The evolution of hod mice

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CH in HOD

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Assume $V = L(\mathbb{R}) + AD$. Then $\mathrm{HOD} \models CH$.

Question: Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Does $\mathrm{HOD} \models GCH$?

Mice and HOD

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Theorem (Steel-Woodin)

Assume $\mathcal{M}_\omega$ exists. Then $L(\mathbb{R}) \vDash x \in OD(y)$ iff $x \in \mathcal{M}_\omega(y)$.

Mice and HOD cont.

Corollary

Assume $\mathcal{M}_\omega$-exists. Let κ be the least measurable of $\mathcal{M}_\omega$ and let μ be the measure on κ in $\mathcal{M}_\omega$. Let $\mathcal{N}$ be the result of iterating μ through the ordinals. Then

$$\mathrm{HOD}|\omega_1 = \mathcal{N}|\omega_1.$$

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Conjecture (The Mouse Set Conjecture, MSC)

Assume AD^+ and that there is no inner model with a superstrong cardinal. Then for $x, y \in \mathbb{R}$,

$$x \in OD(y) \text{ iff } x \text{ is in a } y\text{-mouse.}$$

Summary

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

1. Does $\mathrm{HOD} \vDash GCH$?
2. Is HOD a mouse?
3. Is MSC true?

An answer to 2

Theorem

Assume $V = L(\mathbb{R}) + AD$.

1. *(Steel)* $V_{\Theta}^{\mathrm{HOD}}$ *is a mouse.*

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Theorem

Assume $V = L(\mathbb{R}) + AD$.

1. *(Steel) V_Θ^{HOD} is a mouse.*
2. *(Woodin) HOD isn't a mouse, it is a hybrid mouse. More precisely, it has a fragment of the strategy of V_Θ^{HOD} on its sequence.*

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4. All hod mice have a largest layer for which no strategy is added.
5. For "simple" hod mice, the only layers are its Woodin cardinals and the limit of Woodin cardinals.

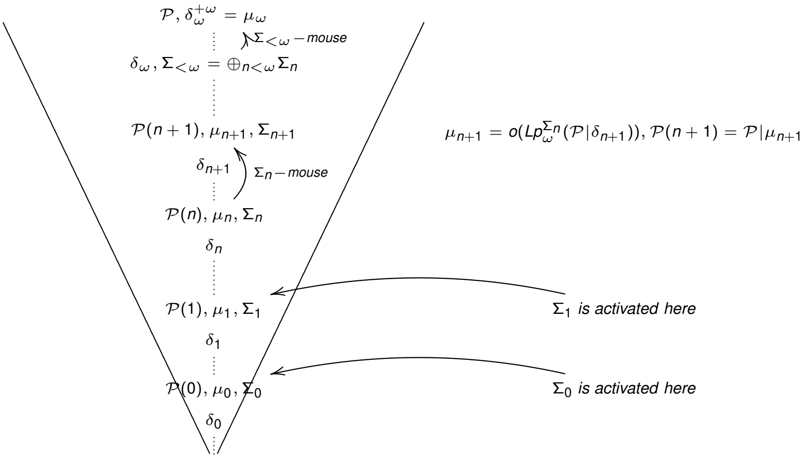

Figure: Hod premouse with $\lambda^{\mathcal{P}} = \omega$.

The Solovay hierarchy

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Then, assuming AD, the Solovay sequence is a closed sequence of ordinals $\langle \theta_\alpha : \alpha \leq \Omega \rangle$ defined by:

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3. $\theta_\lambda = \sup_{\alpha<\lambda} \theta_\alpha$.

The Solovay hierarchy

$$AD^+ + \Theta = \theta_0 <_{con} AD^+ + \Theta = \theta_1 <_{con} ... AD^+ + \Theta = \theta_\omega <_{con} ... AD^+ + \Theta = \theta_{\omega_1} <_{con} AD^+ + \Theta = \theta_{\omega_1+1} <_{con} \cdots$$

The structure of HOD

The motivation behind hod mice is the following theorem.

Theorem (Woodin)

Assume AD^+ and suppose $\theta_{\alpha+1} \leq \Theta$. Then $\mathrm{HOD} \vDash$ "$\theta_{\alpha+1}$ is Woodin".

Important limit points of the Solovay hierarchy

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2. The largest Suslin cardinal is a theta.

$AD_{\mathbb{R}}$ + "Θ is regular" and the Solovay hierarchy

Theorem (Woodin)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $AD_{\mathbb{R}}$ is equivalent to AD^+ + "$\Theta = \theta_\alpha$ for some limit α".

