Fine structure and internal theory of extender models

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Motivation:

1. "Axiomatize" constructions based on forming Skolem hulls
2. Provide a context for such an axiomatization that is as general as possible.

Context: The structes we work with have large degree of rigidity, and the Skolem hulls are definable in the optimal way. We also want to keep as much uniformity in our constructions as possible.

Canonical example: Gödel's constructible universe L.

Structures: J_α.

Uniformity of the construction is guaranteed by

(a) uniform Σ_1-Skolem functions
(b) condentation: If $\sigma : M \to J_\alpha$ is a Σ_1-preserving map and M is transitive then M is of the form $J_{\bar\alpha}$ for some $\bar\alpha \leq \alpha$.

This amount of uniformity suffices for some very basic constructions. More uniformity is achieved by extending the above to formulae of arbitrary complexity. For this purposee, the Lévy hierarchy is not suitable, as pointed out by Jensen.

Jensen's $\mathcal{L}^*$-language.

Typical structures used in fine structure theory are associated with a descending sequence of transitive universes:

$$M = M_0 \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

Example: In the case of L, cosider

- $M = J_\alpha$.
- α_k is the least ordinal $\bar{\alpha}$ such that there is $\mathbf{\Sigma}_k(M)$- definable set A such that $A \cap \bar{\alpha} \notin M$.
- $M_k = J_{\alpha_k}$.

Then M_k constitute a descenting chain with respect to inclusion; actually $M_{k+1} \in M_k$ whenever the inclusion is proper.

In the situation above, the ordinals α_k are called the **projecta** of M and denoted by ϱ_M^k.

So Jensen's $\mathcal{L}^*$-language is designed to express statements about this kind of universes.

For each $k \in \omega$ the language $\mathcal{L}^*$ has variables of type k, so in total we have infinitely many types of variables.

Variables of type k range over the universe M_k.

We write v^k to indicate that a variable of type k.

A quantification $(Qv^k \in w^\ell)$ is <u>bounded of type k</u> iff $\ell \geq k$.

The $\mathcal{L}^*$-hierarchy is defined in the natural way:

- ▶ $\Sigma_0^{(0)}$ are all formulae which have only bounded quantifications of type 0.
- ▶ $\Sigma_0^{(n)}$ is obtained by forming Boolean combinations of $\Sigma_1^{(k)}$ for $k < n$ and applying bounded quantification of type $\leq n$.
- ▶ $\Sigma_m^{(n)}$ is obtained by putting m alternating blocks of quantifiers of type n in front of a $\Sigma_0^{(n)}$-formula, starting with the existential one.

So when we say that a formula is $\Sigma_m^{(n)}$, the subscript m has the "old" meaning: It gives us the number of alternating blocks of quantifiers. The superscript n determines the highest type of quantification occuring in the formula.

Criticism: Having infinitely many types of variables is frightening.

Defence: All notions that can be easily proved for Σ_1 generalize for $\Sigma_1^{(n)}$. Moreover, the model theory for $\mathcal{L}^*$-structures works out very nicely.

We also get a natural notion of $\Sigma_m^{(n)}$ preserving maps $\sigma : M \to N$ where $M = (M_k \mid k \in \omega)$ and $N = (N_k \mid k \in \omega)$ are two $\mathcal{L}^*$-structures.

What else we get:

(a) $\Sigma_1^{(n)}$-satisfaction is uniformly definable by a $\Sigma_1^{(n)}$-fromula

(b) $\Sigma_1^{(n)}$-Skolem h_M^n function has a uniform $\Sigma_1^{(n)}$-definition.

(c) We can define projecta: ϱ^n is the least ordinal $\bar{\varrho}$ such that there is a set A that is $\Sigma_1^{(n-1)}(M)$-definable in some parameter $p \in [\textbf{Ordinals} \cap M]^{<\omega}$ and $A \cap \omega\bar{\varrho} \notin M$.

(d) We can define the standard parameter p_M^n: This is the $<^*$-least parameter p as in (c). Here $<^*$ is the standard well-ordering of $[\textbf{Ordinals}]^{<\omega}$ (lexicographic when sets are viewed as finite decreasing sequences.)

(e) We can define n-soundness: The structure is n-sound iff the Skolem hull $h_M^n(\omega\varrho_M^{n+1} \cup \{p_M^{n+1}\})$ is the entire M.

Notice that the $\mathcal{L}^*$-approach makes it possible to avoid talking about master codes.

In the case of L: Starting from $M = J_\alpha$ we get the $\mathcal{L}^*$-universe where $M_k = J_{\varrho^k}$. Notice this sequence is eventually constant.

Crucial property of the J-hierarchy: The structure J_α is always sound; this means n-sound for all n.

Square in L.

Given a cardinal κ, to each $\nu \in (\kappa, \kappa^+)$ such that $J_\nu \prec J_{\kappa^+}$ assign the **canonical collapsing structure** N_ν.

