Diagonal stationary reflection and generic ultrapowers

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Main Theme

Strong forcing axioms (MM, PFA) imply the existence of ideals with interesting generic ultrapowers; these generic ultrapowers have critical point ω_2.

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Strong forcing axioms (MM, PFA) imply the existence of ideals with interesting generic ultrapowers; these generic ultrapowers have critical point ω_2.

Vaguely: For sufficiently large classes Γ of posets, $MA(\Gamma)$ implies there are ideals I whose positive-set forcings are "almost" in Γ.

Motivation: Condensation

The (duals of the) ideals will concentrate on $M \in \wp_{\omega_2}(H_\theta)$ which have condensation-like properties.

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Gödel's Condensation Lemma for L: whenever $M \prec (H_\theta, \in)$, if $\sigma_M : H_M \to H_\theta$ is the inverse of Mostowski Collapse of M then for every $\alpha \in H_M \cap ORD$: $(L_\alpha)^{H_M} = L_\alpha$.

So the function $\alpha \mapsto L_\alpha$ *condenses on M*.

Motivation: Two condensation-like principles under strong forcing axioms

Strong forcing axioms imply condensation-like properties:

- (Viale-Weiss) Proper Forcing Axiom implies ISP
- (Foreman) Martin's Maximum implies highly simultaneous ("diagonal") stationary set reflection

Motivation: A condensation principle under PFA

Theorem

(Viale/Weiss): Assume PFA and fix regular $\Omega >> \theta \geq \omega_2$. *There are stationarily many* $M \in \wp_{\omega_2}(H_\Omega)$ *such that whenever* $N \mapsto F(N) \subset N$ *is a slender function on* $\wp_{\omega_2}(H_\theta)$ *and* $F \in M$*, then* M *"catches"* F.

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- i.e. if $\sigma : \bar{M} \to M$ is inverse of trans. collapse and $Y := F(M \cap H_\theta)$, then $\sigma^{-1}``Y \subset \bar{M}$ is an element of $\bar{M}$
- So M detects a lot of 2nd order information about itself.

Motivation: A condensation principle under MM

Theorem

(Foreman): Assume MM and suppose $\theta \subset H \subseteq H_\theta$ *where* $|H| = \theta$*. Fix a partition* $\langle R_i \mid i \in H \rangle$ *of* $\theta \cap cof(\omega)$ *into stationary sets.*

Then there are stationarily many (internally approachable) $M \in \wp_{\omega_2}(H)$ such that:

$$R_i \text{ reflects to } \sup(M \cap \theta) \iff i \in M \tag{1}$$

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- This implies: if $\sigma : \bar{M} \to M$ is inverse of collapse map, then $\bar{M}$ is correct about stationarity of every $\sigma^{-1}(R_i)$ (for $R_i \in M$ from the fixed partition).
- Again, M detects some 2nd order information.

Outline

- Notation and background
- Stationary set reflection and connections with:
  - Condensation of NS
  - Generic embeddings with critical point ω_2
- The Diagonal Reflection Principle (DRP)
- Forcing axioms imply DRP
  - And a detour involving $MA(\Gamma)$ and ideals whose positive-set-forcings are in Γ

Notation and background

- $\wp_\kappa(H_\theta) := \{M \prec H_\theta \mid |M| < \kappa \text{ and } M \cap \kappa \in \kappa\}$
- IA_{ω_1} is the class of M such that there is some $\in$-increasing, continuous elementary chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary submodels of M such that
  - $\bigcup_{\alpha<\omega_1} N_\alpha = M$
  - Every proper initial segment of $\vec{N}$ is element of M
- IC_{ω_1} defined similarly, except only require each $N_\alpha \in M$ (equiv: $M \cap [M]^\omega$ contains a club)

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This talk focuses on $\wp_{\omega_2}(H_\theta) \cap \mathrm{IC}_{\omega_1}$.

Stationary set reflection

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A set S *reflects at* γ iff $S \cap \gamma$ is stationary in γ. A set $S \subset \kappa$ *reflects* iff there is a $\gamma < \kappa$ s.t. S reflects at γ.

