Diagonal stationary reflection and generic ultrapowers

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MAMLS 2011, Harvard
February 19, 2011
Main Theme

Strong forcing axioms (MM, PFA) imply the existence of ideals with interesting generic ultrapowers; these generic ultrapowers have critical point $\omega_2$. 
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Strong forcing axioms (MM, PFA) imply the existence of ideals with interesting generic ultrapowers; these generic ultrapowers have critical point $\omega_2$.

Vaguely: For sufficiently large classes $\Gamma$ of posets, $MA(\Gamma)$ implies there are ideals $I$ whose positive-set forcings are “almost” in $\Gamma$. 
Motivation: Condensation

The (duals of the) ideals will concentrate on $M \in \wp_{\omega_2}(H_\theta)$ which have condensation-like properties.
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Gödel’s Condensation Lemma for L: whenever $M \prec (H_\theta, \in)$, if $\sigma_M : H_M \rightarrow H_\theta$ is the inverse of Mostowski Collapse of $M$ then for every $\alpha \in H_M \cap ORD$: $(L_\alpha)^{H_M} = L_\alpha$.

So the function $\alpha \mapsto L_\alpha$ condenses on $M$. 
Motivation: Two condensation-like principles under strong forcing axioms

Strong forcing axioms imply condensation-like properties:

- (Viale-Weiss) Proper Forcing Axiom implies ISP
- (Foreman) Martin’s Maximum implies highly simultaneous ("diagonal") stationary set reflection
Motivation: A condensation principle under PFA

Theorem

(Viale/Weiss): Assume PFA and fix regular $\Omega \gg \theta \geq \omega_2$. There are stationarily many $M \in \wp_{\omega_2}(H_\Omega)$ such that whenever $N \mapsto F(N) \subset N$ is a slender function on $\wp_{\omega_2}(H_\theta)$ and $F \in M$, then $M$ “catches” $F$. 

$\sigma^{-1} Y \subset \bar{M}$ is an element of $\bar{M}$.
Motivation: A condensation principle under PFA

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- i.e. if $\sigma : \tilde{M} \to M$ is inverse of trans. collapse and $Y := F(M \cap H_\theta)$, then $\sigma^{-1}“Y \subset \tilde{M}$ is an element of $\tilde{M}$

- So $M$ detects a lot of 2nd order information about itself.
Motivation: A condensation principle under MM

Theorem
(Foreman): Assume MM and suppose \( \theta \subset H \subseteq H_\theta \) where \( |H| = \theta \). Fix a partition \( \langle R_i \mid i \in H \rangle \) of \( \theta \cap \text{cof}(\omega) \) into stationary sets.

Then there are stationarily many (internally approachable) \( M \in 2^{\omega_2}(H) \) such that:

\[
R_i \text{ reflects to } \sup(M \cap \theta) \iff i \in M
\quad (1)
\]

\( \overset{\text{Again}}{\Rightarrow} \) \( M \) detects some 2nd order information.
Motivation: A condensation principle under MM

Theorem

(Foreman): Assume MM and suppose $\theta \subset H \subseteq H_\theta$ where $|H| = \theta$. Fix a partition $\langle R_i \mid i \in H \rangle$ of $\theta \cap \text{cof}(\omega)$ into stationary sets.

Then there are stationarily many (internally approachable) $M \in \wp(\omega_2)(H)$ such that:

$$R_i \text{ reflects to } \sup(M \cap \theta) \iff i \in M$$  \hspace{1cm} (1)

- This implies: if $\sigma : \tilde{M} \to M$ is inverse of collapse map, then $\tilde{M}$ is correct about stationarity of every $\sigma^{-1}(R_i)$ (for $R_i \in M$ from the fixed partition).
- Again, $M$ detects some 2nd order information.
Outline

- Notation and background
- Stationary set reflection and connections with:
  - Condensation of NS
  - Generic embeddings with critical point $\omega_2$
- The Diagonal Reflection Principle (DRP)
- Forcing axioms imply DRP
  - And a detour involving $MA(\Gamma)$ and ideals whose positive-set-forcings are in $\Gamma$
Notation and background

