Woodin on "The Realm of the Infinite"

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The paper "The Realm of the Infinite" is a tapestry of argumentation that weaves together the argumentation in the papers "The Tower of Hanoi", "The Continuum Hypothesis, the generic-multiverse of sets, and the Ω Conjecture", and "The Transfinite Universe". The last two papers in this trilogy are based on the mathematical results presented in *The Axiom of Determinacy, forcing axioms, and the nonstationary ideal* and *Suitable Extender Models*, respectively. As a result there is a lot going on in the background and it is hard to follow the red thread through the twists and turns of the argumentation. In these comments I aim to bring out that red thread and raise some critical concerns.[1]

The paper can be divided into two general parts. PART I concerns the large finite. Here Woodin addresses a position that we shall call *ultrafinitism*. The ultrafinitist maintains that the "small finite" exists but the "large finite" does not. The motivation for this position is often based on physicalism or considerations of feasibility. The divide between the small finite and the large finite is not a sharp one. But for the purposes of this discussion we can specify numbers that definitely fall on either side. For example, the numbers 10^{24} and 10^{48} will be considered small and we shall specify some large numbers below. The goal of PART I is to show that (1) the non-existence of the large finite is something that can be detected in the realm of the small finite and (2) that using the large finite we can predict that its non-existence will never be detected in the realm of the small finite. This then places the ultrafinitist and the advocate of the large finite in the following dialectical position: We run a search through the realm of the small finite for evidence of the non-existence of the large finite. If over time there is no detection then that tips

[1] These comments were written after my lecture on the topic and after a conversation with Woodin concerning the dialectic of his paper. My comments in the lecture were off target in a number of respects. These comments thus supersede those and have the benefit of discussion with Woodin.

the scales in favour of the advocate of the large finite. If, however, there *is* detection, then that seals the deal for the ultrafinitist (and demolishes most of mathematics).

PART II assumes that outcome of the test in PART I has led to the acceptance of the large finite and then turns to the parallel issue that arises with the transfinite. In other words, the division

$$\frac{\text{Large Finite}}{\text{Small Finite}}$$

that appeared in PART I is now replaced with the division

$$\frac{\text{Large Infinite}}{\text{Small Infinite}}$$

Here the small infinite is the totality of natural numbers (finite objects) and the large infinite is anything beyond. The dialectic of this part involves an exchange between the Skeptic (a finitist) and the Set Theorist (a transfinitist). There are three main steps:

STEP 1: Low-Level Implications of Transfinite: This step parallels PART I. Here the Set Theorist shows that (1) the non-existence of the transfinite is something that can be detected in the realm of the finite and (2) that using the transfinite we can predict that its non-existence will never be detected in the realm of the finite. This then places the Skeptic (finitist) and the Set Theorist (transfinitist) following dialectical position: We run a search through the realm of the finite for evidence of the non-existence of the transfinite. If over time there is no detection then that tips the scales in favour of the advocate of the transfinitist. If, however, there *is* detection, then that seals the deal for the finitist (and demolishes most of higher mathematics).

STEP 2: Accounting for the Low-Level Implications: This step assumes that the outcome of the test in STEP 1 has led to the acceptance of the *consistency of* the transfinite and the challenge to account for this consistency. The Set Theorist accounts fo the consistency by appealing to the *existence* of the transfinite (equivalently, the *truth* of statements asserting that large transfinite objects (large cardinals) exist). The Skeptic accounts for the challenge with the doctrine that if statements asserting the existence of the transfinite (large cardinal axioms) are inconsistent, then there must be a simple proof of the inconsistency. The Skeptic then makes an interesting move, claiming to be in an advantageous position. For there is a *choiceless* large cardinal axiom (that is, a large cardinal axiom that is incompatible with AC) the consistency of which implies the consistency of *every* large cardinal axiom that the set theorist (who is a believer in ZFC) accepts. Thus, the Skeptic just has to account for this *one* prediction. But how is the Set Theorist going to account for this prediction? The trouble is that the set theorist is working in ZFC.

