The HOD Dichotomy

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Definition

Covering holds relative to L if for each $\sigma \subset \mathrm{Ord}$ there exists $\tau \subset \mathrm{Ord}$ such that

1. $\tau \in L$ and $\sigma \subseteq \tau$,
2. $|\tau| \leq |\sigma| + \omega_1$.

Theorem (Jensen Covering Lemma)

One of the following hold.

(1) *Covering holds relative to L.*

(2) $0^{\#}$ *exists.*

The Jensen Dichotomy

One of the following hold.

1. *Every singular cardinal is singular in L and for each singular cardinal γ, $\gamma^+ = (\gamma^+)^L$.*
2. *Every uncountable cardinal is strongly inaccessible in L.*

Question

What generalizations of the Jensen Dichotomy are there?

Key issue: How to generalize L.

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is *ω-strongly measurable in* HOD if there exists $\gamma < \kappa$ such that:

(1) $(2^\gamma)^{\mathrm{HOD}} < \kappa$;

(2) There does not exist a sequence

$$\langle S_\alpha : \alpha < \gamma \rangle \in \mathrm{HOD}$$

of pairwise disjoint subsets of κ such that for each $\alpha < \gamma$, S_α is stationary in $\{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$.

Lemma

Suppose that κ is a regular cardinal which is ω-strongly measurable in HOD.
Then there exists a stationary set $S \subset \kappa$ such that:

(1) $S \subset \{\eta < \kappa \mid \mathrm{cof}(\eta) = \omega\}$;

(2) $S \in \mathrm{HOD}$;

(3) *Let $\mathcal{F}$ be the filter on S generated by the sets $C \cap S$ where C is club in κ. Then $\mathcal{F}$ is an ultrafilter on $\mathcal{P}(S) \cap \mathrm{HOD}$.*

If κ is ω-strongly measurable in HOD then κ is a measurable cardinal in HOD.

The HOD Dichotomy Theorem

Suppose that δ is an extendible cardinal. Then one of the following hold.

(1) *Suppose $\gamma > \delta$ and γ is a singular cardinal. Then γ is singular in HOD and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*

(2) *Every regular cardinal above δ is ω-strongly measurable in HOD.*

Is the HOD Dichotomy a non-trivial dichotomy?

It is not known if any cardinal above a supercompact cardinal can be ω-strongly measurable in HOD.

It is not known if γ^+ can be ω-strongly measurable in HOD if γ is a strong limit cardinal of uncountable cofinality.

It is not known if there can exist more than three cardinals which are *ω-strongly measurable in HOD.*

The HOD Conjecture

There is a proper class of uncountable regular cardinals κ which are **not** *ω-strongly measurable in HOD.*

Consequences of the HOD Conjecture

Theorem (HOD Conjecture)

Suppose that δ is an extendible cardinal. Then the following the following hold.

(1) *Suppose that γ is a singular cardinal and $\gamma > \delta$. Then γ is singular in HOD and*

$$(\gamma^+)^{\mathrm{HOD}} = \gamma^+.$$

(2) *Suppose that $\gamma > \delta$ and*

$$j : \mathrm{HOD} \cap V_{\gamma+1} \to M \subseteq \mathrm{HOD} \cap V_{j(\gamma)+1}$$

is an elementary embedding with $\mathrm{CRT}(j) \geq \delta$. Then $j \in \mathrm{HOD}$.

Further consequences of the HOD Conjecture

Corollary (HOD Conjecture)

Suppose that δ is an extendible cardinal. Then there is no non-trivial elementary embedding,

$$j : \mathrm{HOD} \to \mathrm{HOD}$$

such that $\delta \leq \mathrm{CRT}(j)$.

Theorem (HOD Conjecture)

Suppose that δ is an extendible cardinal. Then there is no non-trivial elementary embedding,

$$j : \mathrm{HOD}_{V_{\lambda+1}} \cap V_{\lambda+2} \to \mathrm{HOD}_{V_{\lambda+1}} \cap V_{\lambda+2}$$

such that $\delta < \lambda$ and such that $\mathrm{CRT}(j) < \lambda$.

The HOD Conjecture and $j : \mathrm{HOD} \to \mathrm{HOD}$

Theorem (HOD Conjecture)

Suppose there is an extendible cardinal. Then there exists $\alpha \in \mathrm{Ord}$ such that there is no non-trivial elementary embedding,

$$j : \mathrm{HOD} \to \mathrm{HOD}$$

such that $j(\alpha) = \alpha$.

Corollary (HOD Conjecture)

Suppose there is an extendible cardinal. Suppose that for each $i < \omega$,

$$j_i : \mathrm{HOD} \to \mathrm{HOD}$$

is a non-trivial elementary embedding.