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If κ is measurable (or just weakly compact), then:

- every stationary subset of κ reflects.
- $V^{Col(\omega_1,<\kappa)} \models$ "every stationary subset of $\omega_2 \cap \mathrm{cof}(\omega)$ reflects"
  - The quoted statement is equiconsistent with a Mahlo cardinal (Harrington/Shelah)

Simultaneous stationary reflection

We can also require distinct sets to have a common reflection point. If κ is measurable, then:

- Every $< \kappa$-sized collection of stationary sets have a common reflection point
- $V^{Col(\omega_1, <\kappa)} \models$ "every ω_1-sized collection of stationary subsets of $\omega_2 \cap \mathrm{cof}(\omega)$ have a common reflection point"
  - The quoted statement is equiconsistent with a weakly compact cardinal (Magidor)

Generalized stationary reflection

For stationary $R \subset [H_\theta]^\omega$, say R reflects to M iff $R \cap [M]^\omega$ is stationary in $[M]^\omega$.

- i.e. for every algebra $\mathcal{A}$ on M, there is an $N \in R$ with $N \prec \mathcal{A}$.

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- i.e. for every algebra $\mathcal{A}$ on M, there is an $N \in R$ with $N \prec \mathcal{A}$.

"For every regular $\theta \geq \omega_2$, every stationary $R \subset [\theta]^\omega$ reflects to an M of size ω_1":

- has powerful consequences if θ is large, e.g. failure of square, NS_{ω_1} is precipitous and more (F-M-S)
- follows from MM (Foreman-Magidor-Shelah)
- holds in $V^{Col(\omega_1, <\kappa)}$ where κ is supercompact

Stationary reflection and condensation of NS

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Stationary reflection and condensation of NS

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Proof: ($\Longrightarrow$): Suppose $R \in M$ reflects to M. Let $\sigma : \bar{H} \to M$ and $\sigma(\bar{R}) = R$. So $\bar{H} \models$ "$\bar{R}$ is stationary."

NTS: $\bar{R}$ is really stationary.

- In V let $\bar{\mathcal{A}} = (\bar{H}, (\bar{f}_n)_{n\in\omega})$
- Need to find a $\bar{N} \prec \bar{\mathcal{A}}$ s.t. $\bar{N} \in \bar{R}$
- Use σ to transfer $\bar{\mathcal{A}}$ to a structure $\mathcal{A} = (M \cap H_\theta, (f_n)_n)$.
- Since $R \cap [M]^\omega$ is stationary and $M \cap [M]^\omega$ contains a club (this is the approachability requirement on M), there is an $N \in R \cap M \cap [M]^\omega$ such that $N \prec \mathcal{A}$.
- Then $\sigma^{-1}(N) \in \bar{R}$ and $\sigma^{-1}(N) \prec \bar{\mathcal{A}}$.

Other characterizations, and generic ultrapowers

Let $R \subset [H_\theta]^\omega$ be stationary, and $Z := \{M \prec H_\Omega \mid M \cap H_\theta \in \mathrm{IC}_{\omega_1}\}$ (where $\Omega >> \theta$; note Z is stationary). TFAE:

1. R reflects to stationarily many $M \in Z$.
2. There are stationarily many $M \in Z$ such that R condenses correctly on M;
3. There is a stationary $S \subset Z$ such that whenever $j : V \to_G ult(V, G)$ is a generic ultrapower with $S \in G$, then R remains stationary in $ult(V, G)$.
   - not necessarily in $V[G]$
   - note: $cr(j) = \omega_2$

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The last characterization can be generalized (i.e. there is some normal filter F extending the club filter such that whenever U is a V-normal ultrafilter extending F, R remains stationary in $ult(V, U)$.)

The Diagonal Reflection Principle (DRP)

Definition

(C.) Let Z be a class of ω_1-sized sets (e.g. $Z = \mathsf{IA}_{\omega_1}$ or $Z = \mathsf{IC}_{\omega_1}$).
$DRP(\theta, Z)$ means that there are stationarily many $M \prec H_{(\theta^\omega)^+}$ such that:

- $M \cap H_\theta \in Z$
- R reflects to M for every stationary $R \subset [H_\theta]^\omega$ which is an element of M.

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For the rest of the talk, we fix $Z = \mathsf{IC}_{\omega_1}$ and omit reference to it. DRP means $DRP(\theta)$ holds for all regular $\theta \geq \omega_2$.

