a projective set which is not Lebesgue measurable:</em></td>
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<td>- <em>there is a projective wellordering of the reals.</em></td>
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<td><strong>(3)</strong> (Scott) <em>There are no measurable cardinals:</em></td>
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<tr>
<td>- <em>there are no (interesting) large cardinals.</em></td>
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<th>(meta) Corollary</th>
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<tr>
<td>$V \neq L$.</td>
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Basic notions: Enlargements of $L$, generalizing the projective sets

**Definition**

1. $L_0(\mathbb{R}) = \mathbb{R}$,
2. (Successor case) $L_{\alpha+1}(\mathbb{R}) = \mathcal{P}_{\text{Def}}(L_\alpha(\mathbb{R}))$,
3. (Limit case) $L_\alpha(\mathbb{R}) = \bigcup\{L_\beta(\mathbb{R}) \mid \beta < \alpha\}$.

$L(\mathbb{R})$ is the class of all sets $a$ such that $a \in L_\alpha(\mathbb{R})$ for some $\alpha$.

- $\mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ is a transfinite extension of the hierarchy of the projective sets.
- Assuming there is a proper class of Woodin cardinals then $L(\mathbb{R}) \models \text{AD}$.

**Suppose** $\Gamma \subseteq \mathcal{P}(\mathbb{R})$. **Then one defines** $L(\Gamma, \mathbb{R})$ **by setting**

$L_0(\Gamma, \mathbb{R}) = \mathbb{R} \cup \Gamma \cup \{\Gamma\}$. 

Forcing axioms

**Definition**

Suppose $\mathbb{B}$ is a complete Boolean algebra and $\kappa$ is a cardinal.

The $\kappa$-Baire Category Theorem holds for $\mathbb{B}$ if the following holds in $\Omega$ where $\Omega$ is the Stone space of $\mathbb{B}$.

- Suppose $\mathcal{A}$ is a family of open dense subsets of $\Omega$ and $|\mathcal{A}| \leq \kappa$. Then $\bigcap \mathcal{A}$ is dense in $\Omega$.

- The $\omega_1$-Baire Category Theorem *cannot* hold for all complete Boolean algebras.

**Question**

*For which complete Boolean algebras $\mathbb{B}$ can the $\omega_1$-Baire Category Theorem hold for $\mathbb{B}$?*
Stationary sets in $\omega_1$

**Definition**

1. A cofinal set $C \subseteq \omega_1$ is *closed and unbounded* if if for all limit ordinals $\alpha < \omega_1$, if $C \cap \alpha$ is cofinal in $\alpha$ then $\alpha \in C$.

2. A set $S \subset \omega_1$ is *stationary* if

   \[ S \cap C \neq \emptyset \]

   for all closed unbounded sets $C \subset \omega_1$.

*Assuming the Axiom of Choice, there exist sets $S \subset \omega_1$ such that both $S$ and $\omega_1 \setminus S$ are stationary.*
Stationary set preserving

**Definition**

A complete Boolean algebra \( \mathbb{B} \) is *stationary set preserving* if the following holds for all \( c \in \mathbb{B} \) with \( c > 0 \), for all sequences \( \langle b_\alpha : \alpha < \omega_1 \rangle \) of elements of \( \mathbb{B} \), and for all stationary sets \( S \subseteq \omega_1 \).

- If \( c \leq \bigvee \{ b_\alpha \mid \beta < \alpha < \omega_1 \} \) for all \( \beta < \omega_1 \), then there exists \( \eta \in S \) and \( 0 < d \leq c \) such that

- \( d \leq \bigvee \{ b_\alpha \mid \beta < \alpha < \eta \} \) for all \( \beta < \eta \).
## Martin’s Maximum

**Theorem (Foreman, Magidor, Shelah)**

Suppose that $\mathbb{B}$ is a complete Boolean algebra and that the $\omega_1$-Baire Category Theorem holds for $\mathbb{B}$. Then $\mathbb{B}$ is stationary set preserving.

**Definition (Foreman, Magidor, Shelah)**

**Martin’s Maximum**: The $\omega_1$-Baire Category Theorem holds for all stationary set preserving complete Boolean algebras.

**Theorem (Foreman, Magidor, Shelah)**

Suppose there is a supercompact cardinal. Then there is a stationary set preserving complete Boolean algebra $\mathbb{B}$ such that $\mathbb{V}^\mathbb{B} \models$ Martin’s Maximum.
Consequences of Martin’s Maximum

Theorem (Foreman, Magidor, Shelah)

Assume Martin’s Maximum. Then $2^{\aleph_0} = \aleph_2$.

Theorem (Martin’s Maximum)

Suppose that

$$\pi : C([0, 1]) \to A$$

is an algebra homomorphism of $C([0, 1])$ into a Banach algebra $A$.