- $\mathcal{P}_\kappa(H_\theta) := \{ M \prec H_\theta \mid |M| < \kappa \text{ and } M \cap \kappa \in \kappa \}$

- $\text{IA}_{\omega_1}$ is the class of $M$ such that there is some $\in$-increasing, continuous elementary chain $\langle N_\alpha \mid \alpha < \omega_1 \rangle$ of countable elementary submodels of $M$ such that
  - $\bigcup_{\alpha<\omega_1} N_\alpha = M$
  - Every proper initial segment of $\vec{N}$ is element of $M$

- $\text{IC}_{\omega_1}$ defined similarly, except only require each $N_\alpha \in M$
  (equiv: $M \cap [M]_\omega$ contains a club)
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- $IC_{\omega_1}$ defined similarly, except only require each $N_\alpha \in M$
  (equiv: $M \cap [M]^{\omega}$ contains a club)

This talk focuses on $\wp_{\omega_2}(H_\theta) \cap IC_{\omega_1}$. 
Stationary set reflection

**Definition**
A set $S$ *reflects at* $\gamma$ iff $S \cap \gamma$ is stationary in $\gamma$. A set $S \subset \kappa$ *reflects* iff there is a $\gamma < \kappa$ s.t. $S$ reflects at $\gamma$.
Stationary set reflection

Definition
A set $S$ reflects at $\gamma$ iff $S \cap \gamma$ is stationary in $\gamma$. A set $S \subset \kappa$ reflects iff there is a $\gamma < \kappa$ s.t. $S$ reflects at $\gamma$.

If $\kappa$ is measurable (or just weakly compact), then:
- every stationary subset of $\kappa$ reflects.
- $\forall \text{Col}(\omega_1, < \kappa) \models \text{“every stationary subset of } \omega_2 \cap \text{cof}(\omega) \text{ reflects”}$
  - The quoted statement is equiconsistent with a Mahlo cardinal (Harrington/Shelah)
Simultaneous stationary reflection

We can also require distinct sets to have a common reflection point. If $\kappa$ is measurable, then:

- Every $< \kappa$-sized collection of stationary sets have a common reflection point

- $V^{Col(\omega_1, < \kappa)} \models$ "every $\omega_1$-sized collection of stationary subsets of $\omega_2 \cap \text{cof}(\omega)$ have a common reflection point"
  - The quoted statement is equiconsistent with a weakly compact cardinal (Magidor)
Generalized stationary reflection

For stationary $R \subseteq [H_\theta]^\omega$, say $R$ reflects to $M$ iff $R \cap [M]^\omega$ is stationary in $[M]^\omega$.

- i.e. for every algebra $\mathcal{A}$ on $M$, there is an $N \in R$ with $N \prec \mathcal{A}$.
Generalized stationary reflection

For stationary $R \subset [H_\theta]^\omega$, say $R$ reflects to $M$ iff $R \cap [M]^\omega$ is stationary in $[M]^\omega$.

- i.e. for every algebra $\mathcal{A}$ on $M$, there is an $N \in R$ with $N \prec \mathcal{A}$.

“For every regular $\theta \geq \omega_2$, every stationary $R \subset [\theta]^\omega$ reflects to an $M$ of size $\omega_1$”:

- has powerful consequences if $\theta$ is large, e.g. failure of square, $NS_{\omega_1}$ is precipitous and more (F-M-S)
- follows from MM (Foreman-Magidor-Shelah)
- holds in $V^{Col(\omega_1,\kappa)}$ where $\kappa$ is supercompact
Stationary reflection and condensation of NS

“$R$ reflects to $M$” is equivalent to saying that the transitive collapse of $M$ is correct about the stationarity of the preimage of $R$. (assuming $M$ is sufficiently approachable)
Stationary reflection and condensation of NS

“$R$ reflects to $M$” is equivalent to saying that the transitive collapse of $M$ is correct about the stationarity of the preimage of $R$. (assuming $M$ is sufficiently approachable)

Proof: ($\iff$): Suppose $R \in M$ reflects to $M$. Let $\sigma : \tilde{H} \to M$ and $\sigma(\tilde{R}) = R$. So $\tilde{H} \models \text{“}\tilde{R} \text{ is stationary.”}$

NTS: $\tilde{R}$ is really stationary.