STEP 3: Accounting for the prediction: The Set Theorist begins by trying two approaches: First, by appealing to the structure theory of large cardinals at the level of inconsistency. Second, by appealing to the generic multiverse. But each approach is problematic. Then the Set Theorist takes an entirely new approach, namely, by arguing that the aforementioned choiceness large cardinal axiom is actually *inconsistent*. This leads to a serious criticism against the Skeptic, something that we shall discuss in detail below.

Part I: The Large Finite

Let us begin by giving examples of the small finite and the large finite. For our purposes the numbers 10^{24} and 10^{48} will serve as examples of the small finite. The examples of the large finite concern the cardinality of finite levels of the cumulative hierarchy of sets, which recall is definite as follows: $V_0 = \emptyset$ and $V_{n+1} = \mathscr{P}(V_n)$. Thus, $|V_0| = 0$ and $|V_{n+1}| = 2^{|V_n|}$. The main example of the large finite that we shall use is this: First consider

$$\begin{aligned} |V_{1000}| &= 2^{|V_{999}|} \\ &= 2^{2^{|V_{998}|}} \\ &\;\;\vdots \\ &= \left.2^{2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}}\right\}999. \end{aligned}$$

The example of the large finite that we shall use is:

$$
\begin{aligned}
|V_{|1000|}| &= 2^{\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}998}\ {}^{V} \\
&= 2^{\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}997}\ {}^{V} \\
&\vdots \\
&= \left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}\left.2^{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}\right\}998.
\end{aligned}
$$

The background theory that the ultrafinitist employs is ZFC_0. This is a version of ZFC with an *anti-infinity* axiom. The intended model is of the form V_n for some n.

Woodin uses the Diagonal Lemma to construct a self-referential sentence Σ that makes reference to this large finite number. The key features of the sentence are as follows:

(1) Σ asserts that there exists a ZFC_0-proof of $\neg\Sigma$ of length less than 10^{24} and that $V_{|1000|}$ exists.

(2) In $\mathrm{ZFC} - \mathrm{Infinity}$ we can prove $\neg\Sigma$. (It proves the second conjunct and refutes the first.)

(3) If there is a ZFC_0-proof of $\neg\Sigma$ of length less than 10^{24} (contra the verdict of $\mathrm{ZFC} - \mathrm{Infinity}$) then $V_{|1000|}$ does not exist. (Notice that the converse to this does not hold: while there can be small-finite-detection of the non-existence of the large finite, there cannot be small-finite-certification of the existence of the large finite).

It is at this stage that Quantum Theory enters the picture. The key feature of Quantum Theory over Classical Theory is that it provides us with a device which is truly random. Suppose you want to produce a binary random sequence of length 10^{25} (used to code a proof of length less than 10^{24} of integers less than 10^{24}). If you use a classical device it will end up following some small algorithm and hence will not produce a truly random sequence—it will just produce something which appears random and is random for practical purposes but which strictly speaking reflects the pattern of the small algorithm used to produce it. However, a Quantum device (assuming that Quantum theory is correct and we devoted tons of global resources to it) can produce (in the next 1000 years say) such a device.

So we have a Quantum device randomly select such a string (from the pool of *all* possible strings of that length) and check whether it codes a proof of $\neg\Sigma$. The advocate of the large finite will be confident that it will not produce such a proof for there is no such proof. What will the ultrafinitist think? Before the experiment is run the ultrafinitist does not have any good reason to think that it will not find a proof.

What is the point of this argument. I see three. First, it forces us to be honest. Do we seriously remain agnostic about such a question? To the extent that we do not (but are confident that there will be no such string) we are revealing our implicit acceptance of the large finite. So the argument serves as an "intuition pump" to indicate to us (by drawing on our expectations with what will happen in the small finite) that we really do believe in the large finite.

Second, it serves to bring down to earth a similar move that is made concerning the transfinite (something we shall discuss in the second part). There are people who point out that there is always the possibility of an inconsistency in the transfinite. This points out that the situation is no different with the large finite.