Here N_ν is the longest level J_β of L in which ν remains a cardinal. Then there is a unique $n = n(\nu)$ such that

$$\omega \varrho_{N_\nu}^{n+1} \leq \kappa < \omega \varrho_{N_\nu}^{n}.$$

Let C_ν be the set of all $\bar\nu$ such that

$$n(\bar\nu) = n$$

and there is a map

$$\sigma : N_{\bar\nu} \to N_\nu$$

satisfying:

(a) $\bar\nu$ is the critical point of σ and $\sigma(\bar\nu) = \nu$.

(b) σ is $\Sigma_0^{(n)}$-preserving.

(c) $\sigma(p_{N_{\bar\nu}}^n) = p_{N_\nu}^n$.

The sets C_ν have the following properties:

  (i) Each C_ν is closed in ν and if the cofinality of ν is uncountable then C_ν is unbounded in ν.

 (ii) (Coherency) If $\bar{\nu} \in C_\nu$ then $C_{\bar{\nu}} = C_\nu \cap \bar{\nu}$.

(iii) There is no club $C \subseteq \kappa^+$ that would thread the sequence (C_ν), that is, that would have the property that $C \cap \nu = C_\nu$ whenever ν is a limit point of C.

The sets C_ν constitute the so-called $\square(\kappa^+)$-sequence.

Here (iii) holds as otherwise the direct limit along the diagram of maps σ would turn out to be a collapsing level of L for κ^{+L}. In fact a set C as in (iii) cannot exist in any generic extension of **V** in which κ^{+L} has uncountable cofinality.

To get a $\square_\kappa$ sequence, that is, to replace (iii) by

$$\text{The order-type of } C_\nu \text{ is } \leq \kappa.$$

we need to thin out the sequence C_ν defined above. This is done by looking at countable Skolem hulls carefully.

The same apporach as above can be done when constructing:

- **Global square sequence:** Here to each singular limit ordinal ν we construct the canonical **singularizing structure** N_ν.

- $\square(\kappa)$-sequence for inaccessible κ that is not weakly compact. Here we first let T be the $<_L$-least κ-tree without cofinal branches. For limit ordinal $\nu < \kappa$ the structure N_ν is the longest level of L such that either ν is inaccessible in N_ν, or else ν is inaccessible in L and the $T|\nu$ has no cofinal branch in N_ν.

- $(\kappa, 1)$ morass. Here to each pair (α, ν) where κ is the largest cardinal in J_α and α is the largest cardinal in J_ν we assign the collapsing level of L for ν.

It is then natural to define the corresponding maps.

Extender models: We use the λ-indexing, due to certain technical advantages for the kind of constructions mentioned above.
So a (κ, λ)-extender is a function F with $\mathrm{dom}(F) \subseteq \mathcal{P}(\kappa)$ and $\mathrm{rng}(F) \subseteq \mathcal{P}(\lambda)$ such that

- F preserves recursive definitions (like $x \cap y$ etc...)
- Every object $[\alpha, f]$ in the ultrapower by F where $\alpha < \lambda$ and $f : \kappa \to \kappa$ is equal to an object of the form $[\beta, \mathrm{id}]$ for some $\beta < \lambda$.

Intuitively: F is a (κ, λ)-extender iff F is the restriction of an ultrapower map with critical point κ to a small domain (so λ is the image of κ).

A structure $M = (J^A_\nu, F)$ is **coherent** iff F is a (κ, λ)-extender, and letting $\tau = \kappa^{+M}$,

$$J^A_\nu = \mathrm{Ult}(J^A_\tau, F).$$

We write $\lambda(F)$ or $\lambda(M)$ to denote λ.

If F is a (κ, λ) extender and $\bar{\lambda} < \lambda$ we write $F|\bar{\lambda}$ to denote the map $x \mapsto F(x) \cap \bar{\lambda}$.

A premouse is a structure of the form $N = (J_\alpha^E, E_{\omega\alpha})$ such that

(a) E is a sequence indexed by ordinals. Each E_α is either empty or an extender.

(b) If $E_{\omega\alpha}$ is an extender then $N||\alpha = (J_\alpha^E, E_{\omega\alpha})$ is a coherent structure.

(c) If $\bar\alpha < \alpha$ then $M||\alpha$ is sound.

(d) Initial segment condition (ISC). If $\bar\alpha < \alpha$ and $\bar\lambda < \lambda(E_{\bar\alpha})$ is such that $E_{\bar\alpha}|\bar\lambda$ is an extender then this extender is on the sequence, i.e. is equal to $E_{\bar\nu}$ for some $\bar\nu < \bar\alpha$.

An extender model is a model of the form $L[E]$ where E is an extender sequence given by premice, i.e. $L[E]||\alpha$ is a premouse for all α.

