Speculations on nature of set theoretic truth: A thesis in 4 parts followed by 4 conjectures

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A thesis in four parts

The axioms of ZFC

Axiom 1 (Extensionality) Two sets A and B are equal if and only if they have the same elements.

Axiom 2 (Pairing) If A and B are sets then there exists a set $C = \{A, B\}$ whose only elements are A and B.

Axiom 3 (Union) If A is a set then there exists a set C whose elements are the elements of the elements of A.

Axiom 4 (Regularity or Foundation) If A is a set then either A is empty (i. e. A has no elements) or there exists an element C of A which is disjoint from A.

Axiom 5 (Comprehension) If A is a set and $\varphi(x)$ formalizes a property of sets then there exists a set C whose elements are the elements of A with this property.

Axiom 6 (Powerset) If A is a set then there exists a set C whose elements are the subsets of A.

Axiom 7 (Axiom of Choice) If A is a set whose elements are pairwise disjoint and each nonempty then there exists a set C which contains exactly one element from each element of A.

Axiom 8 (Replacement) If A is a set and $\varphi(x)$ formalizes a property which defines a function of sets then there exists a set C which contains as elements all the values of this function acting on the elements of A.

Axiom 9 (Infinity) There exists a set W which is nonempty and such that for each element A of W there exists an element B of W such that A is an element of B.

The axioms of ZFC_0: Finite set theory

Axiom 1 (Extensionality) Two sets A and B are equal if and only if they have the same elements.

Axiom 2 (Bounding) There exists a set C such that every set is a subset of C.

Axiom 3 (Union) If A is a set then there exists a set C whose elements are the elements of the elements of A.

Axiom 4 (Regularity) If A is a set then either A is empty or there exists an element C of A which is disjoint from A.

Axiom 5 (Comprehension) If A is a set and $\varphi(x)$ formalizes a property of sets then there exists a set C whose elements are the elements of A with this property.

Axiom (6a) (Powerset) For all sets A one of the following holds.

- $\mathcal{P}(A)$ exists
- there exists a set C such that $V = \mathcal{P}(C)$ and A is not an element of C

Axiom (6b) (Powerset) For all sets A one of the following holds.

- $\mathcal{P}(A)$ exists
- there exist $B \in A$ and $C \subseteq B$ such that $C \notin A$
- $V = \mathcal{P}(A)$

Axiom 7 (Axiom of Finiteness) If A is a nonempty set then there is an element B of A such that for all sets C, if C is an element of A then B is not an element of C.

The cumulative hierarchy

Definition

Define for each ordinal α a set V_α by induction on α.

1. $V_0 = \emptyset$.
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha) = \{X \mid X \subseteq V_\alpha\}$.
3. If β is a limit ordinal then $V_\alpha = \cup\{V_\beta \mid \beta < \alpha\}$.

- It is a consequence of the ZFC axioms that for each set A there exists an ordinal α such that $A \in V_\alpha$.
- The ZFC_0 axioms (1)-(6) imply that $V = V_{\alpha+1}$ for some ordinal α, adding Axiom (7), α is finite.

For each finite ordinal $n > 0$, $V_n \models \mathrm{ZFC}_0$.

- ZFC_0 is a very weak theory.

Theorem

ZFC_0 *proves it own consistency.*

- But ZFC_0 does not prove there is a model of ZFC_0.

The axioms of the form

$$\text{“}V_n \text{ exists”}$$

for specific n are “large cardinal” axioms for ZFC_0.

Gödel sentences and ZFC_0

Theorem (after Gödel)

There is a sentence Φ *such that for all models*

$$(M, E) \models \mathrm{ZFC}_0$$

the following are equivalent:

1. $(M, E) \models \Phi$.
2. $(M, E) \models$ "$\mathrm{ZFC}_0 \vdash (\neg\Phi)$"

The sentence Φ_0

Φ_0 asserts:

1. *V_n exists where $n = |V_{1000}|$.*
2. *There is a proof of $(\neg\Phi_0)$ with length at most 10^{24} from the theory:*

$$\mathrm{ZFC}_0 + \text{“}\, V_n \text{ exists where } n = |V_{1000}|\,\text{”}.$$

Φ_0 *if true is physically* verifiable *from the witness.*

- Φ_0 is a meaningful statement about the actual physical universe.

Question

Is Φ_0 true?

- Of course not. But we have no evidence (physical or mathematical) that Φ_0 is false.