Third, one can modify the set up to bring out an additional point. Suppose that wait a *very* long time and examine *all* of the short proofs. Or suppose that we have access to an oracle that can test all of the proofs. (Each involves a wild idealizing assumption.) At the end of the day if none are found to be a proof of $\neg\Sigma$ that would seem to lend credence to the advocate of the large finite. Along the way the ultrafinitist is agnostic. As this experiment is repeated with longer and longer sequences that agnosticism starts to look peculiar.

Part II: The Transfinite

We now assume that the above considerations have convinced the skeptic that the large finite exists. At this point the background assumptions are that the totality of natural numbers (as a completed infinite) exists and that all statements concerning the natural numbers have a determinate truth-value.

The debate now continues between the Skeptic (who accepts the completed infinity of the natural numbers but denies the existence of the higher infinite) and the Set Theorist (who accepts the existence of the higher infinite). The dialectic can be divided into three steps

Step 1: Low-Level Implications of the Transfinite

The Skeptic maintains:

> The mathematical conception of infinity is meaning- less and without consequence because the entire conception of the universe of sets is a complete fiction. Further, all the theorems of Set Theory are merely finitistic truths, a reflection of the mathematician and not of any genuine mathematical reality.

The Set Theorist responds:

> The development of Set Theory, after Cohen, has led to the realization there is a robust hierarchy of strong axioms of infinity.

The idea here is two-fold: First, there is the remarkable fact that all "natural" theories are well-ordered under the relation of interpretability. Second, there is the remarkable fact that the "principles of pure strength"—which in the higher reaches take the form of large cardinal axioms (strong axioms of infinity)—stand out as the central markers in this well-ordered hierarchy. More precisely, if one is given any two natural theories T_1 and T_2 then to show that they are ordered under the relation of interpretability one proceeds as follows. First, one finds a large cardinal axioms φ_1 and φ_2 such that T_1 is mutually interpretable with $\mathrm{ZFC} + \varphi_1$ and T_2 is mutually interpretable with $\mathrm{ZFC} + \varphi_2$ (and this is established using the dual methods of forcing and inner model theory). Second, one reads of the interpretation of T_1 in T_2 (or conversely) by looking at the natural interpretation of φ_1 in φ_2 (or conversely), the point being that the large cardinal axioms are naturally well-ordered. Thus, the well-ordering of the natural theories boils down to two facts—the fact that each theory is mutually interpretable with a large cardinal axioms and the fact that the large cardinal axioms are naturally well-ordered (under interpretability). A key point here is that there is usually no known way of finding the interpretation of T_1 in T_2 (or conversely) *without* passing through the large cardinal hierarchy. The large cardinal hierarchy thus stands out as a central spine through the interpretability hierarchy and this phenomenon leads to the implication that (in the very least) large cardinal axioms are consistent, something that makes sense to the Skeptic. For more on this subject see my SEP entry "Independence and Large Cardinals".

This leads to the following claim of the Set Theorist (speaking in the voice of Woodin):

> It is only through the calibration by a large cardinal axiom in **conjunction** with our understanding of the hierarchy of such axioms as **true axioms about the universe of sets**, that this prediction; the formal theory ZF+AD is consistent, is justified.

Later in the paper (on page 32) he says:

> To date there is no known (and credible) explanation for these predictions except that they are true *because* the corresponding axioms are true in the universe of sets. (My emphasis)

Now, this is a point on which I disagree. The formulation makes it look as though we have some "sideways on" view of the large cardinal hierarchy and it is this that enables us to be so confident in the consistency claims. But we do not have any such mystical insight. Rather our understanding of the large cardinal hierarchy works from the "ground up" through our investigations of the interpretability hierarchy.

