Speculations on nature of set theoretic truth: A thesis in 4 parts followed by 4 conjectures

W. Hugh Woodin

University of California, Berkeley

September 21, 2011
A thesis in four parts
The axioms of ZFC

**Axiom 1 (Extensionality)** Two sets $A$ and $B$ are equal if and only if they have the same elements.

**Axiom 2 (Pairing)** If $A$ and $B$ are sets then there exists a set $C = \{A, B\}$ whose only elements are $A$ and $B$.

**Axiom 3 (Union)** If $A$ is a set then there exists a set $C$ whose elements are the elements of the elements of $A$.

**Axiom 4 (Regularity or Foundation)** If $A$ is a set then either $A$ is empty (i.e. $A$ has no elements) or there exists an element $C$ of $A$ which is disjoint from $A$.

**Axiom 5 (Comprehension)** If $A$ is a set and $\varphi(x)$ formalizes a property of sets then there exists a set $C$ whose elements are the elements of $A$ with this property.
Axiom 6 (Powerset) If $A$ is a set then there exists a set $C$ whose elements are the subsets of $A$.

Axiom 7 (Axiom of Choice) If $A$ is a set whose elements are pairwise disjoint and each nonempty then there exists a set $C$ which contains exactly one element from each element of $A$.

Axiom 8 (Replacement) If $A$ is a set and $\varphi(x)$ formalizes a property which defines a function of sets then there exists a set $C$ which contains as elements all the values of this function acting on the elements of $A$.

Axiom 9 (Infinity) There exists a set $W$ which is nonempty and such that for each element $A$ of $W$ there exists an element $B$ of $W$ such that $A$ is an element of $B$. 
The axioms of $\text{ZFC}_0$: Finite set theory

Axiom 1 (Extensionality) Two sets $A$ and $B$ are equal if and only if they have the same elements.

Axiom 2 (Bounding) There exists a set $C$ such that every set is a subset of $C$.

Axiom 3 (Union) If $A$ is a set then there exists a set $C$ whose elements are the elements of the elements of $A$.

Axiom 4 (Regularity) If $A$ is a set then either $A$ is empty or there exists an element $C$ of $A$ which is disjoint from $A$.

Axiom 5 (Comprehension) If $A$ is a set and $\varphi(x)$ formalizes a property of sets then there exists a set $C$ whose elements are the elements of $A$ with this property.
Axiom (6a) (Powerset) For all sets $A$ one of the following holds.
  ▶ $\mathcal{P}(A)$ exists
  ▶ there exists a set $C$ such that $V = \mathcal{P}(C)$ and $A$ is not an element of $C$

Axiom (6b) (Powerset) For all sets $A$ one of the following holds.
  ▶ $\mathcal{P}(A)$ exists
  ▶ there exist $B \in A$ and $C \subseteq B$ such that $C \notin A$
  ▶ $V = \mathcal{P}(A)$

Axiom 7 (Axiom of Finiteness) If $A$ is a nonempty set then there is an element $B$ of $A$ such that for all sets $C$, if $C$ is an element of $A$ then $B$ is not an element of $C$. 
The cumulative hierarchy

**Definition**

Define for each ordinal $\alpha$ a set $V_\alpha$ by induction on $\alpha$.

1. $V_0 = \emptyset$.
2. $V_{\alpha + 1} = \mathcal{P}(V_\alpha) = \{X \mid X \subseteq V_\alpha\}$.
3. If $\beta$ is a limit ordinal then $V_\alpha = \bigcup\{V_\beta \mid \beta < \alpha\}$.

- It is a consequence of the ZFC axioms that for each set $A$ there exists an ordinal $\alpha$ such that $A \in V_\alpha$.
- The ZFC$_0$ axioms (1)-(6) imply that $V = V_{\alpha + 1}$ for some ordinal $\alpha$, adding Axiom (7), $\alpha$ is finite.
For each finite ordinal $n > 0$, $V_n \models \text{ZFC}_0$.

- $\text{ZFC}_0$ is a very weak theory.

Theorem

$\text{ZFC}_0$ proves its own consistency.

- But $\text{ZFC}_0$ does not prove there is a model of $\text{ZFC}_0$.