Then $\lim_{n<\omega} \; j_n \circ \cdots \circ j_0(\mathrm{HOD})$ is not wellfounded.

Ω-valid sentences

Definition (ZF)

A sentence φ is Ω-valid from $\mathrm{ZFC} + \Phi$ if for all complete Boolean algebras, $\mathbb{B}$, for all $\alpha \in \mathrm{Ord}$, if

$$V_\alpha^{\mathbb{B}} \models \mathrm{ZFC} + \Phi$$

then $V_\alpha^{\mathbb{B}} \models \varphi$.

Theorem (ZF)

Suppose the HOD *Conjecture is* Ω*-valid from*

$$\mathrm{ZFC} + \text{“There is an extendible cardinal”},$$

δ *is an extendible cardinal, and that there is an extendible cardinal below* δ*.*

Then there exists a transitive class $N \subset V$ *and* $X \in V_\delta$ *such that the following hold.*

(1) $N \models \mathrm{ZFC}$,

(2) N *is* Σ_2*-definable from* X*.*

(3) *There exists a partial order* $\mathbb{P} \in N \cap V_\delta$ *such that for all* $A \subset \mathrm{Ord}$, $A \in N[G]$ *for some* N*-generic filter* $G \subset \mathbb{P}$*.*

Corollary (ZF)

Suppose the HOD *Conjecture is* Ω*-valid from*

$$\mathrm{ZFC} + \text{“There is an extendible cardinal”},$$

δ *is an extendible cardinal, and that there is an extendible cardinal below* δ*.*

Suppose that $\lambda > \delta$ *and*

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

is an elementary embedding.

Then j *is the identity.*

Assume ZF, δ *is an extendible cardinal, there is an extendible cardinal below* δ, *and that the* HOD *Conjecture is* Ω*-valid.*

Let $N \subset V$ *be the inner model of* ZFC *which is close to* V *above* δ *witnessed by* $\mathbb{P}$. *Fix a strongly inaccessible* $\kappa < \delta$ *with* $\mathbb{P} \in V_\kappa$.

Observation

Let $(V^*, N^*) = (V[g], N[g])$ *where* $g \subset \mathrm{Coll}(\omega, \kappa)$ *is* V*-generic.*

Then δ *is an extendible cardinal in* V^* *and (in* V^**) for every set* $A \subset \mathrm{Ord}$, $A \in N^*[c]$ *for some* N^**-generic Cohen real.*

Conjecture (ZF)

Suppose that δ *is an extendible cardinal and that*

$$G \subset \mathrm{Coll}(\omega, V_\delta)$$

is V*-generic. Then* $V[G] \models$ *Axiom of Choice.*

Ω-logic
(The logic of the generic-multiverse)

Definition

Suppose φ is a Π_2-sentence. Then

$$\models_\Omega \varphi$$

if φ holds in all generic extensions of V.

Theorem

Suppose there is a proper class of Woodin cardinals and that φ is a Π_2-sentence.

Then φ is a generic-multiverse truth if and only if $\models_\Omega \varphi$.

Universally Baire sets and strong closure

Definition

A set $A \subset \mathbb{R}$ is *universally Baire* if for all compact Hausdorff spaces, S, and for all continuous functions,

$$F : S \to \mathbb{R},$$

the preimage of A by F has the property of Baire in the space S.

Example: If $A \subseteq \mathbb{R}$ is borel then A is universally Baire.

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that M is a countable transitive model of ZFC.
Then M is *strongly A-closed* if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

The definition of $\vdash_\Omega \varphi$

Definition

Suppose there is a proper class of Woodin cardinals. Suppose that φ is a Π_2-sentence.

Then $\vdash_\Omega \varphi$ if there exists a set $A \subset \mathbb{R}$ such that:

1. A is universally Baire,
2. for all countable transitive models, M, if M is strongly A-closed then
$$M \models \text{“}\models_\Omega \varphi\text{”}.$$

- “$\vdash_\Omega \varphi$” is invariant across the generic-multiverse.

The Ω Conjecture

Theorem (Ω Soundness)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is Π_2-sentence.

If $\vdash_\Omega \varphi$ then $\models_\Omega \varphi$

Definition (Ω Conjecture)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is a Π_2-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

The Ω Conjecture and HOD

Theorem

Suppose that there is proper class of Woodin cardinals and that every OD set $A \subseteq \mathbb{R}$ is universally Baire.

Then $\mathrm{HOD} \models$ The Ω Conjecture.

Observation

Assume the HOD *Conjecture and that there is an extendible cardinal. Then* HOD *is "universal" for large cardinals.*

Example:

Theorem

Assume the HOD *Conjecture and that there is an extendible cardinal. Suppose that there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that $\mathrm{CRT}(j) < \lambda$ *and such that* $V_\lambda \prec V$.