Here is how I see the case for large cardinal axioms: For definiteness let me concentrate on the case of axioms at the level of "There exist infinitely many Woodin cardinals". First, we make a case for axioms like $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$. This involves looking at the structure theory. More precisely, one investigates statements that are low in the evidentiary order in the degree of ZFC+ "$There exists in finitely many Woodin cardinals''$ and the inter-relations among those statements and the outcome is that one has a strong case for $\mathrm{ZFC} + \mathrm{AD}^{L(\mathbb{R})}$. (For a detailed discussion of the evidentiary order and the case for $\mathrm{AD}^{L(\mathbb{R})}$ see my SEP entry "Large Cardinals and Determinacy".) Second, one notes that this principle gives the consistency of the large cardinals in question; in fact, it gives canonical inner models of such large cardinals (a very *strong* form of consistency). Third, one argues that large cardinals are the sort of thing which if consistent (especially in this strong sense) must exist.

I refer to the aforementioned paper for steps one and two. Let me here say a little more about step three. The general principle "If it is consistent that A exists then A exists". For example, working over ZFC we have that it is consistent that there exists a one-to-one between the real numbers and ω_1 (an object witnessing CH) and also that it is consistent that there exists a set of real numbers which cannot be put into one-to-one correspondence with either the natural numbers or the real numbers (an object witnessing $\neg$CH). Yet, clearly *both* of these objects cannot exist. So the general principle "Consistency implies Existence" is false. Instead one would have to maintain a restricted principle, something of the form "Consistency + X implies Existence". And the question, of course, is "What is X?" Now I don't plan to answer that question. But I do think that large cardinal axioms have the feature X and this it tied to the fact that they are *principles of pure existence*. As such they are naturally well-ordered. They stand as central markers in the hierarchy of interpretability and, as such, with regard to them there is no incompatibility issue like the one that arose with the above example of CH and $\neg$CH.

I would take me too far afield to elaborate on this. But let me boost your intuitions. Imagine dying and going to heaven and meeting God. God tells you that the large cardinal axioms "There exists infinitely many Woodin cardinals" is consistent, in fact, it is consistent in a strong sense: There are ω-models, β-models, and even inner models (in the sense of modern inner model theory). But then God says that these large cardinal axioms do not *in fact* exist. This is something I do not think that we could possibly believe. It buys into a picture of mathematics that involves a mythical model in the sky (to use Tait's phase) and a sideways on view of that model (to use McDowell's phrase). But there is no such model and no such view. Our access to mathematical truth is more down to earth. And the only obstacle in the way of the existence of an object like a large cardinal (something that arises from a principle asserting that there are very large levels of the universe of sets) is inconsistency. If such objects are consistent (especially in a strong sense) then they exist.

In discussion Woodin agreed with this point of view. So his main point can be put thus: *Once one examines all the evidence (structure theory, etc.) the only reasonable outcome is a view in which large cardinals exist.*

Step 2: Accounting for the Low-Level Implications

At this stage we assume that the outcome of the exchange in Step 1 has led to the acceptance of the *consistency of* the transfinite and the challenge to account for this consistency.

The Set Theorist accounts fo the consistency by appealing (in Woodin's original formulation) to the *existence* of the transfinite (equivalently, the *truth* of statements asserting that large transfinite objects (large cardinals) exist), or, even better (in the modified formulation) to the three-step case made above.

The Skeptic accounts for the challenge with the doctrine that *if statements asserting the existence of the transfinite (large cardinal axioms) are inconsistent, then there must be a simple proof of the inconsistency.* The Skeptic then makes an interesting move, claiming to be in an advantageous position. For there is a *choiceness* large cardinal axiom (that is, a large cardinal axiom that is incompatible with AC, in this case, a *weak Reinhardt cardinal*) the consistency of which implies the consistency of *every* large cardinal axiom that the set theorist (who is a believer in ZFC) accepts. Thus, the Skeptic just has to account for this *one* prediction. But how is the Set Theorist going to account for this prediction? The trouble is that the set theorist is working in ZFC.