Claim

Any coherent basis (at present) for the assertion that Φ_0 is false must also yield that the conception of V_n where $n = |V_{1000}|$ is meaningful.

Thesis: Part 1

Any coherent basis for the mathematical claim of the consistency of a formal (recursive) theory T must be paired ultimately with a conception of mathematical objects with structure, whose existence implies the consistency of T.

Consistency and independence in ZFC

- ω_1 is the least uncountable ordinal
  - it is the set of all countable ordinals.

Definition

1. A set $C \subseteq \omega_1$ is *closed* if for all $\alpha < \omega_1$ if $C \cap \alpha$ is cofinal in α then $\alpha \in C$.
2. A set $S \subseteq \omega_1$ is *stationary* if $S \cap C \neq \emptyset$ for all closed, cofinal, sets $C \subseteq \omega_1$.

- The sets, $S \subset \omega_1$, which are stationary and co-stationary are the simplest manifestation of the Axiom of Choice.
- How complicated is the structure of the stationary, co-stationary, subsets of ω_1?
  - Can exist a small generating family for these sets?

The combinatorics of stationary subsets of ω_1

A precise question along these lines is the following:

Question

Can there exist ω_1 many stationary sets, $\langle S_\alpha : \alpha < \omega_1 \rangle$, such that for every *stationary set $S \subseteq \omega_1$, there exists $\alpha < \omega_1$ such that $S_\alpha \subseteq S$ modulo a non-stationary set?*

The assertion that $S_\alpha \subseteq S$ modulo a non-stationary set is simply the assertion that the set,

$$S_\alpha \backslash S = \{\beta < \omega_1 \mid \beta \in S_\alpha \text{ and } \beta \notin S\},$$

is not stationary.

Observation

Such a sequence, $\langle S_\alpha : \alpha < \omega_1 \rangle$, of stationary subsets of ω_1 would give in a natural sense, a basis *for the stationary subsets of ω_1 which is of cardinality ω_1.*

Infinite games on ω and Determinacy Axioms

- Associated to a set $A \subseteq \mathbb{R}$ is an infinite game involving two players, Player I and Player II. The players construct a function, $f : \omega \to \{0, 1\}$, in stages,

(Stage 0) : Player I specifies $f(0)$,

(Stage 1) : Player II specifies $f(1)$,

(Stage 2) : Player I specifies $f(2)$,

. . . .

After infinitely many stages a function $f : \omega \to \{0, 1\}$ is constructed.

- Player I wins this run of the game if

$$\sum_{k=0}^{\infty} f(k) 2^{-(k+1)} \in A,$$

otherwise Player II wins.

Strategies

A strategy is a function

$$\tau : \{s : k \to \{0,1\} \mid k \in \omega\} \to \{0,1\}$$

and a player follows τ *in a run of the game yielding* f *if at each stage* k *for that player,* $f(k) = \tau(f|k)$*.*

Definition (Mycielski, Steinhaus: 1961)

The *Axiom of Determinacy*, AD, is the axiom which asserts that for *all* sets $A \subseteq \mathbb{R}$ there is a winning strategy for either Player I or Player II in the game given by A.

- AD contradicts the Axiom of Choice.

Question

Is the Axiom of Choice necessary to construct a set $A \subseteq \mathbb{R}$ *for which the corresponding game is not determined?*

Large Cardinal Axioms

Basic template for (modern) large cardinal axioms

A cardinal κ is a large cardinal *if there exist an ordinal α, a transitive set M, and an elementary embedding,*

$$j : V_\alpha \to M$$

such that κ is the least ordinal such that $j(\beta) \neq \beta$.

- $\mathrm{CRT}(j)$ denotes the least ordinal β such that $j(\beta) \neq \beta$.
  - If j is the identity on α then j is the identity on V_α.
- One can require more sets to belong to M, possibly in a way that depends on the action of j on the ordinals.
  - A hierarchy of notions.
  - (Axiom of Choice) If $M = V_\alpha$ then either $\alpha = \lambda$ or $\alpha = \lambda + 1$ where λ is the supremum of $\langle \kappa_i : i < \omega \rangle$, $\kappa_0 = \mathrm{CRT}(j)$ and for all $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$.

Three theories

Theory 1

ZFC + " There exist ω_1 many stationary sets, $\langle S_\alpha : \alpha < \omega_1 \rangle$, such that for every stationary set $S \subseteq \omega_1$, there exists $\alpha < \omega_1$ such that $S_\alpha \subseteq S$ modulo a non-stationary set".