Then $\pi$ is continuous.
### Lemma

There are 2 infinite total orders such that every infinite total order contains an isomorphic copy of them.

- \((\mathbb{Z}^+, <)\) and \((\mathbb{Z}^-, <)\)

### Theorem (Martin’s Maximum: J. Moore)

There are 5 uncountable total orders such that every uncountable total order contains an isomorphic copy of one of them.
The Brown-Douglas-Filmore Problem

Question (Brown-Douglas-Filmore)

Suppose that

$$\pi : \mathcal{B}(H)/\mathcal{K}(H) \to \mathcal{B}(H)/\mathcal{K}(H)$$

is an automorphism. Must $\pi$ be an inner automorphism?

Theorem (CH: Phillips and Weaver)

There is an automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ which is not an inner automorphism.

Theorem (Martin's Maximum: I. Farah)

Every automorphism of $\mathcal{B}(H)/\mathcal{K}(H)$ is an inner automorphism.
Is Martin’s Maximum true?

**Definition**

Suppose that $\kappa$ is a infinite regular cardinal. $H(\kappa)$ is the set of all sets $X$ such that there is a transitive set $Y$ such that

1. $X \in Y$,
2. $|Y| < \kappa$.

Several interesting cases.

1. $H(\omega_1)$. This is logically equivalent to $V_{\omega+1}$.
2. $H(c^+)$. This is logically equivalent to $V_{\omega+2}$.
3. $H(\omega_2)$.
   - Assuming CH, $H(\omega_2)$ is logically equivalent to $V_{\omega+2}$.
   - Assuming Martin’s Maximum, $H(\omega_2)$ is *not* logically equivalent to $V_{\omega+2}$.
The structure \((H(\omega_2), I_{\text{NS}})\)

**Definition**

\(I_{\text{NS}}\) is the ideal of all non-stationary subsets of \(\omega_1\).

**Lemma (Martin’s Maximum)**

*Suppose that \(\varphi\) is a \(\Pi_2\) sentence and that there is a stationary set preserving Boolean algebra \(B\) such that*

\[
V^B \models “H(\omega_2) \models \varphi”
\]

*Then \(H(\omega_2) \models \varphi.\)*
Observation

*Martin’s Maximum is attempting to maximize the $\Pi_2$-theory of $(H(\omega_2), I_{NS})$.***

Definition

A $\Pi_2$-sentence $\varphi$ is $\Omega$-satisfiable for $(H(\omega_2), I_{NS})$ if there is a complete Boolean algebra $\mathbb{B}$ such that

\[
V^{\mathbb{B}} \models " (H(\omega_2), I_{NS}) \models \varphi ".
\]

- No requirement that $\mathbb{B}$ be stationary set preserving.
Theorem (\(\Pi_2\)-Maximality)

Assume there is a proper class of Woodin cardinals. There is a partial order \(\mathbb{P}_{\text{max}} \in L(\mathbb{R})\) such that the following hold.

1. \(\mathbb{P}_{\text{max}}\) is homogeneous and \(\omega\)-closed.
2. \(L(\mathbb{R})^{\mathbb{P}_{\text{max}}} \models \text{ZFC}\).
3. Suppose that \(\varphi\) is a \(\Pi_2\)-sentence which is \(\Omega\)-satisfiable for \((H(\omega_2), I_{\text{NS}})\). Then
\[
L(\mathbb{R})^{\mathbb{P}_{\text{max}}} \models "(H(\omega_2), I_{\text{NS}}) \models \varphi".
\]

The Axiom (*)

1. \(L(\mathbb{R}) \models \text{AD}\).
2. There is an \(L(\mathbb{R})\)-generic filter \(G \subset \mathbb{P}_{\text{max}}\) such that
\[
H(\omega_2) \subset L(\mathbb{R})[G].
\]
Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}^n$ is *universally* Baire if for all topological spaces $\Omega$ and for all continuous functions $\pi : \Omega \rightarrow \mathbb{R}^n$, the preimage of $A$ by $\pi$ has the property of Baire in the space $\Omega$.

Theorem

*Suppose that there is a proper class of Woodin cardinals and suppose $A \subseteq \mathbb{R}$ is universally Baire.*

*Then every set $B \in L(A, \mathbb{R}) \cap \mathcal{P}(\mathbb{R})$ is universally Baire.*
An abstract generalization of the projective sets

Theorem (Martin-Steel, Woodin)

Suppose that there is a proper class of Woodin cardinals and suppose \( A \subseteq \mathbb{R} \) is universally Baire.

Then \( A \) is determined.

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and suppose \( A \subseteq \mathbb{R} \times \mathbb{R} \) is universally Baire.

Then \( A \) can be uniformized by a universally Baire function.
The ultimate forcing axiom?