Characterizations of DRP

Theorem

TFAE:

1. $DRP(\theta)$
2. There are stationarily many M such that $NS \restriction [M \cap H_\theta]^\omega$ condenses correctly via M.
3. There is a stationary S such that whenever $j : V \to_G ult(V, G)$ is a generic ultrapower with $S \in G$, then all stationary subsets of $[H_\theta]^\omega$ from V remain stationary in $ult(V, G)$
   - not necessarily in $V[G]$!
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   - *not necessarily in $V[G]$!*
   - *again, $cr(j) = \omega_2$*

So $DRP(\theta)$ is a weaker version of the following statement:

"There is an ideal I such that $(I^+, \subset)$ is a proper forcing."

Proper ideal forcings and well-determined generic embeddings with critical point ω_2

Suppose I is an ideal over $\wp_{\omega_2}(H_\theta)$ such that $(I^+, \subset)$ is proper; it is known this implies I is precipitous. Let $j : V \to_G ult(V, G)$ be generic ultrapower; note $cr(j) = \omega_2$. Let $\tilde{\theta} := sup(j``\theta)$.

Let $\vec{S}$ partition $\theta \cap \text{cof}(\omega)$ into θ stationary sets, and *assume* $j(\vec{S}) \in V$. Then $j \restriction \theta \in V$.

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- Because $j``\theta = \{\eta < j(\theta) \mid j(\vec{S})_\eta$ reflects at $\tilde{\theta}\}$; this is primarily because:
  - properness of I^+ implies for every $i < \theta$, S_i remains stationary in $ult(V, G)$ and so $j``S_i$ is stationary there as well (note $j \restriction H_\theta \in ult(V, G)$).

DRP and well-determined generic embeddings with critical point ω_2

Aside from assuming $j(\vec{S}) \in V$ *and some degree of precipitousness*, the points from the previous slide used only very weak consequences of "$(I^+, \subset)$ is proper."

In particular, minor variations of DRP can be used instead of the "$(I^+, \subset)$ is proper" assumption.

- And variations of DRP follow from MM (later).

Chang ideals and DRP

(maybe skip)...

Forcing Axioms and "Plus" versions

Definition

$MA^{+\alpha}(\mathbb{P})$ means for every ω_1-sized collection $\mathcal{D}$ of dense sets and every α-sequence $\mathcal{S} = \langle \dot{S}_i \mid i < \alpha \rangle$ of $\mathbb{P}$-names of stationary subsets of ω_1, there is a filter F which:

- meets every set in $\mathcal{D}$
- evaluates each name in $\mathcal{S}$ as a stationary set (i.e. $(\dot{S}_i)_F := \{\beta < \omega_1 \mid (\exists q \in F)(q \Vdash \check{\beta} \in \dot{S}_i)\}$ is stationary for each $i < \alpha$).

$MA^{+\alpha}(\Gamma)$ means $MA^{+\alpha}(\mathbb{P})$ holds for every $\mathbb{P} \in \Gamma$.

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$MA^{+\alpha}(\Gamma)$ means $MA^{+\alpha}(\mathbb{P})$ holds for every $\mathbb{P} \in \Gamma$.

- $MA^{+}(\Gamma)$ means $MA^{+1}(\Gamma)$.
- What I'm calling $MA^{+\omega_1}(\Gamma)$ appears sometimes in the literature as $MA^{++}(\Gamma)$ (and in Baumgartner's original article as just one "plus"...)

Forcing Axioms and reflection

Theorem

(Baumgartner) $MA^{+\omega_1}(\sigma$-closed posets) implies that for every regular $\theta \geq \omega_2$, every ω_1-sized collection of stationary subsets of $\theta \cap cof(\omega)$ have a common reflection point of cofinality ω_1.

- Even for just $\theta = \omega_3$ the consistency strength of this kind of reflection is not known, but requires at least measurable cardinals of high Mitchell order.

Theorem

(Foreman-Magidor-Shelah): MM implies every stationary $R \subset [H_\theta]^\omega$ reflects to stationarily many sets in IA_{ω_1}.

Forcing Axioms and DRP

Theorem

(C.) Assume $MA^{+\omega_1}$(σ-closed posets). Then DRP(θ) holds for every regular $\theta \geq \omega_2$.