The axioms of the form

“$V_n$ exists”

for specific $n$ are “large cardinal” axioms for $\text{ZFC}_0$. 
Theorem (after Gödel)

There is a sentence $\Phi$ such that for all models $(M, E) \models ZFC_0$

the following are equivalent:

1. $(M, E) \models \Phi$.
2. $(M, E) \models \text{“ZFC}_0 \vdash (\neg \Phi)\text{”}$
The sentence $\Phi_0$

$\Phi_0$ asserts:

1. $V_n$ exists where $n = |V_{1000}|$.
2. There is a proof of ($\neg \Phi_0$) with length at most $10^{24}$ from the theory:

$$ZFC_0 + \text{"} V_n \text{ exists where } n = |V_{1000}| \text{"}.$$
$\Phi_0$ if true is physically verifiable from the witness.

- $\Phi_0$ is a meaningful statement about the actual physical universe.

**Question**

*Is $\Phi_0$ true?*

- Of course not. But we have no evidence (physical or mathematical) that $\Phi_0$ is false.

**Claim**

Any coherent basis (at present) for the assertion that $\Phi_0$ is false must also yield that the conception of $V_n$ where $n = |V_{1000}|$ is meaningful.
Any coherent basis for the mathematical claim of the consistency of a formal (recursive) theory $T$ must be paired ultimately with a conception of mathematical objects with structure, whose existence implies the consistency of $T$. 
Consistency and independence in ZFC

- $\omega_1$ is the least uncountable ordinal
  - it is the set of all countable ordinals.

**Definition**

1. A set $C \subseteq \omega_1$ is **closed** if for all $\alpha < \omega_1$ if $C \cap \alpha$ is cofinal in $\alpha$ then $\alpha \in C$.

2. A set $S \subseteq \omega_1$ is **stationary** if $S \cap C \neq \emptyset$ for all closed, cofinal, sets $C \subseteq \omega_1$.

- The sets, $S \subset \omega_1$, which are stationary and co-stationary are the simplest manifestation of the Axiom of Choice.
- How complicated is the structure of the stationary, co-stationary, subsets of $\omega_1$?
  - Can exist a small generating family for these sets?
The combinatorics of stationary subsets of $\omega_1$

A precise question along these lines is the following:

**Question**

*Can there exist $\omega_1$ many stationary sets, $\langle S_\alpha : \alpha < \omega_1 \rangle$, such that for every stationary set $S \subseteq \omega_1$, there exists $\alpha < \omega_1$ such that $S_\alpha \subseteq S$ modulo a non-stationary set?*

The assertion that $S_\alpha \subseteq S$ modulo a non-stationary set is simply the assertion that the set,

$$S_\alpha \setminus S = \{ \beta < \omega_1 \mid \beta \in S_\alpha \text{ and } \beta \notin S \},$$

is not stationary.

**Observation**

*Such a sequence, $\langle S_\alpha : \alpha < \omega_1 \rangle$, of stationary subsets of $\omega_1$ would give in a natural sense, a basis for the stationary subsets of $\omega_1$ which is of cardinality $\omega_1$.***
Infinite games on $\omega$ and Determinacy Axioms

- Associated to a set $A \subseteq \mathbb{R}$ is an infinite game involving two players, Player I and Player II. The players construct a function, $f : \omega \to \{0, 1\}$, in stages,

  (Stage 0) : Player I specifies $f(0)$,
  (Stage 1) : Player II specifies $f(1)$,
  (Stage 2) : Player I specifies $f(2)$,

  .............

  After infinitely many stages a function $f : \omega \to \{0, 1\}$ is constructed.

- Player I wins this run of the game if

  $$\sum_{k=0}^{\infty} f(k)2^{-(k+1)} \in A,$$

  otherwise Player II wins.
Strategies

A strategy is a function

\[ \tau : \{ s : k \rightarrow \{0, 1\} \mid k \in \omega \} \rightarrow \{0, 1\} \]

and a player follows \( \tau \) in a run of the game yielding \( f \) if at each stage \( k \) for that player, \( f(k) = \tau(f|k) \).

Definition (Mycielski, Steinhaus: 1961)

The Axiom of Determinacy, AD, is the axiom which asserts that for all sets \( A \subseteq \mathbb{R} \) there is a winning strategy for either Player I or Player II in the game given by \( A \).

- AD contradicts the Axiom of Choice.