Theory 2

ZF + AD

Theory 3

ZFC + "There exist infinitely many Woodin cardinals".

Theorem

These three theories are equiconsistent.

Thesis: Part 2

The conception of the universe of sets with the structure from large cardinals can account for all possible consistency claims.

Two large cardinal axioms in ZF

Definition: κ is an Enormous Cardinal

There exist $\kappa < \lambda < \gamma$ and an elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

such that

1. $\kappa = \mathrm{CRT}(j)$ *and* $\lambda > \kappa$ *is least such that* $j(\lambda) = \lambda$,
2. $V_\lambda \prec V_\gamma$.

Definition: κ is a Weak Reinhardt Cardinal

There exist $\kappa < \lambda < \gamma$ and an elementary embedding

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

such that

1. $\kappa = \mathrm{CRT}(j)$ *and* $\lambda > \kappa$ *is least such that* $j(\lambda) = \lambda$,
2. $V_\lambda \prec V_\gamma$.

Theorem (Kunen)

Assume the Axiom of Choice. Suppose λ is an ordinal and

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

is an elementary embedding. Then j is the identity.

Corollary (ZFC)

There are no Weak Reinhardt Cardinals.

Theorem (ZF)

Assume there is a Weak Reinhardt Cardinal. Then

$$\mathrm{ZFC} + \text{“There is a proper class of Enormous Cardinals”}$$

is consistent.

- Must the Axiom of Choice be abandoned for Thesis (Part 2)?

The effective cumulative hierarchy: L

The definable power set

For each set X, $\mathcal{P}_{\mathrm{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that X is logically definable in the structure $(X, \in)$ from parameters in X.

- (Axiom of Choice) $\mathcal{P}_{\mathrm{Def}}(X) = \mathcal{P}(X)$ if and only if X is finite.

Gödel's constructible universe, L

Define L_α by induction on α as follows.

1. $L_0 = \emptyset$,
2. (Successor case) $L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha)$,
3. (Limit case) $L_\alpha = \cup\{L_\beta \mid \beta < \alpha\}$.

L is the class of all sets X such that $X \in L_\alpha$ for some ordinal α.

- (Scott) Assume $V = L$. There are no (modern) large cardinals.

Definition

For each ordinal α, $\mathrm{HOD}_{\alpha+1}$ is the set of all sets $a \subseteq V_\alpha$ such that:

1. a is definable in V_α from ordinal parameters.
2. If $b \in \mathrm{TC(a)}$ then b is definable in V_α from ordinal parameters.

where for each set a, $\mathrm{TC(a)}$ is the smallest transitive set M with $a \in M$.

- The definition of $\mathrm{HOD}_{\alpha+1}$ is a mixture of the definition of $L_{\alpha+1}$ and $V_{\alpha+1}$.

Definition

HOD be the class of all sets a such that $a \in \mathrm{HOD}_{\alpha+1}$ for some α.

- If the existence of a proper class of Enormous Cardinals is consistent then the existence is consistent with $V = \mathrm{HOD}$.

Definition

A class N is Σ_2-definable if there is a formula $\varphi(x_0)$ such that

$$N = \cup\{a \mid V_\alpha \models \varphi[a] \text{ for some ordinal } \alpha\}.$$

- L is Σ_2-definable.
- $\langle V_\alpha : \alpha \in \mathrm{Ord}\rangle$ is Σ_2-definable.
- HOD is Σ_2-definable.

Definition

A class N is $\underset{\sim}{\Sigma}_2$-definable if there is a formula $\varphi(x_0, x_1)$ and a set b such that

$$N = \cup\{a \mid V_\alpha \models \varphi[a, b] \text{ for some ordinal } \alpha\}.$$

Definition: $N[X]$

Suppose that N is a transitive class and X is a transitive set. Then $N[X]$ is the smallest transitive class M such that

1. $N \subseteq M$ *and* $X \cap M \in M$
2. $M \models \mathrm{ZF}$

Definition: $N(X)$

Suppose that N is a transitive class and X is a transitive set. Then $N(X)$ is the smallest transitve class M such that

1. $N \subseteq M$ *and* $X \in M$
2. $M \models \mathrm{ZF}$

Lemma

Suppose N is $\underset{\sim}{\Sigma}_2$-definable. Then $N(X)$ and $N[X]$ are each $\underset{\sim}{\Sigma}_2$-definable.