The Axiom \((\ast)^+\)

*There is a proper class of measurable Woodin cardinals. Let \(\Gamma\) be the collection of all universally Baire sets.*

1. \(\Gamma = L(\Gamma) \cap P(\mathbb{R})\).
2. *There is an \(L(\Gamma)\)-generic filter \(G \subseteq \mathbb{P}_{\text{max}}\) such that*

\[
H(c^+) \subseteq L(\Gamma)[G].
\]

- The Axiom \((\ast)^+\) implies the Axiom \((\ast)\).
Measuring the complexity of universally Baire sets: calibrating Axiom \((\ast)^+\)

**Definition**

Suppose \(A\) and \(B\) are subsets of \(\mathbb{R}\).

1. \(A\) is *borel reducible* to \(B\), \(A \leq_{\text{borel}} B\), if there is a borel function \(\pi : \mathbb{R} \to \mathbb{R}\) such that
   - either \(A = \pi^{-1}[B]\) or \(A = \mathbb{R} \setminus \pi^{-1}[B]\).
2. \(A\) and \(B\) are *borel bi-reducible* if \(A \leq_{\text{borel}} B\) and \(B \leq_{\text{borel}} A\).
3. The *borel degree* of \(A\) is the equivalence class of all sets which are borel bi-reducible with \(A\).

**Theorem (Martin-Steel, Martin, Wadge)**

Assume there is a proper class of Woodin cardinals.

*Then the borel degrees of the universally Baire sets are linearly ordered by borel reducibility and moreover this is a wellorder.*
Claim

The Axiom (*) is the ultimate forcing axiom as far as the structure of H(ω₂) is concerned. This is certified by:
- The Π₂-Maximality Theorem.
- The extension of (*) to (**)⁺.

Theorem

Assume there is a proper class of Woodin cardinals and that the Axiom (*) holds.

Then H(ω₂) is logically bi-interpretable with H(ω₁).

Conclusion

The ultimate forcing axiom logically reduces H(ω₂) to H(ω₁).
In search of \( V \) ... a generic-multiverse of sets?

Suppose that \( M \) is a countable transitive set and that

\[
M \models \text{ZFC.}
\]

Let \( \mathbb{V}_M \) be the smallest set of countable transitive sets such that

\[
M \in \mathbb{V}_M
\]

and such that for all pairs, \((M_1, M_2)\), of countable transitive sets, if

1. \( M_1 \models \text{ZFC} \),
2. \( M_2 \) is a generic extension of \( M_1 \),
3. \( M_1 \in \mathbb{V}_M \) or \( M_2 \in \mathbb{V}_M \),

then both \( M_1 \) and \( M_2 \) are in \( \mathbb{V}_M \).

**Definition**

\( \mathbb{V}_M \) is the *generic-multiverse* generated from \( M \).
Theorem

For each sentence $\varphi$ there is a sentence $\varphi^*$ such that for all countable transitive sets $M$ if

$$M \models \text{ZFC}$$

then the following are equivalent.

1. $M \models \varphi^*$,
2. For all $N \in \mathcal{V}_M$, $N \models \varphi$. 
The generic-multiverse view of truth

A $\Pi_2$-sentence, $\varphi$, is a generic-multiverse truth if $\varphi$ holds in each universe of the generic-multiverse generated by $V$.

Theorem

Suppose there is a proper class of strongly inaccessible cardinals. Then the following are equivalent in the generic-multiverse view of truth (each if true implies the truth of the other).

1. $L(\mathbb{R}) \models AD$.
2. $L(\mathbb{R}) \not\models$ Axiom of Choice.
## Definition

Suppose $\varphi$ is a $\Pi_2$-sentence. Then

$$\models_{\Omega} \varphi$$

if $\varphi$ holds in all generic extensions of $\mathcal{V}$.

## Theorem

*Suppose there is a proper class of Woodin cardinals and that $\varphi$ is a $\Pi_2$-sentence. Then $\varphi$ is a generic-multiverse truth if and only if $\models_{\Omega} \varphi$.**
**Universally Baire sets and strong closure**

**Definition**

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that $M$ is a countable transitive set such that $M \models \text{ZFC}$. Then $M$ is **strongly $A$-closed** if for all countable transitive sets $N$ such that $N$ is a generic extension of $M$,

$$A \cap N \in N.$$
The definition of $\vdash_\Omega \varphi$

**Definition**

Suppose there is a proper class of Woodin cardinals. Suppose that $\varphi$ is a $\Pi^0_2$-sentence.

Then $\vdash_\Omega \varphi$ if there exists a set $A \subset \mathbb{R}$ such that:

1. $A$ is universally Baire,
2. for all countable transitive sets, $M \models \text{ZFC}$, if $M$ is strongly $A$-closed then
   \[ M \models "\vdash_\Omega \varphi". \]
The **Ω Conjecture**

**Theorem (Ω Soundness)**

Suppose that there exists a proper class of Woodin cardinals and suppose that $\varphi$ is $\Pi_2$-sentence.