Question

Is the Axiom of Choice necessary to construct a set \( A \subseteq \mathbb{R} \) for which the corresponding game is not determined?
Large Cardinal Axioms

Basic template for (modern) large cardinal axioms

A cardinal $\kappa$ is a large cardinal if there exist an ordinal $\alpha$, a transitive set $M$, and an elementary embedding,

$$j : V_\alpha \rightarrow M$$

such that $\kappa$ is the least ordinal such that $j(\beta) \neq \beta$.

▶ $\text{CRT}(j)$ denotes the least ordinal $\beta$ such that $j(\beta) \neq \beta$.
  ▶ If $j$ is the identity on $\alpha$ then $j$ is the identity on $V_\alpha$.
  ▶ One can require more sets to belong to $M$, possibly in a way that depends on the action of $j$ on the ordinals.
    ▶ A hierarchy of notions.
    ▶ (Axiom of Choice) If $M = V_\alpha$ then either $\alpha = \lambda$ or $\alpha = \lambda + 1$ where $\lambda$ is the supremum of $\langle \kappa_i : i < \omega \rangle$, $\kappa_0 = \text{CRT}(j)$ and for all $i < \omega$, $\kappa_{i+1} = j(\kappa_i)$. 
Three theories

| Theory 1 | ZFC + “There exist $\omega_1$ many stationary sets, $\langle S_\alpha : \alpha < \omega_1 \rangle$, such that for every stationary set $S \subseteq \omega_1$, there exists $\alpha < \omega_1$ such that $S_\alpha \subseteq S$ modulo a non-stationary set”.
 |
| Theory 2 | ZF + AD |
| Theory 3 | ZFC + “There exist infinitely many Woodin cardinals”. |

Theorem

*These three theories are equiconsistent.*
The conception of the universe of sets with the structure from large cardinals can account for all possible consistency claims.
Two large cardinal axioms in $\text{ZF}$

**Definition:** $\kappa$ is an Enormous Cardinal

There exist $\kappa < \lambda < \gamma$ and an elementary embedding

$$j : V_{\lambda+1} \rightarrow V_{\lambda+1}$$

such that

1. $\kappa = \text{CRT}(j)$ and $\lambda > \kappa$ is least such that $j(\lambda) = \lambda$,
2. $V_\lambda \prec V_\gamma$.

**Definition:** $\kappa$ is a Weak Reinhardt Cardinal

There exist $\kappa < \lambda < \gamma$ and an elementary embedding

$$j : V_{\lambda+2} \rightarrow V_{\lambda+2}$$

such that

1. $\kappa = \text{CRT}(j)$ and $\lambda > \kappa$ is least such that $j(\lambda) = \lambda$,
2. $V_\lambda \prec V_\gamma$. 
Theorem (Kunen)

Assume the Axiom of Choice. Suppose $\lambda$ is an ordinal and

$$j : V_{\lambda+2} \rightarrow V_{\lambda+2}$$

is an elementary embedding. Then $j$ is the identity.

Corollary (ZFC)

There are no Weak Reinhardt Cardinals.

Theorem (ZF)

Assume there is a Weak Reinhardt Cardinal. Then

$$ZFC + \text{“There is a proper class of Enormous Cardinals”}$$

is consistent.

- Must the Axiom of Choice be abandoned for Thesis (Part 2)?
The effective cumulative hierarchy: \( L \)

### The definable power set

For each set \( X \), \( \mathcal{P}_{\text{Def}}(X) \) denotes the set of all \( Y \subseteq X \) such that \( X \) is logically definable in the structure \((X, \in)\) from parameters in \( X \).

- (Axiom of Choice) \( \mathcal{P}_{\text{Def}}(X) = \mathcal{P}(X) \) if and only if \( X \) is finite.

### Gödel's constructible universe, \( L \)

Define \( L_\alpha \) by induction on \( \alpha \) as follows.

1. \( L_0 = \emptyset \),
2. (Successor case) \( L_\alpha + 1 = \mathcal{P}_{\text{Def}}(L_\alpha) \),
3. (Limit case) \( L_\alpha = \bigcup\{L_\beta \mid \beta < \alpha\} \).

\( L \) is the class of all sets \( X \) such that \( X \in L_\alpha \) for some ordinal \( \alpha \).

- (Scott) Assume \( V = L \). There are no (modern) large cardinals.
Definition

For each ordinal $\alpha$, $\text{HOD}_{\alpha+1}$ is the set of all sets $a \subseteq V_\alpha$ such that:

1. $a$ is definable in $V_\alpha$ from ordinal parameters.
2. If $b \in \text{TC}(a)$ then $b$ is definable in $V_\alpha$ from ordinal parameters.

where for each set $a$, $\text{TC}(a)$ is the smallest transitive set $M$ with $a \in M$.