Vopenka's Theorem

Theorem (Vopenka)

For each transitive set X:

1. $\mathrm{HOD}[X]$ *is a generic extension of* HOD*;*
2. $\mathrm{HOD}(X)$ *is a symmetric generic extension of* HOD*.*

Corollary (ZF)

Suppose that κ is a Weak Reinhardt Cardinal. Then κ is a Weak Reinhardt Cardinal in some symmetric generic extension of HOD*.*

Speculation

Perhaps a multiverse *conception based on generic extensions could provide a framework for truth which:*

- *Accounts for the consistency of Weak Reinhardt Cardinals,*
  - *avoiding having to abandon the Axiom of Choice.*
- *Avoids having to settle the Continuum Hypothesis.*

Definition

Suppose that M is a countable transitive set and that

$$M \models \mathrm{ZFC}.$$

The *generic-multiverse* generated by M is the smallest set $\mathbb{V}_M$ of countable transitive sets such that for all pairs (N_0, N_1) of countable transitive sets if

1. N_1 is a generic extension of N_0
2. either $N_0 \in \mathbb{V}_M$ or $N_1 \in \mathbb{V}_M$

then both $N_0 \in \mathbb{V}_M$ and $N_1 \in \mathbb{V}_M$.

(meta) Definition

The Generic-Multiverse is the generic-multiverse generated by V.

The Generic-Multiverse and truth

The generic-multiverse view of truth

A sentence φ *is a* Generic-Multiverse truth *if* φ *holds in each universe of the Generic-Multiverse.*

- This can be formally reduced to truth within V.

There is a (recursive) transformation of sentences giving φ^* from φ such that:

Theorem

For all countable transitive sets M *the following are equivalent.*

1) $M \models \varphi^*$.
2) $N \models \varphi$ *for each* $N \in \mathbb{V}_M$.

The Resurrection Theorem

- A Σ_2-sentence is a sentence of the form:

  "There exists α such that $V_\alpha \models \psi$"
- A Π_2-sentence is a sentence of the form:

  "For all α, $V_\alpha \models \psi$"

Theorem (Resurrection Theorem)

Suppose there is a proper class of Woodin cardinals and that φ is a Σ_2-sentence true in V. Then for each universe N of the Generic-Multiverse, φ is true in some generic extension of N.

Corollary

Suppose that there is a proper class of Woodin cardinals and that φ is a Π_2-sentence. Then the following are equivalent.

1. *φ is true in all generic extensions of V.*
2. *φ is true in all universes of the Generic-Multiverse.*

Example

Suppose ψ is a sentence and consider the Π_2-sentences

- *"$V_{\omega+2} \models \psi$"*
- *"$V_{\omega+2} \models (\neg\psi)$"*

Suppose there is a proper class of Woodin cardinals and that neither of the sentences a Generic-Multiverse truth.

Suppose N is a universe of the Generic-Multiverse. Then:

- *There are generic extensions of N in which "$V_{\omega+2} \models \psi$".*
- *There are generic extensions of N in which "$V_{\omega+2} \models (\neg\psi)$".*

- $V_{\omega+2}$ can be replaced by $V_{\omega+3}$, $V_{\omega+10000}$, V_{δ_0+1} where δ_0 denotes the least Woodin cardinal, etc.

Thesis: Part 3 in two parts

Part A

The conception of Π_2 truth must be at least as strong as that given by the Generic-Multiverse.

Part B

For the conception of Π_2 truth, the only possibilities are

1. *V (Set Theoretic Platonism)*
   - *For each sentence ψ, the sentence "$V_{\omega+2} \models \psi$" has determinate truth value etc.*
2. *The Generic-Multiverse.*

Another consequence of the Resurrection Theorem

Assume ZF and that there is a Weak Reinhardt Cardinal. Suppose that

$$\mathrm{HOD} \models \text{“There is a proper class of Woodin cardinals.”}$$

Consider the generic multiverse generated by HOD.

- If N is a universe of this generic-multiverse then there is a symmetric extension of N in which there is a Weak Reinhardt Cardinal.