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Theorem (Woodin)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$. Then $AD_{\mathbb{R}}$ is equivalent to AD^+ + "$\Theta = \theta_\alpha$ for some limit α". Hence, $AD_{\mathbb{R}}$ + "Θ is regular" is equivalent to AD^+ + "$\Theta = \theta_\Theta$ and Θ is regular"

Hod mice below $AD_{\mathbb{R}}$ + "Θ is regular"

Q is a shortening of a hod premouse if there is a hod premouse $\mathcal{P}$ such that $Q = \mathcal{P}|\eta$ where η is the largest layer of $\mathcal{P}$.

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Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and that there is no inner model of $AD_{\mathbb{R}}$ + "Θ is regular". Then

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Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and that there is no inner model of $AD_{\mathbb{R}}$ + "Θ is regular". Then

1. *$V_{\Theta}^{\mathrm{HOD}}$ is a shortening of a hod premouse.*
2. $\mathrm{HOD} \vDash$ *GCH.*
3. *MSC holds.*

Large cardinals and $AD_{\mathbb{R}}$ + "Θ is regular"

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Suppose there is a Woodin limit of Woodins with a measurable above. Then there is a proper class inner model containing the reals and satisfying $AD_{\mathbb{R}}$ + "Θ is regular.

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Theorem (Steel, building on a prior work of Woodin)

Assume $AD_{\mathbb{R}}$. Then there is an inner model in which there is λ which is a limit of Woodin cardinals and $< \lambda$-strong cardinals.

Large cardinals and $AD_{\mathbb{R}}$ + "Θ is regular" cont.

Remark: One should be able to get many more Woodins.

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Large cardinals and $AD_{\mathbb{R}}$ + "Θ is regular" cont.

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Remark: One should be able to get many more Woodins. However, the exact large cardinal corresponding to $AD_{\mathbb{R}}$ + "Θ is regular" isn't known and has been rather mysterious. Perhaps that isn't even the right question to ask.
Question: Is every model of AD^+ a derived model of a mouse?

One application

Theorem (S.)

Suppose $\neg\Box_\kappa$ for some singular strong limit κ. Then there is a proper class model containing $\mathbb{R}$ and satisfying $AD_{\mathbb{R}}$ + "Θ is regular".

HOD below $AD_{\mathbb{R}}$ + "Θ is regular"

Theorem (S.)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and that there is no inner model of $AD_{\mathbb{R}}$ + "Θ is regular". Then V_Θ^{HOD} is a shortening of a hod premouse.

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Definition

Suppose $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ are two hod pairs. Then comparison holds for $(\mathcal{P}, \Sigma)$ and $(\mathcal{Q}, \Lambda)$ if there are $(\mathcal{R}, \Psi)$ and $(\mathcal{S}, \Phi)$ such that

1. $\mathcal{R}$ and $\mathcal{S}$ are respectively Σ-iterate of $\mathcal{P}$ and a Λ-iterate of $\mathcal{Q}$,
2. Ψ and Φ are the corresponding tails of respectively Σ and Λ.
3. $\mathcal{R} \trianglelefteq_{hod} \mathcal{S}$ and $\Phi_{\mathcal{R}} = \Psi$ or $\mathcal{S} \trianglelefteq_{hod} \mathcal{R}$ and $\Psi_{\mathcal{S}} = \Phi$.

Directed system associated to hod pairs

Suppose $(\mathcal{P}, \Sigma)$ is a hod pair. Let

$$\mathcal{F} = \{\mathcal{Q} : \mathcal{Q} \text{ is a } \Sigma\text{-iterate of } \mathcal{P}\}.$$

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Define $\leq$ on $\mathcal{F}$ by $\mathcal{Q} \leq \mathcal{R}$ iff $\mathcal{R}$ is a $\Sigma_{\mathcal{Q}}$-iterate of $\mathcal{Q}$. Let $i_{\mathcal{Q},\mathcal{R}} : \mathcal{Q} \to \mathcal{R}$ be the iteration embedding.

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Let

$$\mathcal{M}_\infty(\mathcal{P}, \Sigma) = dirlim(\mathcal{F}, \leq)$$

under $i_{\mathcal{Q},\mathcal{R}}$'s.

Proof cont.

One then shows that for every α such that $\theta_\alpha < \Theta$, there is a hod pair $(\mathcal{P}, \Sigma)$ such that

$$V_{\theta_\alpha}^{\mathrm{HOD}} = \mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\alpha.$$

Proof cont.