Then in HOD, *there exists an elementary embedding*

$$j : L(V_{\lambda+1}) \to L(V_{\lambda+1})$$

such that $\mathrm{CRT}(j) < \lambda$ *and such that* $V_\lambda \prec V$.

Another HOD Dichotomy?

Suppose some large cardinal axiom refutes the Ω Conjecture. Then this large cardinal axiom (in conjunction with the existence of an extendible cardinal) must imply that one of the following hold.

1. There is an OD-set $A \subseteq \mathbb{R}$ which is not universally Baire.
2. All sufficiently large regular cardinals are ω-strongly measurable in HOD.
   - i.e., the HOD Conjecture fails.

If the HOD Conjecture is provable then this large cardinal axiom must imply that there is an OD-set $A \subseteq \mathbb{R}$ which is **not universally Baire***.*

The HOD Conjecture and the ultimate version of L

Definition

Suppose N is a transitive class, $\mathrm{Ord} \subset N$, and $N \models \mathrm{ZFC}$.

Then N is a *weak extender model* for δ is supercompact if for all $\gamma > \delta$, there is a normal fine δ-complete ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such that

1. $\mathcal{P}_\delta(\gamma) \cap N \in U$,
2. $U \cap N \in N$.

Covering and weak extender models

Covering Theorem

Suppose N is a weak extender model for δ is supercompact.

Suppose $\gamma > \delta$ and γ is a singular cardinal.

Then γ is singular in N and $\gamma^+ = (\gamma^+)^N$.

The universality of weak extender models

Universality Theorem

Suppose N is a weak extender model for δ is supercompact.

Suppose F is an extender of strong limit length κ and

(i) $j_F(N) \cap V_{\kappa+1} \subset N$,

(ii) $\mathrm{CRT}(j_F) \geq \delta$,

where $j_F : V \to M_F \cong \mathrm{Ult}(V, F)$ is the ultrapower embedding.

Then $F \cap N \in N$.

Theorem

Suppose N is a weak extender model for δ is supercompact. There is no elementary embedding, $j : N \to N$, with $\mathrm{CRT}(j) \geq \delta$.

A weak extender model for δ is supercompact has the closure properties of HOD assumimg δ is extendible and that the HOD Conjecture holds.

An example

Let U be a normal, κ-complete, uniform ultrafilter on κ and let

$$j_0 : V \to M_1 \cong \mathrm{Ult}(V, U)$$

be the associated ultrapower embedding.

Let M_ω be the ω-th iterate of V and let

$$N = M_\omega[\langle \kappa_i : i < \rangle] = \cap_{i<\omega} M_i$$

where for each $i < \omega$, $\kappa_i = \mathrm{CRT}(j_i)$ and (M_i, j_i) is the i-th iterate of

$$(M_0, j_0) = (V, j_0).$$

Theorem

Suppose $\delta > \kappa$ and δ is supercompact. Then the following hold.

(1) *$N^\omega \subset N$ and $j_0(N) = N$.*

(2) *N is a weak extender model for δ is supercompact.*

Weak extender models and the HOD Conjecture

Speculation

The extension of Inner Model Theory to the level one supercompact cardinal should yield as a theorem that if δ is supercompact then there exists

$$N \subseteq \mathrm{HOD}$$

such that N is a weak extender model for δ is supercompact.

Theorem

Suppose that δ is an extendible cardinal. Then the following are equivalent.

1. *The HOD Conjecture.*
2. *There is a weak extender model N for δ is supercompact such that $N \subset \mathrm{HOD}$.*

The axiom for ultimate L

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ is a Mitchell-Steel inner model.

- This shows that the Mitchell-Steel construction really is canonical: at least at the level of Woodin cardinals.

Theorem

Suppose that there is a proper class of Woodin cardinals.

Then $\mathrm{HOD}^{L(\mathbb{R})}$ is **not** *a Mitchell-Steel inner model.*

- There is another class of inner models
  - previously unknown.

(Conjecture) The axiom for ultimate L

There is a proper class of Woodin cardinals. Further for each sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

- This axiom implies the Continuum Hypothesis.
- This axiom settles (modulo axioms of infinity) *all* sentences about $\mathcal{P}(\mathbb{R})$ which have been shown to be independent by Cohen's method.

(meta) Conjecture

This axiom will be validated on the basis of compelling and accepted principles of infinity just as the axiom PD *has been.*

- *The natural variations will reduce all questions of Set Theory to axioms of infinity.*

Reference

- Suitable Extender Models I
  - To appear, Journal of Mathematical Logic (2011)