This suggests that the Generic-Multiverse conception of truth might be able to account for the consistency with ZF *of the existence of Weak Reinhardt Cardinals.*

The first multiverse law

Definition (For a given multiverse)

1. For any universe N, $(\delta_0)^N$ denotes the first Woodin cardinal of N.
2. A sentence φ is a multiverse truth of V_{δ_0+1} if for each universe N of the multiverse,

$$(V_{\delta_0+1})^N \models \varphi.$$

The First Multiverse Law

The set of Π_2-sentences which are multiverse truths is not recursive in the set of multiverse truths of V_{δ_0+1}.

- The multiverse given by all ω-models

  $N \models \mathrm{ZFC} +$ "There is a proper class of Woodin cardinals"

  violates the First Multiverse Law.

The second multiverse law

Definition (For a given multiverse)

A set $X \subseteq V_\omega$ is definable in V_{δ_0+1} across the multiverse if for each universe N of the multiverse, X is logically definable in $(V_{\delta_0+1})^N$ without parameters.

The Second Multiverse Law

The set of Π_2-sentences which are multiverse truths, is not definable in V_{δ_0+1} across the multiverse.

- The multiverse given by all ω-models

  $$N \models \mathrm{ZFC} + \text{"There is a proper class of Woodin cardinals"}$$

  satisfies the Second Multiverse Law.

Thesis: Part 4

The conception of a multiverse of sets should not violate both the multiverse laws.

Therefore if the Generic-Multiverse is to be the basis for Π_2 truth:

The Generic-Multiverse must satisfy one of the multiverse laws.

- The multiverse laws are really a family of laws indexed by (Σ_2-definitions of) δ:
  - here δ is the first Woodin cardinal.
- The "smaller" the choice of δ, the "weaker" the laws.
- Taking $\delta = \omega + 2$ yields the weakest (plausible) versions, these are the Weak Multiverse Laws.

Predictions

Ω-logic
(The logic of the Generic-Multiverse)

Definition

Suppose φ is a Π_2-sentence. Then

$$\models_\Omega \varphi$$

if φ holds in all generic extensions of V.

Theorem

Suppose there is a proper class of Woodin cardinals and that φ is a Π_2-sentence.

Then φ is a Generic-Multiverse truth if and only if $\models_\Omega \varphi$.

Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is *universally Baire* if for all topological spaces, S, and for all continuous functions,

$$F : S \to \mathbb{R},$$

the preimage of A by F has the property of Baire in the space S (differs from an open set by a meager set).

Example: If $A \subseteq \mathbb{R}$ is borel then A is universally Baire.

Theorem (after et al)

Suppose there is a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is universally Baire. Then

1. $L(A, \mathbb{R}) \models \mathrm{AD}$,
2. *Every set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.*

Strong closure

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that M is a countable transitive model of ZFC.

Then M is *strongly A-closed* if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

- If M is strongly A-closed then $A \cap M \in M$. But this alone does not suffice.

The definition of $\vdash_\Omega \varphi$

Definition

Suppose there is a proper class of Woodin cardinals. Suppose that φ is a Π_2-sentence.

Then $\vdash_\Omega \varphi$ if there exists a set $A \subseteq \mathbb{R}$ such that:

1. A is universally Baire,
2. for all countable transitive models, M, if M is strongly A-closed then
$$M \models \text{“}\models_\Omega \varphi\text{”}.$$

- "$\vdash_\Omega \varphi$" is invariant across the Generic-Multiverse.

The Ω Conjecture

Theorem (Ω Soundness)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is Π_2-sentence.

If $\vdash_\Omega \varphi$ then $\models_\Omega \varphi$

Definition (Ω Conjecture)

Suppose that there exists a proper class of Woodin cardinals and suppose that φ is a Π_2-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

- The Ω Conjecture is invariant across the Generic-Multiverse.

Theorem (Fundamental Theorem of Ω-logic)

Suppose there is a proper class of Woodin cardinals and that the Ω Conjecture holds.

Let $\mathcal{V}_\Omega$ be the set of all Π_2-sentences φ such that $\models_\Omega \varphi$. Then:

1. *$\mathcal{V}_\Omega$ is recursive in the set of Generic-Multiverse truths of V_{δ_0+1}.*
2. *$\mathcal{V}_\Omega$ is definable in V_{δ_0+1}.*

- (1) can be strengthened to $\mathcal{V}_\Omega$ is recursive in the set of Generic-Multiverse truths of $V_{\omega+2}$ (which is best possible).

Corollary

Suppose there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the Generic-Multiverse violates both multiverse laws.