Forcing Axioms and DRP

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(C.) Assume $MA^{+\omega_1}$*(*σ*-closed posets). Then DRP(*θ*) holds for every regular* $\theta \geq \omega_2$*.*

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A nice characterization of forcing axioms

Theorem

(Woodin) TFAE for any separative poset $\mathbb{P}$ *(here* $\theta >> |\mathbb{P}|$*):*

1. $MA(\mathbb{P})$
2. $S_{\mathbb{P}}$ *is stationary, where* $S_{\mathbb{P}} := \{M \prec H_\theta \mid \omega_1 \subset M \text{ and } (\exists g)(g \text{ is an } (M, \mathbb{P})\text{-generic filter})\}$

(similiar version for $MA^{+\alpha}(\mathbb{P})$)

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(similiar version for $MA^{+\alpha}(\mathbb{P})$)

- In particular, if say *PFA* holds then for every proper $\mathbb{P}$ there is a normal filter $F_{\mathbb{P}}$ concentrating on $S_{\mathbb{P}}$.
- (Shelah) However in ZFC there are proper $\mathbb{P}$, $\mathbb{Q}$ such that $S_{\mathbb{P}} \cap S_{\mathbb{Q}}$ is nonstationary.

$MA(\Gamma)$ and ideals whose associated posets are in Γ

QUESTION: Is $MA(\Gamma)$ consistent with the existence of ideals such that $(I^+, \subset) \in \Gamma$?

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It is well-known that in $V^{Col(\omega_1, <\kappa)}$ where κ is supercompact:

- $MA^{+\omega_1}(\sigma\text{-closed})$ holds
- There are filters F on $\wp_{\omega_2}(H_\theta)$ such that $(F^+, \subset)$ is equivalent to a σ-closed forcing.

$MA(\Gamma)$ and ideals whose associated posets are in Γ

QUESTION: Is $MA(\Gamma)$ consistent with the existence of ideals such that $(I^+, \subset) \in \Gamma$?

It is well-known that in $V^{Col(\omega_1, <\kappa)}$ where κ is supercompact:

- $MA^{+\omega_1}(\sigma\text{-closed})$ holds
- There are filters F on $\wp_{\omega_2}(H_\theta)$ such that $(F^+, \subset)$ is equivalent to a σ-closed forcing.

But also:

Theorem

(C.) It is consistent with a superhuge cardinal that PFA holds and for each proper $\mathbb{P}$ there is a normal filter $F_{\mathbb{P}}$ concentrating on $S_{\mathbb{P}}$ such that $(F_{\mathbb{P}}^+, \subset)$ is a proper forcing.

When $(I^+, \subset)$ completely embeds into another ideal forcing

By the Woodin characterization of $MA(\Gamma)$, if $MA(\Gamma)$ holds and I is an ideal such that $(I^+, \subset) \in \Gamma$, then $(I^+, \subset)$ completely embeds into another ideal forcing.

- namely, into the poset for $NS \restriction S_{(I^+, \subset)}$

It is natural to ask if this complete embedding can be the same as the "lifting" map in the Rudin-Keisler sense.

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It is natural to ask if this complete embedding can be the same as the "lifting" map in the Rudin-Keisler sense.

Partial positive answer: there is a model of PFA starting from a super-2-huge cardinal, where there are I, J where both $(I^+, \subset)$ and $(J^+, \subset)$ are proper, I is the projection of J in the Rudin-Keisler sense, and this projection is also a *forcing* projection.

(I don't know if we can arrange that J is the NS ideal...)

Proof: $PFA^{+\omega_1}$ (just $MA^{+\omega_1}$(σ-closed)) implies $DRP(\theta)$.

$\mathbb{Q}$:= continuous countable chains of models from $H_{(\theta^\omega)^+}$, ordered by end-extension.

Let $G \subset \mathbb{Q}$ be generic and $\langle N^G_\alpha \mid \alpha < \omega_1 \rangle$ the generic object. Note that in $V[G]$: $|H^V_{(\theta^\omega)^+}| = \omega_1$.

Let $\langle \dot{R}_\alpha \mid \alpha < \omega_1 \rangle$ be a name for enumeration of all stationary subsets of $[\theta]^\omega$ from the ground model.