Construction of extender models is one of the most important open questions in inner model theory (set theory?). These were constructed by Dodd-Jensen, Mitchell, Jensen, Mitchell-Steel,Steel, Schimmerling-Steel, Andretta-Neeman-Steel, and Neeman.

Our analysis does not use the construction of a model itself, but only the internal properties which are independent of the construction, but do depend to some extent on the indexing.

Condensation Lemma: Assume N is a level of $L[E]$, $\bar{N}$ is premouse and $\sigma : \bar{N} \to N$ is a map such that

(i) Critical point of σ is ν.

(ii) $h_{\bar{N}}^n(\nu \cup \{p_{\bar{N}}^{n+1}\})$ is the entire $\bar{N}$.

(iii) σ is $\Sigma_0^{(n)}$-preserving.

Then one of the following holds:

(a) $\bar{N}$ is the core of N and σ is the core map.

(b) $\bar{N}$ is a proper initial segment of N.

(c) ν has a cardinal predecessor in $\bar{N}$; denote it by κ. Then $\bar{N}$ is the fine ultrapower of the collapsing level of N for ν by an extender on the N-sequence with critical point κ with the single generator κ.

(d) $\bar{N}$ is a proper initial segment of $\mathrm{Ult}(N, E_\nu^N)$.

Now we can imitate Jensen's constructions and generalize them from L to $L[E]$. Everything becomes a bit more technical, with some additional technical issues, so it looka like everything generalizes in the straightforward way.

But ...

Proposition (Burke,Jensen) If the set

$$S_\kappa = \{x \in [\kappa^+]^{<\kappa} \mid x \cap \kappa \in \kappa \ \& \ \text{ordertype}(x) \text{ is a cardinal}\}$$

is stationary then $\square_\kappa$ fails.

So there must be a real obstacle for constructing square sequences in extender models.

A crucial step in the proof that the sets C_ν are unbounded for ν of uncountable cofinality uses the so called Interpolation Lemma, Due to Jensen. The lemma says that if $\sigma : \bar{N} \to N$ is a $\Sigma_1^{(n)}$-preserving map and $\sigma(\bar{\nu}) = \nu$ we can interpolate, or factor the map σ through a structure $\tilde{N}$, so we have

$$\tilde{\sigma} : \bar{N} \to \tilde{N} \quad \text{and} \quad \sigma' : \tilde{N} \to N$$

where both $\tilde{\sigma}, \sigma'$ are both $\Sigma_0^{(n)}$ and $\tilde{\sigma}$ is continuous at $\bar{\nu}$, that is it maps $\bar{\nu}$ continuously into $\tilde{\sigma}(\bar{\nu})$.

Point: If $n = 0$ then $\tilde{\sigma}$ is not sufficiently preserving to guarantee that $\tilde{N}$ is a premouse, although it is still a coherent structure.

Such a structure $\tilde{N}$ is called a **protomouse**.

But protomice are not levels of $L[E]$ — what now?

Answer: There does not seem to be a different way than using protomice in the analogous way as premice.

Plus side: Every protomouse coming from the above construction is associated to precisely one level of $L[E]$.

Minus side: Levels of $L[E]$ may have many protomice, so which one to choose?

Answer:

(i) Restrict to protomice with good preservation properties.

(ii) Pick one with the largest critical point.

It can be proved that protomice chosen this way are canonical in the sense that they are preserved under weakly preserving embeddings.

So we split the ordinals into two disjoint sets S^0, S^1 and perform the construction on each of these sets — fortunately, they don't interfere with each other.

Some theorems.

Theorem (Schimmerling-Z.2000?)

In any $L[E]$-model with λ-indexing, the following are equivalent:

(a) $\square_\kappa$ *holds*

(b) S_κ *is nonstationary*

(c) *The set of all $\nu < \kappa^+$ that index an extender is nonstationary.*

Theorem (Z.,2000?)

In any L[E]-model with λ-indexing, global $\square$ holds on singular cardinals.

Above, the sets C_ν are defined for singular cardinals only. There is no limitation given by large cardinals. So this form of global square is compatible with any large cardinals that can be defined in terms of extenders in our sense.

Theorem (Schimmerling-Z.,2008)

In any L[E]-model with λ-indexing, $(\kappa, 1)$-morass exists for every cardinal κ. In fact, there is a global Gap-1 morass.

However, unlike in the construction of a square sequence, here the two parts of the construction — the one with premice and one with protomice — interfere.

Theorem (Kypriotakis,Z.,2009)

In any L[E]-model with λ-indexing, if κ is an uncoutable cardinal and there is a nonreflecting stationary subset of κ^+ concentrating on points of cofinality smaller than κ then $\square(\kappa^+)$ holds.

Theorem (Z.,2010)

In any L[E]-model with λ-indexing, given an inaccessible κ, the following are equivalent.

(a) *κ is weakly compact.*

(b) *Nonreflecting stationary subsets of κ are dense in the poset of all stationary subsets of κ. In fact, a strong form of $\square(\kappa)$ holds.*