Predictions: Conjecture 1

The Ω Conjecture is a theorem of ZFC.

Supercompact and extendible cardinals

Definition: δ is a supercompact cardinal

For each ordinal $\gamma > \delta$ there exist $\bar{\delta} < \bar{\gamma} < \delta$ and an elementary embedding

$$j : V_{\bar{\gamma}+1} \to V_{\gamma+1}$$

such that $\mathrm{CRT}(j) = \bar{\delta}$ and $j(\bar{\delta}) = \delta$.

Definition: δ is an extendible cardinal

For each ordinal $\gamma > \delta$ there exists an elementary embedding

$$j : V_{\gamma+1} \to V_{j(\gamma)+1}$$

such that $\mathrm{CRT}(j) = \delta$ and such that $j(\delta) > \gamma$.

Definition: Extenders

Suppose that

$$j : V_{\delta+1} \to M$$

is an elementary embedding. Suppose $\mathrm{CRT}(j) < \alpha$ *and* $V_{\alpha+1} \subset M$.

- *The extender* E *of strength* α *derived from* j *is the function*

$$E : V_{\delta+1} \to V_{\alpha+1}$$

  defined by $E(a) = j(b) \cap V_\alpha$.

- Define $\mathrm{CRT}(E)$ to be the least ordinal ξ such that $E(\xi) \neq \xi$.
- Suppose that E is the extender of strength α derived from j. Then $\mathrm{CRT}(j) = \mathrm{CRT}(E)$.

Weak extender models

Definition

A $\underset{\sim}{\Sigma}_2$-definable transitive class N is a weak extender model of δ is supercompact if δ is supercompact in N and this is witnessed by the class of $F \in N$ such that

1. F is an extender in N,
2. there is an extender E of V such that $E \cap N = F$.

The Inner Model Program is the attempt to build weak extender models for various large cardinal notions–subject to non-triviality requirements.

- *The models produced are generalizations of L.*

Mitchell-Steel models

Theorem (after Mitchell-Steel, Steel)

Suppose there is a proper class of Woodin cardinals. Then there is a $\underset{\sim}{\Sigma}_2$-definable transitive class N such that:

1. *N is a weak extender model of δ is a Woodin cardinal for a proper class of δ.*
2. $N \subseteq \mathrm{HOD}$.
3. *Every set* $A \subset \mathrm{Ord}$ *is generic over N.*
4. *Suppose that there is a supercompact cardinal. Then every set* $A \in N(\mathbb{R})$ *is universally Baire.*

- The supercompact is far stronger than necessary but some additional hypothesis is necessary.

The Universality Theorem

Theorem (Universality Theorem)

Suppose that N is a weak extender model of δ is supercompact. Suppose that F is an extender such that:

- $\mathrm{CRT}(F) \geq \delta$ *and N is closed under F.*

Then $F \cap N \in N$.

- For any extender F, L is closed under F but $F \cap L \notin L$.

Corollary

Suppose that N is a weak extender model of δ is supercompact and that $\kappa > \delta$ is an extendible cardinal. Then N is a weak extender model of κ is supercompact.

- Any weak extender model of δ is supercompact *inherits* all large cardinals from V which occur above δ.

Theorem

Suppose there is a proper class of supercompact cardinals and that N is a weak extender model of δ is supercompact such that

1. *Every set $A \in N(\mathbb{R})$ is universally Baire.*
2. *Every set bounded $A \subset \delta$ is generic over N.*

Then $N \models$ "The Ω Conjecture".

- By the Universality Theorem one cannot hope to prove the existence of such N from any large cardinal hypothesis.
  - The correct conjecture would be that such N exist in some generic extension of V.

Predictions: Conjecture 2

Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:*

1. *N is a weak extender model of δ is supercompact.*
2. *Every bounded set $A \subset \delta$ is generic over N.*
3. *$N \subseteq \mathrm{HOD}$ and N is Σ_2-definable from δ.*
4. *$N \models$ "The Ω Conjecture".*

- The conjecture implies that *no* large cardinal hypothesis can refute the Ω Conjecture.

By the Universality Theorem:

The successful extension of the Inner Model Program to the level of exactly one supercompact cardinal yields an **ultimate** *version of L.*

- Finding the corresponding ultimate version of the axiom, "$V = L$", is possibly a **much harder problem**.

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes HOD as defined in $L(A, \mathbb{R})$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.

The axiom for $V = \text{Ultimate-L}$?