$\dot{S}_\alpha$:= indices of the models in $\dot{R}_\alpha$.

Proof: $PFA^{+\omega_1}$ (just $MA^{+\omega_1}(\sigma\text{-closed})$) implies $DRP(\theta)$.

$\mathbb{Q} :=$ continuous countable chains of models from $H_{(\theta^\omega)^+}$, ordered by end-extension.

Let $G \subset \mathbb{Q}$ be generic and $\langle N_\alpha^G \mid \alpha < \omega_1 \rangle$ the generic object. Note that in $V[G]$: $|H^V_{(\theta^\omega)^+}| = \omega_1$.

Let $\langle \dot{R}_\alpha \mid \alpha < \omega_1 \rangle$ be a name for enumeration of all stationary subsets of $[\theta]^\omega$ from the ground model.

$\dot{S}_\alpha :=$ indices of the models in $\dot{R}_\alpha$.

Each $\dot{S}_\alpha$ names a stationary subset of ω_1 (b/c $\mathbb{Q}$ is σ-closed so the set named by $\dot{R}_\alpha$ remains stationary).

proof, cont.

Let $S_{\mathbb{Q}} \subset \wp_{\omega_2}(H_{(\theta^\omega)^+})$ be the stationary set from the characterization of $MA^{+\omega_1}$.

So for every $M \in S_{\mathbb{Q}}$: $\omega_1 \subset M$ and there is a g which is $(M, \mathbb{Q})$-generic such that $(\dot{S}_\alpha)_g$ is stationary for every $\alpha < \omega_1$.

proof, cont.

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So for every $M \in S_{\mathbb{Q}}$: $\omega_1 \subset M$ and there is a g which is $(M, \mathbb{Q})$-generic such that $(\dot{S}_\alpha)_g$ is stationary for every $\alpha < \omega_1$.

Fix such an M and g.

Note $\vec{N}^g$ witnesses that a large initial segment of M is internally approachable. (density argument)

Let $R \in M$ be a stationary subset of $[H_\theta]^\omega$. Then $R = \dot{R}^g_\alpha$ for some α.

So $R \cap [M]^\omega$ contains the models in the generic chain indexed by S^g_α.

MM implies wDRP

Sketch: (forcing and proof rely heavily on Foreman's paper): Conditions of the form $\langle f(\beta), N_\beta \mid \beta \leq \delta \rangle$ where (fix some maximal antichain $\langle T_\alpha \mid \alpha < \omega_1 \rangle$ which is pairwise disjoint):

1. $\delta < \omega_1$
2. $\vec{N}$ continuous $\in$-chain
3. $f : \delta + 1 \to H_{\theta^+}$
4. For every $\beta < \delta$:
   - If $f(\beta)$ is a stationary subset of $\theta \cap \mathrm{cof}(\omega)$, then for all limit $\beta' \in (\beta, \delta] \cap T_\beta$ require that $\sup(N_{\beta'} \cap \theta) \in f(\beta)$.

outline of proof

- $D_R := \{(f, \vec{N}) | R \in \text{range}(f)\}$ is dense
- $D_\alpha := \{q \in \mathbb{Q} | \alpha < \delta^q\}$ is dense for each $\alpha < \omega_1$
- stationary set preservation

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- stationary set preservation

Then let $S_{\mathbb{Q}}$ be the stationary set of $M \in \wp_{\omega_2}(H_{\theta^+})$ for which a generic exists.

- Every $R \in M$ is of the form $f^{g_M}(\beta)$ some $\beta < \omega_1$
- So the points in the generic chain indexed by T_β (above β) witness that R reflects to $\sup(M \cap \theta)$.

Final remarks

Corollary

Strong forcing axioms imply there are generic embeddings which weakly resemble generic embeddings by proper ideal forcings.

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Similar ideas can use $MM^{+\omega_1}$ to form a kind of product of certain s.s.p. forcings.

Final remarks

Assume MM and that for each stationary set preserving $\mathbb{P}$ there is a precipitous ideal whose dual concentrates on $S_{\mathbb{P}}$.

- What more can we say about these generic embeddings?
- e.g. when $\mathbb{P}$ is the s.s.p. poset from above used to show MM implies diagonal reflection?