▶ The definition of $\text{HOD}_{\alpha+1}$ is a mixture of the definition of $L_{\alpha+1}$ and $V_{\alpha+1}$.

Definition

$\text{HOD}$ be the class of all sets $a$ such that $a \in \text{HOD}_{\alpha+1}$ for some $\alpha$.

▶ If the existence of a proper class of Enormous Cardinals is consistent then the existence is consistent with $V = \text{HOD}$.
A class $N$ is $\Sigma_2$-definable if there is a formula $\varphi(x_0)$ such that

$$N = \bigcup \{ a \mid V_\alpha \models \varphi[a] \text{ for some ordinal } \alpha \}.$$ 

$L$ is $\Sigma_2$-definable.

$\langle V_\alpha : \alpha \in \text{Ord} \rangle$ is $\Sigma_2$-definable.

HOD is $\Sigma_2$-definable.

A class $N$ is $\Sigma_2$-definable if there is a formula $\varphi(x_0, x_1)$ and a set $b$ such that

$$N = \bigcup \{ a \mid V_\alpha \models \varphi[a, b] \text{ for some ordinal } \alpha \}.$$
**Definition: $N[X]$**

Suppose that $N$ is a transitive class and $X$ is a transitive set. Then $N[X]$ is the smallest transitive class $M$ such that

1. $N \subseteq M$ and $X \cap M \in M$
2. $M \models ZF$

**Definition: $N(X)$**

Suppose that $N$ is a transitive class and $X$ is a transitive set. Then $N(X)$ is the smallest transitive class $M$ such that

1. $N \subseteq M$ and $X \in M$
2. $M \models ZF$

**Lemma**

Suppose $N$ is $\Sigma_2$-definable. Then $N(X)$ and $N[X]$ are each $\Sigma_2$-definable.
### Vopenka’s Theorem

**Theorem (Vopenka)**

*For each transitive set $X$:

1. $\text{HOD}[X]$ is a generic extension of $\text{HOD}$;
2. $\text{HOD}(X)$ is a symmetric generic extension of $\text{HOD}$.*

**Corollary (ZF)**

*Suppose that $\kappa$ is a Weak Reinhardt Cardinal. Then $\kappa$ is a Weak Reinhardt Cardinal in some symmetric generic extension of $\text{HOD}$.*

**Speculation**

*Perhaps a multiverse conception based on generic extensions could provide a framework for truth which:

- Accounts for the consistency of Weak Reinhardt Cardinals,
  - avoiding having to abandon the Axiom of Choice.
- Avoids having to settle the Continuum Hypothesis.*
Definition

Suppose that $M$ is a countable transitive set and that $M \models \text{ZFC}$.

The *generic-multiverse* generated by $M$ is the smallest set $\mathcal{V}_M$ of countable transitive sets such that for all pairs $(N_0, N_1)$ of countable transitive sets if

1. $N_1$ is a generic extension of $N_0$
2. either $N_0 \in \mathcal{V}_M$ or $N_1 \in \mathcal{V}_M$

then both $N_0 \in \mathcal{V}_M$ and $N_1 \in \mathcal{V}_M$.

(meta) Definition

*The Generic-Multiverse is the generic-multiverse generated by $V$.***
The Generic-Multiverse and truth

The generic-multiverse view of truth

A sentence $\varphi$ is a Generic-Multiverse truth if $\varphi$ holds in each universe of the Generic-Multiverse.

This can be formally reduced to truth within $V$.

There is a (recursive) transformation of sentences giving $\varphi^*$ from $\varphi$ such that:

Theorem

For all countable transitive sets $M$ the following are equivalent.

1) $M \models \varphi^*$.

2) $N \models \varphi$ for each $N \in \mathcal{V}_M$. 
The Resurrection Theorem

- A $\Sigma_2$-sentence is a sentence of the form:
  
  "There exists $\alpha$ such that $V_\alpha \models \psi$"

- A $\Pi_2$-sentence is a sentence of the form:
  
  "For all $\alpha$, $V_\alpha \models \psi$"

**Theorem (Resurrection Theorem)**

*Suppose there is a proper class of Woodin cardinals and that $\varphi$ is a $\Sigma_2$-sentence true in $V$. Then for each universe $N$ of the Generic-Multiverse, $\varphi$ is true in some generic extension of $N$.*

**Corollary**

*Suppose that there is a proper class of Woodin cardinals and that $\varphi$ is a $\Pi_2$-sentence. Then the following are equivalent.