One then shows that for every α such that $\theta_\alpha < \Theta$, there is a hod pair $(\mathcal{P}, \Sigma)$ such that

$$V_{\theta_\alpha}^{\mathrm{HOD}} = \mathcal{M}_\infty(\mathcal{P}, \Sigma)|\theta_\alpha.$$

Remark: It follows from comparison that $\mathcal{M}_\infty(\mathcal{P}, \Sigma)$ is independent of $(\mathcal{P}, \Sigma)$.

Largest Suslin is a theta

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LST: $AD^+ + \Theta = \theta_{\alpha+1}$ + "θ_α is the largest Suslin cardinal below Θ".

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Fact

LST $\rightarrow$ *Con*($AD_{\mathbb{R}}$ + "Θ *is regular*")

Hod mice below *LST*

Theorem (S-Steel)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and that there is no proper class inner model containing the reals and satisfying LST. Then

1. V_Θ^{HOD} *is a shortening of a hod premouse.*
2. $\mathrm{HOD} \models GCH$
3. *MSC holds.*

Large cardinals and *LST*

Theorem (S.-Steel)

Suppose there is a Woodin limit of Woodins and a measurable cardinal above. Then there is a hod mouse satisfying LST.

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Conjecture

LST *is equiconsistent with Woodin limit of Woodins.*

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LST is equiconsistent with Woodin limit of Woodins.

The conjecture is important as it will probably lead to showing that *Con*($\neg\Box_\kappa$ for singular strong limit κ) is stronger then *Con*(Woodin limit of Woodins).

Two limitations

Theorem (Woodin)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

1. *If* $\Theta = \theta_\alpha$ *for some limit* α *then* Θ *isn't Woodin in* HOD.

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Theorem (Woodin)

Assume $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$.

1. *If* $\Theta = \theta_\alpha$ *for some limit* α *then* Θ *isn't Woodin in* HOD.
2. *If* $\theta_{\alpha+2} \leq \Theta$ *and* θ_α *is the largest Suslin cardinal below* $\theta_{\alpha+1}$ *then, in* HOD, θ_α *isn't* κ*-strong where* κ *is the least Suslin above* $\theta_{\alpha+1}$.

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1. $\mathcal{P}$ has a largest Woodin cardinal δ,
2. if κ is the least cardinal which is $< \delta$-strong then κ is a limit of Woodins,
3. δ is only overlapped by extenders with critical point κ,
4. For every $\eta < \delta$ if η isn't a cutpoint then there is $\mathcal{Q} \trianglelefteq \mathcal{P}$ such that $\mathcal{Q} \models$ "η isn't Woodin" and in $\mathcal{Q}$, all the extenders overlapping η have the same critical point.

Comparison theory for minimally overlapped *LST*'s

Theorem (S.)

Assume MSC. Then comparison holds for 1-overlapped LST pairs.

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Assume MSC. Then comparison holds for 1-overlapped LST pairs.

Remark: This isn't the theorem we want to prove as it assumes *MSC*. There are also arguments for proving the theorem without assuming *MSC* but again they are not fully worked out.

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2. There seem to be much more serious problems after Woodin limit of Woodins.
3. However, maybe as far as superstrongs are concerned all we need is the theory up to Woodin limit of Woodins.

The superstrong question

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Remark: In general, a Woodin in a hod mouse is much stronger than a Woodin in a mouse. Recall $AD_{\mathbb{R}}$-equiconsistency result.

The inner model problem

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Remark: Neeman solved it for a Woodin limit of Woodins, but the solution cannot be used for lower bound evaluations. Neeman's result is the best partial result.

Large cardinals and the Solovay hierarchy

One approach to IMP has been via descriptive set theory. One approach using DST has been to consider HOD of models of $AD^+ + V = L(\mathcal{P}(\mathbb{R}))$ and translate the strategies into extenders to get an equivalent model with large cardinals.

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Theorem (Steel)

Assume $V \vDash$ "I am the minimal model of $AD_{\mathbb{R}}$" and let $\mathcal{P} = \mathrm{HOD}$. Then $\mathcal{P}^S \vDash AD_{\mathbb{R}}$-hypo.

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For this to work in general, we need that the Solovay hierarchy catches up with the large cardinal hierarchy.

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Remark: Even the case of $\phi =_{def}$ "δ is Woodin and there is a δ-strong cardinal" is open. This isn't a superficial way of strengthening the Solovay hierarchy. It seems that any reasonable approach to the problems outlines in this talk will inevitably lead to the study of axiom like S_ϕ.