If $\not\vdash_\Omega \varphi$ then $\not\models_\Omega \varphi$.

**Definition (Ω Conjecture)**

Suppose that there exists a proper class of Woodin cardinals and suppose that $\varphi$ is a $\Pi_2$-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

**Theorem**

*The Ω Conjecture is invariant across the generic-multiverse.*
The $\Omega$ Conjecture and generic-multiverse view of truth

**Definition**

1. $T_0$ is the set of sentences $\varphi$ such that $\models_\Omega \text{“} H(\omega_2) \models \varphi \text{”}$.
2. $T$ is the set of $\Pi_2$-sentences $\varphi$ such that $\models \varphi$.

**Theorem**

*Suppose that there is a proper class of Woodin cardinals and that the $\Omega$ Conjecture holds.*

*Then $T$ is (recursively) reducible to $T_0$.***
Claim

*If there is a proper class of Woodin cardinals and the $\Omega$ Conjecture holds then the generic-multiverse view of truth is not viable.*

The generic-multiverse view of truth is simply a form of formalism; it reduces truth to Third Order Number Theory.

Claim

*If there is a proper class of Woodin cardinals and the $\Omega$ Conjecture holds then there is no (mathematical) evidence that the Continuum Hypothesis has no answer.*
Gödel’s transitive class HOD

Definition

HOD is the class of all sets $X$ such that there exist $\alpha \in \text{Ord}$ and $Y \subseteq \alpha$ such that

1. $Y$ is definable in $V_\alpha$ without parameters,
2. $X \in L[Y]$.

$(\text{ZF})$ The Axiom of Choice holds in HOD.
HOD and the Ω Conjecture

**Definition**

A set $A \subseteq \mathbb{R}$ is *ordinal definable* if there exists an ordinal $\alpha$ such that $A$ is definable in $V_\alpha$ without parameters.

**Theorem**

*Suppose that there is a proper class of Woodin cardinals and that for all sets $A \subseteq \mathbb{R}$, if $A$ is ordinal definable then $A$ is universally Baire.*

*Then* $\text{HOD} \models \text{“The } \Omega \text{ Conjecture ”}$. 
**HOD\(^{(A,R)}\) and large cardinal axioms**

**Definition**

Suppose that \( A \subseteq \mathbb{R} \). Then \( \text{HOD}\(^{(A,R)}\) \) is the class HOD as defined within \( L(A, \mathbb{R}) \).

**Definition**

Suppose that \( A \subseteq \mathbb{R} \) is universally Baire.

Then \( \Theta\(^{(A,R)}\) \) is the supremum of the ordinals \( \alpha \) such that there is a surjection, \( \pi : \mathbb{R} \to \alpha \), such that \( \pi \in L(A, \mathbb{R}) \).

**Theorem**

*Suppose that there is a proper class of Woodin cardinals and that \( A \) is universally Baire.*

*Then \( \Theta\(^{(A,R)}\) \) is a Woodin cardinal in \( \text{HOD}\(^{(A,R)}\) \).*
**Theorem (Steel)**

*Suppose that there is a proper class of Woodin cardinals and let* 
\[ \delta = \Theta^{L(\mathbb{R})}. \]

*Then* \( HOD^{L(\mathbb{R})} \cap V_\delta \) *is a Mitchell-Steel inner model.*

**Theorem**

*Suppose that there is a proper class of Woodin cardinals. Then* \( HOD^{L(\mathbb{R})} \) *is not a Mitchell-Steel inner model.*
(Conjecture) The axiom scheme for $V = \text{ultimate } L$:

There is a proper class of Woodin cardinals. Further for each sentence $\varphi$, if $\varphi$ holds in $V$ then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\text{HOD}^{L(A, \mathbb{R})} \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^{L(A, \mathbb{R})}$.
Two possible futures

Future 1: The axiom “\( V = \text{ultimate } L \)” is false

Assume there is a proper class of supercompact cardinals. The following must hold.

- There is a ordinal definable set \( A \subseteq \mathbb{R} \) which is not universally Baire.
- Let \( \Gamma \) be the set of all universally Baire sets. Then \( \Gamma = L(\Gamma) \cap \mathcal{P}(\mathbb{R}) \).

Future 2: The axiom “\( V = \text{ultimate } L \)” is possibly true

Assume there is a proper class of supercompact cardinals. The following must hold.

- There exists an infinite cardinal \( \kappa \) such that
  \[
  \kappa^+ = (\kappa^+)^{\text{HOD}}
  \]
This axiom "V = ultimate L" will be validated on the basis of compelling and accepted principles of infinity just as the axiom PD has been.

- This axiom will reduce all questions of Set Theory to axioms of infinity,
- ending the age of (forcing) independence.