1. $\varphi$ is true in all generic extensions of $V$.
2. $\varphi$ is true in all universes of the Generic-Multiverse.*
Example

Suppose \( \psi \) is a sentence and consider the \( \Pi_2 \)-sentences

- “\( V_{\omega+2} \models \psi \)”
- “\( V_{\omega+2} \models (\neg \psi) \)”

Suppose there is a proper class of Woodin cardinals and that neither of the sentences a Generic-Multiverse truth.

Suppose \( N \) is a universe of the Generic-Multiverse. Then:

- There are generic extensions of \( N \) in which “\( V_{\omega+2} \models \psi \)”.
- There are generic extensions of \( N \) in which “\( V_{\omega+2} \models (\neg \psi) \)”.

- \( V_{\omega+2} \) can be replaced by \( V_{\omega+3}, V_{\omega+10000}, V_{\delta_0+1} \) where \( \delta_0 \) denotes the least Woodin cardinal, etc.
Thesis: Part 3 in two parts

Part A

The conception of $\Pi_2$ truth must be at least as strong as that given by the Generic-Multiverse.

Part B

For the conception of $\Pi_2$ truth, the only possibilities are

1. $V$ (Set Theoretic Platonism)
   - For each sentence $\psi$, the sentence “$V_{\omega+2} \models \psi$” has determinate truth value etc.

2. The Generic-Multiverse.
Another consequence of the Resurrection Theorem

Assume ZF and that there is a Weak Reinhardt Cardinal. Suppose that

\[ \text{HOD} \models \text{"There is a proper class of Woodin cardinals."} \]

Consider the generic multiverse generated by HOD.

- If \( N \) is a universe of this generic-multiverse then there is a symmetric extension of \( N \) in which there is a Weak Reinhardt Cardinal.

This suggests that the Generic-Multiverse conception of truth might be able to account for the consistency with ZF of the existence of Weak Reinhardt Cardinals.
The first multiverse law

**Definition (For a given multiverse)**

1. For any universe $N$, $(\delta_0)^N$ denotes the first Woodin cardinal of $N$.
2. A sentence $\varphi$ is a multiverse truth of $V_{\delta_0+1}$ if for each universe $N$ of the multiverse,

$$
(V_{\delta_0+1})^N \models \varphi.
$$

**The First Multiverse Law**

The set of $\Pi_2$-sentences which are multiverse truths is not recursive in the set of multiverse truths of $V_{\delta_0+1}$.

- The multiverse given by all $\omega$-models $N \models \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$ violates the First Multiverse Law.
The second multiverse law

Definition (For a given multiverse)

A set \( X \subseteq V_\omega \) is definable in \( V_{\delta_0+1} \) across the multiverse if for each universe \( N \) of the multiverse, \( X \) is logically definable in \((V_{\delta_0+1})^N\) without parameters.

The Second Multiverse Law

The set of \( \Pi_2 \)-sentences which are multiverse truths, is not definable in \( V_{\delta_0+1} \) across the multiverse.

- The multiverse given by all \( \omega \)-models

  \[ N \models \text{ZFC + “There is a proper class of Woodin cardinals”} \]

  satisfies the Second Multiverse Law.
The conception of a multiverse of sets should not violate both the multiverse laws.

Therefore if the Generic-Multiverse is to be the basis for $\Pi_2$ truth:

The Generic-Multiverse must satisfy one of the multiverse laws.

- The multiverse laws are really a family of laws indexed by ($\Sigma_2$-definitions of) $\delta$:
  - here $\delta$ is the first Woodin cardinal.
- The “smaller” the choice of $\delta$, the “weaker” the laws.
- Taking $\delta = \omega + 2$ yields the weakest (plausible) versions, these are the Weak Multiverse Laws.
Predictions
Ω-logic
(The logic of the Generic-Multiverse)

**Definition**
Suppose $\varphi$ is a $\Pi_2$-sentence. Then

$$\models_\Omega \varphi$$

if $\varphi$ holds in all generic extensions of $V$.