$\quad \Rightarrow$ In $V$ let $\tilde{A} = (\tilde{H}, (\tilde{f}_n)_{n \in \omega})$

$\quad \Rightarrow$ Need to find a $\tilde{N} \prec \tilde{A}$ s.t. $\tilde{N} \in \tilde{R}$

$\quad \Rightarrow$ Use $\sigma$ to transfer $\tilde{A}$ to a structure $A = (M \cap H_\theta, (f_n)_n)$.

$\quad \Rightarrow$ Since $R \cap [M]^{\omega}$ is stationary and $M \cap [M]^{\omega}$ contains a club (this is the approachability requirement on $M$), there is an $N \in R \cap M \cap [M]^{\omega}$ such that $N \prec A$.

$\quad \Rightarrow$ Then $\sigma^{-1}(N) \in \tilde{R}$ and $\sigma^{-1}(N) \prec \tilde{A}$. 
Other characterizations, and generic ultrapowers

Let $R \subset [H_\theta]^\omega$ be stationary, and $Z := \{ M \prec H_\Omega \mid M \cap H_\theta \in IC_{\omega_1} \}$ (where $\Omega > \theta$; note $Z$ is stationary). TFAE:

1. $R$ reflects to stationarily many $M \in Z$.
2. There are stationarily many $M \in Z$ such that $R$ condenses correctly on $M$;
3. There is a stationary $S \subset Z$ such that whenever $j : V \to_G ult(V, G)$ is a generic ultrapower with $S \in G$, then $R$ remains stationary in $ult(V, G)$.
   - not necessarily in $V[G]$
   - note: $cr(j) = \omega_2$
Other characterizations, and generic ultrapowers

Let $R \subset [H_\theta]^\omega$ be stationary, and $Z := \{M \prec H_\Omega \mid M \cap H_\theta \in IC_{\omega_1}\}$ (where $\Omega \gg \theta$; note $Z$ is stationary). TFAE:

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   ▶ not necessarily in $V[G]$  
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The last characterization can be generalized (i.e. there is some normal filter $F$ extending the club filter such that whenever $U$ is a $V$-normal ultrafilter extending $F$, $R$ remains stationary in $ult(V, U)$.)
The Diagonal Reflection Principle (DRP)

Definition
(C.) Let $Z$ be a class of $\omega_1$-sized sets (e.g. $Z = \text{IA}_{\omega_1}$ or $Z = \text{IC}_{\omega_1}$). $\text{DRP}(\theta, Z)$ means that there are stationarily many $M \prec H(\theta \omega)^+$ such that:

- $M \cap H_\theta \in Z$
- $R$ reflects to $M$ for every stationary $R \subset [H_\theta]^{\omega}$ which is an element of $M$. 

For the rest of the talk, we fix $Z = \text{IC}_{\omega_1}$ and omit reference to it.

$\text{DRP}$ means $\text{DRP}(\theta)$ holds for all regular $\theta \geq \omega_2$.
The Diagonal Reflection Principle (DRP)

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For the rest of the talk, we fix $Z = \mathbf{IC}_{\omega_1}$ and omit reference to it. $\text{DRP}$ means $\text{DRP}(\theta)$ holds for all regular $\theta \geq \omega_2$. 
Characterizations of DRP

Theorem
TFAE:

1. \( \text{DRP}(\theta) \)
2. There are stationarily many \( M \) such that \( \text{NS} \upharpoonright [M \cap H_\theta]^\omega \) condenses correctly via \( M \).
3. There is a stationary \( S \) such that whenever \( j : V \to G \) is a generic ultrapower with \( S \in G \), then all stationary subsets of \( [H_\theta]^\omega \) from \( V \) remain stationary in \( \text{ult}(V, G) \)
   - not necessarily in \( V[G] \)!
   - again, \( cr(j) = \omega_2 \)
Characterizations of DRP