Here the dialectic gets a bit puzzling. Why is the Skeptic's move even a good one? What reason is there to believe that if a large cardinal axiom is inconsistent then there must be a simple proof of inconsistency? In mainstream large cardinal theory there has only been one inconsistency, namely, Kunen's proof that Reinhardt cardinals are inconsistent. That proof is indeed (relatively) simple. But if the Skeptic is basing his considerations on that alone then it would seem to be a case of induction on one.

Step 3: Accounting for the Prediction

The Set Theorist begins by trying two approaches: First, by appealing to the structure theory of large cardinals at the level of inconsistency. Second, by appealing to the generic multiverse. But each approach is problematic.

The first approach is problematic for the following reason: It is true that there is a rich structure theory at the level of the largest large cardinal axioms (compatible with AC). For example, the structure theory of axioms asserting that there is a non-trivial elementary embedding from $L(V_{\lambda+1})$ to itself provides good evidence that these axioms are consistent. Moreover, this can be pushed further into the higher reaches, approaching the boarder of inconsistency. For more on this see my SEP entry "The Continuum Hypothesis" and Woodin's forthcoming "Suitable Extender Models II". But the problem is that these axioms do not reach the level of a weak Reinhardt cardinal.

The second approach requires some background concerning two fundamental theorems. The first is a theorem of Vopenka:

Theorem 0.1 (Vopenka)**.** *Assume* ZF*. For every ordinal α,* $\mathrm{HOD}(V_\alpha)$ *is a symmetric generic extension of* HOD*.*

The second theorem is due to Woodin:

Theorem 0.2 (Woodin)**.** *Assume* ZFC *and that there is a proper class of Woodin cardinals. Then for each Σ_2-sentence φ, the statement that φ holds in a generic extension of V is itself absolute to all forcing extensions of V.*

To see just how surprising the above result is, consider the following: Assume ZFC and that there is a proper class of Woodin cardinals. Suppose that there is a huge cardinal in V (this is expressible by a Σ_2-statement). Now let $V^{\mathbb{B}}$ be a generic extension in which the huge cardinal is collapsed. The model V satisfies that there is a generic extension (namely the trivial extension) in which there is a huge cardinal. So, by the above theorem, $V^{\mathbb{B}}$ also satisfies this. Thus, there is a generic extension $V^{\mathbb{B}*\mathbb{C}}$ containing a huge cardinal. The heritage V of $V^{\mathbb{B}}$ is leveraged to "resurrect" a huge cardinal. The large cardinal axiom has in effect been "resurrected".

These two theorems can be applied in our present setting as follows: Recall that the Set Theorist is trying to account for the consistency of a weak Reinhardt cardinal (a choiceless notion) but in the setting of ZFC (where one has choice). Assume that V satisfies ZFC and that there is a weak Reinhardt cardinal. Since being a weak Reinhardt cardinal is a Σ_2-notion (and this is the point of working with a *weak* Reinhardt cardinal as opposed to a full Reinhardt cardinal) there exists an α such that V_α satisfies that there is a weak Reinhardt cardinal. Now, by Vopenka, $\mathrm{HOD}(V_\alpha)$ is a generic extension of HOD (where both of these HOD's are computed in V. So HOD is a model of ZFC which has a generic extension $\mathrm{HOD}(V_\alpha)$ that contains a weak Reinhardt cardinal. The first idea of the Set Theorist is this: Perhaps the forcing is so canonical that there is a tight connection between the ZFC model HOD and the ZF model $\mathrm{HOD}(V_\alpha)$ that the structural features of the weak Reinhardt cardinal in $\mathrm{HOD}(V_\alpha)$ are reflected in HOD in such a way that the reflected structure theory provides us with a certificate for the consistency of a weak Reinhardt cardinal.