**Theorem**

*Suppose there is a proper class of Woodin cardinals and that $\varphi$ is a $\Pi_2$-sentence.*

*Then $\varphi$ is a Generic-Multiverse truth if and only if $\models_\Omega \varphi$.***
Universally Baire sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is universally Baire if for all topological spaces, $S$, and for all continuous functions,

$$F : S \to \mathbb{R},$$

the preimage of $A$ by $F$ has the property of Baire in the space $S$ (differs from an open set by a meager set).

Example: If $A \subseteq \mathbb{R}$ is borel then $A$ is universally Baire.

Theorem (after et al)

Suppose there is a proper class of Woodin cardinals and $A \subseteq \mathbb{R}$ is universally Baire. Then

1. $L(A, \mathbb{R}) \models AD$,

2. Every set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.
Strong closure

**Definition**

Suppose that $A \subseteq \mathbb{R}$ is universally Baire and suppose that $M$ is a countable transitive model of ZFC.

Then $M$ is *strongly A-closed* if for all countable transitive sets $N$ such that $N$ is a generic extension of $M$,

$$A \cap N \in N.$$ 

- If $M$ is strongly $A$-closed then $A \cap M \in M$. But this alone does not suffice.
The definition of $\models_\Omega \varphi$

**Definition**

Suppose there is a proper class of Woodin cardinals. Suppose that $\varphi$ is a $\Pi_2$-sentence.

Then $\models_\Omega \varphi$ if there exists a set $A \subseteq \mathbb{R}$ such that:

1. $A$ is universally Baire,

2. for all countable transitive models, $M$, if $M$ is strongly $A$-closed then

   $M \models "\models_\Omega \varphi"$.

$\models_\Omega \varphi$ is invariant across the Generic-Multiverse.
The Ω Conjecture

Theorem (Ω Soundness)
Suppose that there exists a proper class of Woodin cardinals and suppose that $\varphi$ is $\Pi_2$-sentence.

If $\vdash_\Omega \varphi$ then $\models_\Omega \varphi$

Definition (Ω Conjecture)
Suppose that there exists a proper class of Woodin cardinals and suppose that $\varphi$ is a $\Pi_2$-sentence.

Then $\models_\Omega \varphi$ if and only if $\vdash_\Omega \varphi$.

- The Ω Conjecture is invariant across the Generic-Multiverse.
Theorem (Fundamental Theorem of Ω-logic)

Suppose there is a proper class of Woodin cardinals and that the Ω Conjecture holds.

Let $\mathcal{V}_\Omega$ be the set of all $\Pi_2$-sentences $\varphi$ such that $\models_{\Omega} \varphi$. Then:

1. $\mathcal{V}_\Omega$ is recursive in the set of Generic-Multiverse truths of $V_{\delta_0+1}$.
2. $\mathcal{V}_\Omega$ is definable in $V_{\delta_0+1}$.

(1) can be strengthened to $\mathcal{V}_\Omega$ is recursive in the set of Generic-Multiverse truths of $V_{\omega+2}$ (which is best possible).

Corollary

Suppose there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the Generic-Multiverse violates both multiverse laws.
Predictions: Conjecture 1

The $\Omega$ Conjecture is a theorem of ZFC.
Supercompact and extendible cardinals

**Definition:** \( \delta \) is a supercompact cardinal

For each ordinal \( \gamma > \delta \) there exist \( \bar{\delta} < \bar{\gamma} < \delta \) and an elementary embedding

\[ j : V_{\bar{\gamma}+1} \rightarrow V_{\gamma+1} \]

such that \( \text{CRT}(j) = \bar{\delta} \) and \( j(\bar{\delta}) = \delta \).

**Definition:** \( \delta \) is an extendible cardinal

For each ordinal \( \gamma > \delta \) there exists an elementary embedding

\[ j : V_{\gamma+1} \rightarrow V_{j(\gamma)+1} \]

such that \( \text{CRT}(j) = \delta \) and such that \( j(\delta) > \gamma \).
Definition: Extenders

Suppose that

\[ j : V_{\delta+1} \to M \]

is an elementary embedding. Suppose \( \text{CRT}(j) < \alpha \) and \( V_{\alpha+1} \subset M \).

- The extender \( E \) of strength \( \alpha \) derived from \( j \) is the function
  \[ E : V_{\delta+1} \to V_{\alpha+1} \]
  defined by \( E(a) = j(b) \cap V_\alpha \).