Theorem

TFAE:

1. \( DRP(\theta) \)
2. There are stationarily many \( M \) such that \( NS \upharpoonright [M \cap H_{\theta}]^{\omega} \)
   condenses correctly via \( M \).
3. There is a stationary \( S \) such that whenever \( j : V \rightarrow G \) ult\((V, G)\) is a generic ultrapower with \( S \in G \), then all stationarily subsets of \([H_{\theta}]^{\omega}\) from \( V \) remain stationary in
   ult\((V, G)\)
   ▶ not necessarily in \( V[G]! \)
   ▶ again, \( cr(j) = \omega_2 \)

So \( DRP(\theta) \) is a weaker version of the following statement:

“There is an ideal \( I \) such that \((I^+, \subseteq)\) is a proper forcing.”
Proper ideal forcings and well-determined generic embeddings with critical point $\omega_2$

Suppose $I$ is an ideal over $\wp_{\omega_2}(H_\theta)$ such that $(I^+, \subset)$ is proper; it is known this implies $I$ is precipitous. Let $j : V \to G \ult(V, G)$ be generic ultrapower; note $cr(j) = \omega_2$. Let $\tilde{\theta} := sup(j``\theta)$.

Let $\tilde{S}$ partition $\theta \cap \text{cof}(\omega)$ into $\theta$ stationary sets, and assume $j(\tilde{S}) \in V$. Then $j \upharpoonright \theta \in V$. 

Proper ideal forcings and well-determined generic embeddings with critical point $\omega_2$

Suppose $I$ is an ideal over $\varnothing_{\omega_2}(H_\theta)$ such that $(I^+, \subset)$ is proper; it is known this implies $I$ is precipitous. Let $j : V \rightarrow G \ ult(V, G)$ be generic ultrapower; note $cr(j) = \omega_2$. Let $\tilde{\theta} := sup(j^{``}\theta)$.

Let $\tilde{S}$ partition $\theta \cap cof(\omega)$ into $\theta$ stationary sets, and assume $j(\tilde{S}) \in V$. Then $j \upharpoonright \theta \in V$.

Because $j^{``}\theta = \{\eta < j(\theta) \mid j(\tilde{S})_\eta \text{ reflects at } \tilde{\theta}\}$; this is primarily because:

- properness of $I^+$ implies for every $i < \theta$, $S_i$ remains stationary in $ult(V, G)$ and so $j^{``}S_i$ is stationary there as well (note $j \upharpoonright H_\theta \in ult(V, G)$).
DRP and well-determined generic embeddings with critical point $\omega_2$

 Aside from assuming $j(\tilde{S}) \in V$ and some degree of precipitousness, the points from the previous slide used only very weak consequences of "$(I^+, \subset)$ is proper."

 In particular, minor variations of DRP can be used instead of the "$(I^+, \subset)$ is proper" assumption.

 ▶ And variations of DRP follow from MM (later).
Chang ideals and DRP

(maybe skip)
Forcing Axioms and “Plus” versions

Definition

$MA^{+\alpha}(\mathbb{P})$ means for every $\omega_1$-sized collection $\mathcal{D}$ of dense sets and every $\alpha$-sequence $S = \langle \dot{S}_i | i < \alpha \rangle$ of $\mathbb{P}$-names of stationary subsets of $\omega_1$, there is a filter $F$ which:

- meets every set in $\mathcal{D}$

- evaluates each name in $S$ as a stationary set (i.e. $(\dot{S}_i)_F := \{ \beta < \omega_1 | (\exists q \in F)(q \Vdash \dot{\beta} \in \dot{S}_i) \}$ is stationary for each $i < \alpha$).