That is the Set Theorist's *first* idea. On reflection, however, the Set Theorist realizes that the connection is probably not tight enough and, moreover, that the model HOD is probably not canonical enough. To remedy this the Set Theorist considers the *generic multiverse* generated by HOD. We will not go into the details of this—they can be found in Woodin's paper and my SEP entry "The Continuum Hypothesis". Suffice it to say that the generic multiverse of HOD is obtained by successively taking generic extensions and generic refinements. The *generic multiverse conception of truth* is that a statement of set theory is true iff it is true in all universes of the generic multiverse. To set this up one works in the background theory of ZFC+ "There is a proper class of Woodin cardinals". It turns out that the generic multiverse conception of truth is a robust notion with regard to Π_2-statements, that is, all universes in the generic multiverse agree on the Π_2-generic multiverse truths. Now, all of the models in the generic multiverse are ZFC models. But by the theorem of Woodin above they all satisfy that there is a ZF generic extension that contains a weak Reinhardt cardinal. There is thus, as it were, a generic multiverse of ZF models attached to the generic multiverse. And by the canonicity of the generic multiverse ZFC (Π_2) truths one might hope that the structure theory provides a certificate for the weak Reinhardt cardinal (living as it were in the attached ZF generic multiverse).

That is the gist of the second approach. The trouble is that Woodin shows that assuming the Ω Conjecture (a conjecture of ZFC for which there is evidence) the generic multiverse conception of truth violates two rather plausible Transcendence Principles. This undermines the second approach.

So what is the Set Theorist to do? Here there is a very interesting twist. Recent work in inner model theory actually provides reasons for thinking that weak Reinhardt cardinals are *inconsistent*. It would take us too far afield to go into the details. Suffice it to say that the results show that if inner model theory (in anything like its current form) can reach the level of a supercompact cardinal then it goes all of the way and accomodate all large cardinal axioms. Should this turn out to be the case then there are three substantive consequences that bear on the present discussion.

First, this leads to evidence for the conjecture that weak Reinhardt cardinals are inconsistent, thus undermining the Skeptic's strategy of resting the case on that single large cardinal axiom and placing us in a setting where we have to argue for the consistency of large cardinal axioms in a stepwise fashion, working our way upward from below and not singling out a large cardinal at the top. Second, it shows that the mere existence of an inner model of a large cardinal is not sufficient to certify its consistency (since it will be precisely through inner model theory that we will come to detect inconsistency). Thus, something else will have to certify consistency. The most natural candidate is structure theory. But, in fact, in the one major place where we have detracted from Woodin above, we have argued that all along it was structure theory that certified consistency and not any "sideways on" view of the large cardinal hierarchy. Third, and finally, the advances in inner model theory lead to a new axiom—$V = L^{\Omega}_{S}$—which, along with large cardinal axioms, may very well wipe out independence and serve as the ultimate completion of the axioms of ZFC.

In summary: The first part puts pressure on the denier of the large finite. The second part puts pressure on the denier of the transfinite. Eventually the Skeptic takes up the challenge of accounting for the consistency of large cardinal axioms. The Skeptic does this through the doctrine that if a large cardinal axiom is inconsistent then there must be a simple proof and, picking a single such axiom—a weak Reinhardt cardinal—then turns the tables, challenging the Set Theorist to account for *that*. After two failed attempts the Set Theorist discovers evidence that (a) weak Reinhardt cardinals are in fact inconsistent, (b) the proof of this is (contra the Skeptic) a very complicated proof, and (c) the situation that emerges is one were one probes the border of inconsistency by approaching it from above with ever more complicated inconsistency proofs and securing it from below by rich structure theory.

In closing we shall address one question that Woodin does not address, namely, how could the Skeptic respond to this? One thing the Skeptic could do is "piggy-back" on the reasons for consistency that the Set Theorist provides. The Set Theorist views the structure theory as not just providing evidence of *consistency* but as providing evidence of *truth*. The Skeptic could resist this last step and regard the reasons only as evidence for consistency. The response to this is (in my view) the following: The structure theory is *on its face* evidence for more than consistency. To regard it only as evidence for consistency is analogous to the instrumentalist stance in physics that refuses to take observational evidence as evidence for theoretical propositions. This, however, is a large topic and it is best to stop here.[2]

[2]For further discussion of this parallel see my "Truth in Mathematics: The Question of Pluralism"