A sentence φ is a Σ_3-sentence if it is of the form:

- There exists α such that $V_\alpha \models \psi$ and such that $V_\alpha \prec_{\Sigma_2} V$;

for some sentence ψ.

(meta) Conjecture: The axiom for $V = \text{Ultimate-L}$

There is a proper class of Woodin cardinals. Further for each Σ_3-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\mathrm{HOD}^{L(A,\mathbb{R})} \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^{L(A,\mathbb{R})}$.

- This axiom settles (modulo axioms of infinity) *all* sentences about $\mathcal{P}(\mathbb{R})$ (and much more) which have been shown to be independent by Cohen's method.

Consequences of $V = \text{Ultimate-L}$

Theorem ($V = \text{Ultimate-L}$)

The Continuum Hypothesis holds.

Theorem ($V = \text{Ultimate-L}$)

The Ω Conjecture holds.

Theorem ($V = \text{Ultimate-L}$)

V is the minimum universe of the Generic-Multiverse.

In contrast:

Suppose N is a Mitchell-Steel extender model and there is a Woodin cardinal in N then:

- *N is* **not** *the minimum universe of the generic-multiverse generated by N even restricting to Mitchell-Steel extender models in that multiverse.*

Predictions: Conjecture 3

Ultimate-L Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:*

1. *N is a weak extender model of δ is supercompact.*
2. *Every bounded set $A \subset \delta$ is generic over N.*
3. *$N \subseteq \mathrm{HOD}$ and N is Σ_2-definable from δ.*
4. *$N \models$ "$V = \mathrm{Ultimate\text{-}L}$".*

Theorem (ZF)

Suppose that the Ultimate-L Conjecture is provable from ZFC. Then there are no Weak Reinhardt Cardinals.

Claim

The structure involved in the construction of Mitchell-Steel weak extender models at the level of Woodin cardinals arguably suffices to validate consistency claims at the level of Woodin cardinals.

A claim and a (serious) problem

The structure involved in the construction of the weak extender model N witnessing Ultimate-L Conjecture **cannot suffice to validate consistency claims beyond the level of one supercompact cardinal.**

Theorem (Ultimate-L Conjecture)

Suppose that δ extendible cardinal. Then there is a weak extender model N witnessing the Ultimate-L *Conjecture at δ and a class generic extension $V[G]$ such that for all α:*

(1) $V[G]_\alpha \not\models \mathrm{ZFC} +$ *"There is a supercompact cardinal".*

(2) *In $V[G]$, N satisfies all the requirements of the* Ultimate-L *Conjecture at δ.*

Predictions: Conjecture 4

Conjecture

($V = \mathrm{Ultimate\text{-}L}$) *Suppose that $\lambda > \omega$ is a cardinal such that*

$$L(V_{\lambda+1}) \not\models \textit{Axiom of Choice.}$$

Then there exists a non-trivial elementary embedding

$$j : V_{\lambda+1} \to V_{\lambda+1}.$$

- The existence of a Very Enormous Cardinal implies the existence of λ such that

$$L(V_{\lambda+1}) \not\models \text{Axiom of Choice.}$$

- Assuming Conjecture 3 and the consistency of the existence of a Very Enormous Cardinal, Conjecture 4 cannot be vacuously true.
  - It **is** vacuously true for Mitchell-Steel extender models.

Claim

Conjecture 4 is a conjecture of rich structure associated in Ultimate-L *to large cardinals.*

- There is a generic extension of L in which:

$$L(\mathbb{R}) \not\models \text{"Axiom of Choice"}.$$

Theorem (Steel)

Suppose N is a Mitchell-Steel weak extender model. Then the following are equivalent in N:

(1) $L(\mathbb{R}) \not\models$ *Axiom of Choice.*

(2) $L(\mathbb{R}) \models \mathrm{AD}$.

The ultimate (meta) conjecture

(meta) Conjecture

Large cardinal axioms above the level of one supercompact cardinal will be validated by their structural consequences for $\mathrm{Ultimate\text{-}L}$*.*

- $\mathrm{Ultimate\text{-}L}$ *has the ultimate structure associated to large cardinals.*
- *This structure is equivalent in* $\mathrm{Ultimate\text{-}L}$ *to the occurrence of large cardinals.*
- *This structure* **implies** $V = \mathrm{Ultimate\text{-}L}$ *in the context of large (enough) cardinals.*