$MA^{+\alpha}(\Gamma)$ means $MA^{+\alpha}(\mathbb{P})$ holds for every $\mathbb{P} \in \Gamma$. 
Forcing Axioms and “Plus” versions

Definition

\( MA^{+\alpha}(\mathbb{P}) \) means for every \( \omega_1 \)-sized collection \( D \) of dense sets and every \( \alpha \)-sequence \( S = \langle \dot{S}_i \mid i < \alpha \rangle \) of \( \mathbb{P} \)-names of stationary subsets of \( \omega_1 \), there is a filter \( F \) which:

- meets every set in \( D \)
- evaluates each name in \( S \) as a stationary set (i.e. \( (\dot{S}_i)_F := \{ \beta < \omega_1 \mid (\exists q \in F)(q \Vdash \check{\beta} \in \dot{S}_i) \} \) is stationary for each \( i < \alpha \)).

\( MA^{+\alpha}(\Gamma) \) means \( MA^{+\alpha}(\mathbb{P}) \) holds for every \( \mathbb{P} \in \Gamma \).

- \( MA^{+}(\Gamma) \) means \( MA^{+1}(\Gamma) \).

- What I’m calling \( MA^{+\omega_1}(\Gamma) \) appears sometimes in the literature as \( MA^{++}(\Gamma) \) (and in Baumgartner’s original article as just one “plus”...).
Forcing Axioms and reflection

Theorem

(Baumgartner) $\text{MA}^+\omega_1(\sigma\text{-closed posets})$ implies that for every regular $\theta \geq \omega_2$, every $\omega_1$-sized collection of stationary subsets of $\theta \cap \text{cof}(\omega)$ have a common reflection point of cofinality $\omega_1$.

- Even for just $\theta = \omega_3$ the consistency strength of this kind of reflection is not known, but requires at least measurable cardinals of high Mitchell order.

Theorem

(Foreman-Magidor-Shelah): $\text{MM}$ implies every stationary $R \subset [H_\theta]^\omega$ reflects to stationarily many sets in $\text{IA}_{\omega_1}$.
Theorem

(C.) Assume $\text{MA}^+\omega_1 (\sigma$-closed posets). Then $\text{DRP} (\theta)$ holds for every regular $\theta \geq \omega_2$. 
Forcing Axioms and DRP

Theorem
(C.) Assume $\text{MA}^+\omega_1$ ($\sigma$-closed posets). Then $\text{DRP}(\theta)$ holds for every regular $\theta \geq \omega_2$.

Theorem
(C.) Assume $\text{MM}$. Then $\text{wDRP}(\theta)$ holds for every regular $\theta \geq \omega_2$. 
A nice characterization of forcing axioms

Theorem
(Woodin) TFAE for any separative poset $\mathbb{P}$ (here $\theta \gg |\mathbb{P}|$):

1. $\text{MA}(\mathbb{P})$

2. $S_\mathbb{P}$ is stationary, where $S_\mathbb{P} := \{ M \prec H_\theta \mid \omega_1 \subset M \text{ and } (\exists g)(g \text{ is an } (M, \mathbb{P})\text{-generic filter)}\}$

(similar version for $\text{MA}^{+\alpha}(\mathbb{P})$)
A nice characterization of forcing axioms

Theorem

(Woodin) TFAE for any separative poset \( P \) (here \( \theta \gg |P| \)):

1. \( MA(P) \)
2. \( S_P \) is stationary, where \( S_P := \{ M \prec H_\theta \mid \omega_1 \subset M \text{ and } (\exists g)(g \text{ is an } (M, P)-\text{generic filter}) \} \)

(similiar version for \( MA^+\alpha(P) \))

- In particular, if say PFA holds then for every proper \( P \) there is a normal filter \( F_P \) concentrating on \( S_P \).
- (Shelah) However in ZFC there are proper \( P, Q \) such that \( S_P \cap S_Q \) is nonstationary.
MA(Γ) and ideals whose associated posets are in Γ

QUESTION: Is MA(Γ) consistent with the existence of ideals such that \((I^+, \subseteq) \in \Gamma\)?
QUESTION: Is $MA(\Gamma)$ consistent with the existence of ideals such that $(I^+, \subset) \in \Gamma$?