- Define \( \text{CRT}(E) \) to be the least ordinal \( \xi \) such that \( E(\xi) \neq \xi \).
- Suppose that \( E \) is the extender of strength \( \alpha \) derived from \( j \). Then \( \text{CRT}(j) = \text{CRT}(E) \).
Weak extender models

Definition

A \( \Sigma_2 \)-definable transitive class \( N \) is a weak extender model of \( \delta \) is supercompact if \( \delta \) is supercompact in \( N \) and this is witnessed by the class of \( F \in N \) such that

1. \( F \) is an extender in \( N \),
2. there is an extender \( E \) of \( V \) such that \( E \cap N = F \).

The Inner Model Program is the attempt to build weak extender models for various large cardinal notions—subject to non-triviality requirements.

- The models produced are generalizations of \( L \).
Mitchell-Steel models

Theorem (after Mitchell-Steel, Steel)

Suppose there is a proper class of Woodin cardinals. Then there is a $\Sigma_2$-definable transitive class $N$ such that:

1. $N$ is a weak extender model of $\delta$ is a Woodin cardinal for a proper class of $\delta$.
2. $N \subseteq \text{HOD}$.
3. Every set $A \subseteq \text{Ord}$ is generic over $N$.
4. Suppose that there is a supercompact cardinal. Then every set $A \in N(\mathbb{R})$ is universally Baire.

The supercompact is far stronger than necessary but some additional hypothesis is necessary.
The Universality Theorem

Theorem (Universality Theorem)

Suppose that $N$ is a weak extender model of $\delta$ is supercompact. Suppose that $F$ is an extender such that:

- $\text{CRT}(F) \geq \delta$ and $N$ is closed under $F$.

Then $F \cap N \in N$.

- For any extender $F$, $L$ is closed under $F$ but $F \cap L \notin L$.

Corollary

Suppose that $N$ is a weak extender model of $\delta$ is supercompact and that $\kappa > \delta$ is an extendible cardinal. Then $N$ is a weak extender model of $\kappa$ is supercompact.

- Any weak extender model of $\delta$ is supercompact inherits all large cardinals from $V$ which occur above $\delta$.
Theorem

Suppose there is a proper class of supercompact cardinals and that $N$ is a weak extender model of $\delta$ is supercompact such that

1. Every set $A \in N(\mathbb{R})$ is universally Baire.
2. Every set bounded $A \subset \delta$ is generic over $N$.

Then $N \models \text{“The } \Omega \text{ Conjecture”}$. 

By the Universality Theorem one cannot hope to prove the existence of such $N$ from any large cardinal hypothesis.

- The correct conjecture would be that such $N$ exist in some generic extension of $V$. 
Predictions: Conjecture 2

Conjecture

(ZFC) Suppose that $\delta$ is an extendible cardinal. Then there is a transitive class $N$ such that:

1. $N$ is a weak extender model of $\delta$ is supercompact.
2. Every bounded set $A \subset \delta$ is generic over $N$.
3. $N \subseteq \text{HOD}$ and $N$ is $\Sigma_2$-definable from $\delta$.
4. $N \models \text{“The } \Omega \text{ Conjecture”}$.

- The conjecture implies that no large cardinal hypothesis can refute the $\Omega$ Conjecture.
By the Universality Theorem:

The successful extension of the Inner Model Program to the level of exactly one supercompact cardinal yields an ultimate version of $L$.

- Finding the corresponding ultimate version of the axiom, “$V = L$”, is possibly a much harder problem.

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^L(A, \mathbb{R})$ is the supremum of the ordinals $\alpha$ such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\text{HOD}^L(A, \mathbb{R})$ denotes HOD as defined in $L(A, \mathbb{R})$.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A$ is universally Baire.

Then $\Theta^L(A, \mathbb{R})$ is a Woodin cardinal in $\text{HOD}^L(A, \mathbb{R})$. 
The axiom for $V = \text{Ultimate-L}$?

A sentence $\varphi$ is a $\Sigma_3$-sentence if it is of the form:

- There exists $\alpha$ such that $V_\alpha \models \psi$ and such that $V_\alpha \prec_{\Sigma_2} V$; for some sentence $\psi$.