It is well-known that in $V^{Col(\omega_1,<\kappa)}$ where $\kappa$ is supercompact:

- $MA^{+\omega_1}(\sigma\text{-closed})$ holds
- There are filters $F$ on $\wp_\omega(H_\theta)$ such that $(F^+, \subset)$ is equivalent to a $\sigma\text{-closed}$ forcing.
MA(Γ) and ideals whose associated posets are in Γ

QUESTION: Is $MA(Γ)$ consistent with the existence of ideals such that $(I^+, ⊂) ∈ Γ$?

It is well-known that in $V^{Col(ω₁,<κ)}$ where $κ$ is supercompact:

- $MA^{+ω₁}(σ$-closed) holds
- There are filters $F$ on $犷_{ω₂}(Hθ)$ such that $(F^+, ⊂)$ is equivalent to a $σ$-closed forcing.

But also:

Theorem
(C.) It is consistent with a superhuge cardinal that PFA holds and for each proper $P$ there is a normal filter $F_P$ concentrating on $S_P$ such that $(F_P^+, ⊂)$ is a proper forcing.
When \((I^+, \subset)\) completely embeds into another ideal forcing

By the Woodin characterization of \(MA(\Gamma)\), if \(MA(\Gamma)\) holds and \(I\) is an ideal such that \((I^+, \subset) \in \Gamma\), then \((I^+, \subset)\) completely embeds into another ideal forcing.

- namely, into the poset for \(NS \upharpoonright S_{(I^+, \subset)}\)

It is natural to ask if this complete embedding can be the same as the “lifting” map in the Rudin-Keisler sense.
When \((I^+, \subset)\) completely embeds into another ideal forcing

By the Woodin characterization of \(MA(\Gamma)\), if \(MA(\Gamma)\) holds and \(I\) is an ideal such that \((I^+, \subset) \in \Gamma\), then \((I^+, \subset)\) completely embeds into another ideal forcing.

- namely, into the poset for \(NS \upharpoonright S(I^+, \subset)\)

It is natural to ask if this complete embedding can be the same as the “lifting” map in the Rudin-Keisler sense.

Partial positive answer: there is a model of PFA starting from a super-2-huge cardinal, where there are \(I, J\) where both \((I^+, \subset)\) and \((J^+, \subset)\) are proper, \(I\) is the projection of \(J\) in the Rudin-Keisler sense, and this projection is also a \textit{forcing} projection.

(I don’t know if we can arrange that \(J\) is the NS ideal...
Proof: $PFA^{\omega_1}$ (just $MA^{\omega_1}(\sigma\text{-closed})$) implies $DRP(\theta)$.

$\mathbb{Q} :=$ continuous countable chains of models from $H(\theta \omega)^+$, ordered by end-extension.

Let $G \subset \mathbb{Q}$ be generic and $\langle N^G_\alpha \mid \alpha < \omega_1 \rangle$ the generic object. Note that in $V[G]$: $|H^V(\theta \omega)^+| = \omega_1$.

Let $\langle \dot{R}_\alpha \mid \alpha < \omega_1 \rangle$ be a name for enumeration of all stationary subsets of $[\theta]^\omega$ from the ground model.

$\dot{S}_\alpha :=$ indices of the models in $\dot{R}_\alpha$.
Proof: $PFA^{+\omega_1}$ (just $MA^{+\omega_1}(\sigma\text{-closed})$) implies $DRP(\theta)$.

$\mathbb{Q} :=$ continuous countable chains of models from $H(\theta^{\omega})^+$, ordered by end-extension.

Let $G \subset \mathbb{Q}$ be generic and $\langle N^G_\alpha \mid \alpha < \omega_1 \rangle$ the generic object. Note that in $V[G]: |H(V)^{\theta^{\omega}}| = \omega_1$.

Let $\langle \dot{R}_\alpha \mid \alpha < \omega_1 \rangle$ be a name for enumeration of all stationary subsets of $[\theta]^\omega$ from the ground model.

$\dot{S}_\alpha :=$ indices of the models in $\dot{R}_\alpha$.

Each $\dot{S}_\alpha$ names a stationary subset of $\omega_1$ (b/c $\mathbb{Q}$ is $\sigma$-closed so the set named by $\dot{R}_\alpha$ remains stationary).
proof, cont.