(meta) Conjecture: The axiom for $V = \text{Ultimate-L}$

There is a proper class of Woodin cardinals. Further for each $\Sigma_3$-sentence $\varphi$, if $\varphi$ holds in $V$ then there is a universally Baire set $A \subseteq \mathbb{R}$ such that

$$\text{HOD}^L(A,\mathbb{R}) \cap V_\Theta \models \varphi$$

where $\Theta = \Theta^L(A,\mathbb{R})$.

- This axiom settles (modulo axioms of infinity) all sentences about $\mathcal{P}(\mathbb{R})$ (and much more) which have been shown to be independent by Cohen’s method.
Consequences of $V = \text{Ultimate-L}$

<table>
<thead>
<tr>
<th>Theorem ($V = \text{Ultimate-L}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The Continuum Hypothesis holds.</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem ($V = \text{Ultimate-L}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>The $\Omega$ Conjecture holds.</strong></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Theorem ($V = \text{Ultimate-L}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>$V$ is the minimum universe of the Generic-Multiverse.</strong></td>
</tr>
</tbody>
</table>

**In contrast:**

*Suppose $N$ is a Mitchell-Steel extender model and there is a Woodin cardinal in $N$ then:*

- $N$ is **not** the minimum universe of the generic-multiverse generated by $N$ even restricting to Mitchell-Steel extender models in that multiverse.*
Predictions: Conjecture 3

Ultimate-L Conjecture

(ZFC) Suppose that $\delta$ is an extendible cardinal. Then there is a transitive class $N$ such that:

1. $N$ is a weak extender model of $\delta$ is supercompact.
2. Every bounded set $A \subset \delta$ is generic over $N$.
3. $N \subseteq \text{HOD}$ and $N$ is $\Sigma_2$-definable from $\delta$.
4. $N \models \text{"V} = \text{Ultimate-L"}$.

Theorem (ZF)

Suppose that the Ultimate-L Conjecture is provable from ZFC. Then there are no Weak Reinhardt Cardinals.
Claim

The structure involved in the construction of Mitchell-Steel weak extender models at the level of Woodin cardinals arguably suffices to validate consistency claims at the level of Woodin cardinals.

A claim and a (serious) problem

The structure involved in the construction of the weak extender model $N$ witnessing Ultimate-L Conjecture cannot suffice to validate consistency claims beyond the level of one supercompact cardinal.

Theorem (Ultimate-L Conjecture)

Suppose that $\delta$ extendible cardinal. Then there is a weak extender model $N$ witnessing the Ultimate-L Conjecture at $\delta$ and a class generic extension $V[G]$ such that for all $\alpha$:

1. $V[G]_\alpha \not\models \text{ZFC + "There is a supercompact cardinal"}$.
2. In $V[G]$, $N$ satisfies all the requirements of the Ultimate-L Conjecture at $\delta$. 
Conjecture

\((V = \text{Ultimate-L})\) Suppose that \(\lambda > \omega\) is a cardinal such that

\[ L(V_{\lambda + 1}) \nsubseteq \text{Axiom of Choice}. \]

Then there exists a non-trivial elementary embedding

\[ j : V_{\lambda + 1} \rightarrow V_{\lambda + 1}. \]

- The existence of a Very Enormous Cardinal implies the existence of \(\lambda\) such that
  \[ L(V_{\lambda + 1}) \nsubseteq \text{Axiom of Choice}. \]
- Assuming Conjecture 3 and the consistency of the existence of a Very Enormous Cardinal, Conjecture 4 cannot be vacuously true.
  - It is vacuously true for Mitchell-Steel extender models.
Conjecture 4 is a conjecture of rich structure associated in Ultimate-L to large cardinals.

- There is a generic extension of $L$ in which:

  $$L(\mathbb{R}) \not\models \text{“Axiom of Choice”}.$$ 

Theorem (Steel)

Suppose $N$ is a Mitchell-Steel weak extender model. Then the following are equivalent in $N$:

1. $L(\mathbb{R}) \not\models \text{Axiom of Choice}.$
2. $L(\mathbb{R}) \models \text{AD}.$
The ultimate (meta) conjecture

(meta) Conjecture

Large cardinal axioms above the level of one supercompact cardinal will be validated by their structural consequences for Ultimate-L.

- Ultimate-L has the ultimate structure associated to large cardinals.
- This structure is equivalent in Ultimate-L to the occurrence of large cardinals.
- This structure implies $V = \text{Ultimate-L}$ in the context of large (enough) cardinals.