Let $S_Q \subset \wp_{\omega_2}(H(\theta \omega)^+)$ be the stationary set from the characterization of $MA^{+\omega_1}$.

So for every $M \in S_Q$: $\omega_1 \subset M$ and there is a $g$ which is $(M, \mathbb{Q})$-generic such that $(\dot{S}_\alpha)_g$ is stationary for every $\alpha < \omega_1$. 

Fix such an $M$ and $g$. Note $\vec{N}_g$ witnesses that a large initial segment of $M$ is internally approachable. (density argument)

Let $R \in M$ be a stationary subset of $[H(\theta \omega)]^{\omega_1}$. Then $R = \dot{R}_g$ for some $\alpha$. So $R \cap [M]^{\omega_1}$ contains the models in the generic chain indexed by $S_g \alpha$. 
proof, cont.

Let $S_Q \subset \rho_{\omega_2}(H(\theta \omega)^+)$ be the stationary set from the characterization of $MA^{+\omega_1}$.

So for every $M \in S_Q$: $\omega_1 \subset M$ and there is a $g$ which is $(M, Q)$-generic such that $(\dot{S}_\alpha)_g$ is stationary for every $\alpha < \omega_1$.

Fix such an $M$ and $g$.

Note $\vec{N}^g$ witnesses that a large initial segment of $M$ is internally approachable. (density argument)

Let $R \in M$ be a stationary subset of $[H_\theta]^\omega$. Then $R = \dot{R}_\alpha^g$ for some $\alpha$.

So $R \cap [M]^\omega$ contains the models in the generic chain indexed by $S_\alpha^g$. 
MM implies wDRP

Sketch: (forcing and proof rely heavily on Foreman’s paper): Conditions of the form \( \langle f(\beta), N_\beta \mid \beta \leq \delta \rangle \) where (fix some maximal antichain \( \langle T_\alpha \mid \alpha < \omega_1 \rangle \) which is pairwise disjoint):

1. \( \delta < \omega_1 \)
2. \( \vec{N} \) continuous \( \in \)-chain
3. \( f : \delta + 1 \rightarrow H_{\theta^+} \)
4. For every \( \beta < \delta \):
   - If \( f(\beta) \) is a stationary subset of \( \theta \cap \text{cof}(\omega) \), then for all limit \( \beta' \in (\beta, \delta] \cap T_\beta \) require that \( \sup(N_{\beta'} \cap \theta) \in f(\beta) \).
outline of proof

- $D_R := \{(f, \vec{N}) | R \in \text{range}(f)\}$ is dense
- $D_\alpha := \{q \in \mathbb{Q} | \alpha < \delta^q\}$ is dense for each $\alpha < \omega_1$
- stationary set preservation
outline of proof

- \( D_R := \{(f, \vec{N})| R \in \text{range}(f)\} \) is dense
- \( D_\alpha := \{q \in \mathbb{Q}| \alpha < \delta^q\} \) is dense for each \( \alpha < \omega_1 \)
- stationary set preservation

Then let \( S_\mathcal{Q} \) be the stationary set of \( M \in \wp_\omega_2(H_{\theta^+}) \) for which a generic exists.

- Every \( R \in M \) is of the form \( f^{g_M}(\beta) \) some \( \beta < \omega_1 \)
- So the points in the generic chain indexed by \( T_\beta \) (above \( \beta \)) witness that \( R \) reflects to \( \text{sup}(M \cap \theta) \).
Final remarks

Corollary

*Strong forcing axioms imply there are generic embeddings which weakly resemble generic embeddings by proper ideal forcings.*
Final remarks

Corollary

Strong forcing axioms imply there are generic embeddings which weakly resemble generic embeddings by proper ideal forcings.

Similar ideas can use $MM^{+\omega_1}$ to form a kind of product of certain s.s.p. forcings.
Assume MM and that for each stationary set preserving $\mathbb{P}$ there is a precipitous ideal whose dual concentrates on $S_\mathbb{P}$.

- What more can we say about these generic embeddings?
- e.g. when $\mathbb{P}$ is the s.s.p. poset from above used to show MM implies diagonal reflection?