Chapter 1

The realm of the infinite

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1.1 Introduction

The 20$^{\text{th}}$ century witnessed the development and refinement of the mathematical notion of infinity. Here of course I am referring primarily to the development of Set Theory which is that area of modern mathematics devoted to the study of infinity. This development raises an obvious question: Is there a non-physical realm of infinity?

As is customary in modern Set Theory, V denotes the universe of sets. The purpose of this notation is to facilitate the (mathematical) discussion of Set Theory—it does not presuppose any meaning to the concept of the universe of sets.

The basic properties of V are specified by the ZFC axioms. These axioms allow one to infer the existence of a rich collection of sets, a collection which is complex enough to support all of modern mathematics (and this according to some is the only point of the conception of the universe of sets).

I shall assume familiarity with elementary aspects of Set Theory. The *ordinals* calibrate V through the definition of the cumulative hierarchy of sets, [17]. The relevant definition is given below.

Definition 1. Define for each ordinal α a set V_α by induction on α.

(1) $V_0 = \emptyset$.

(2) $V_{\alpha+1} = \mathcal{P}(V_\alpha) = \{X \mid X \subseteq V_\alpha\}$.

(3) If β is a limit ordinal then $V_\alpha = \cup \left\{ V_\beta \mid \beta < \alpha \right\}$. □

There is a much more specific version of the question raised above concerning the existence of a non-physical realm of infinity: Is the universe of sets a non-physical realm? It is this latter question that I shall focus on.

There are a number of serious challenges to the claim that the answer is yes. But where do these issues arise? More precisely for which ordinals α is the conception of V_α meaningful?

The first point that I wish to make is that for a rather specific *finite* value of n, the claim that V_n exists is a falsifiable claim and moreover that this "possibility" is consistent with our collective (formal) experience in Mathematics to date. The details are the subject of the next section and this account is a variation of that given in [12]. I will continue the narative bringing in the basic arguments of [13] and [15], ultimately defining a position on mathematical truth which is the collective conclusion of these three papers.

1.2 The realm of the finite

By Gödel's Second Incompleteness Theorem any system of axioms of *reasonable* expressive power is subject to the possibility of inconsistency, including of course the axioms for number theory. A natural question is how profound an effect could an inconsistency have on our view of mathematics or indeed on our view of physics.

For each finite integer n, $|V_{n+1}| = 2^{|V_n|}$ and so even for relatively small values of n, V_n is quite large. Is the conception of V_{1000} meaningful? What about the conception of V_n where $n = |V_{1000}|$?

By a routine Gödel sentence construction I produce a formula in the language of set theory which implicitly defines a property for finite sequences of length at most 10^{24}. For a given sequence this property is easily decided; if s is a sequence with this property then s is a sequence of non-negative integers each less than 10^{24} and the verification can be completed (with appropriate inputs) in significantly fewer than 10^{48} steps.

If there exists a sequence with this property then the conception of V_n is meaningless where $n = |V_{1000}|$. I use the bound 10^{24} in part because the verification that a candidate sequence of length at most 10^{24} has the indicated property, is arguably physically feasible. The question of whether or not there is a sequence of length 10^{24} concerns only the realm of all sequences of length 10^{24} and so is certainly a meaningful question for this realm. The existence of such a sequence has implications for nonexistence of V_n where $n = |V_{1000}|$ which is a vastly larger realm. This is entirely analogous to the well known and often discussed situation of the number theoretic statement,

"ZFC is formally inconsistent".

This statement concerns only V_ω (i.e., the realm of Number Theory) and yet its truth has implications for the nonexistence of the universe of sets, again a vastly larger realm.

The philosophical consequences of the existence of a sequence of length 10^{24} as described above, are clearly profound for it would demonstrate the necessity of the finiteness of the universe. Clearly such a sequence does not exist. However this property has the feature that if arbitrarily large sets *do* exist then there is no proof of *length* less than 10^{24} that no such sequence of length at most 10^{24} can have this property. I shall make these claims more precise.

Is the existence of such a sequence a meaningful question for our actual physical universe? A consequence of quantum mechanics (as opposed to classical mechanics) is that one could really build (on Earth, today) a device with a *nonzero* (though ridiculously small) chance of finding such a sequence if such a sequence exists, which is the other reason for the explicit bound of 10^{24}. So the claim that no such sequence exists is a prediction about our world.

Now the claim that there is no such sequence is analogous to the claim that there is no formal contradiction in Set Theory or in Set Theory together with large cardinal axioms. I do not see any credible argument at present for the former claim other than the claim that the conception of V_n is meaningful where $n = |V_{1000}|$ (though in $2^{10^{26}}$ years there will be such a credible argument). But then what can possibly provide the basis for the latter claim other than some version of the belief that the conception of the universe of sets is also meaningful?

1.2.1 Preliminaries

I shall assume familiarity with set theory at a naive level and below list informally the axioms. I do this because I will need a variation of this system of axioms and this variation is not a standard one.

Axiom 0 There exists a set.

Axiom 1 (Extensionality) Two sets A and B are equal if and only if they have the same elements.

Axiom 2 (Pairing) If A and B are sets then there exists a set $C = \{A, B\}$ whose only elements are A and B.

Axiom 3 (Union) If A is a set then there exists a set C whose elements are the elements of the elements of A.

Axiom 4 (Powerset) If A is a set then there exists a set C whose elements are the subsets of A.

Axiom 5 (Regularity or Foundation) If A is a set then either A is empty (i. e. A has no elements) or there exists an element C of A which is disjoint from A.

Axiom 6 (Comprehension) If A is a set and $P(x)$ formalizes a property of sets then there exists a set C whose elements are the elements of A with this property.

Axiom 7 (Axiom of Choice) If A is a set whose elements are pairwise disjoint and each nonempty then there exists a set C which contains exactly one element from each element of A.

Axiom 8 (Replacement) If A is a set and $P(x)$ formalizes a property which defines a function of sets then there exists a set C which contains as elements all the values of this function acting on the elements of A.

Axiom 9 (Infinity) There exists a set W which is nonempty and such that for each element A of W there exists an element B of W such that A is an element of B.

I make some remarks. **Axiom 6** and **Axiom 8** are really infinite lists or schemata corresponding to the possibilities of the *acceptable properties*. These axioms are vague in that it may not be clear what an acceptable property is. Intuitively these properties are those that can be expressed using only the fundamental relationships of equality and set membership and is made mathematically precise through the use of formal mathematical logic.

Axioms 0-8 are (essentially) a reformulation of the axioms of number theory. It is the **Axiom of Infinity** that takes one from number theory to set theory. An exact reformulation of the number theory is given by **Axioms 0-8** together the negation of **Axiom 9**. Mathematical constructions specify objects in the universe of sets, this is the informal point of view I shall adopt. For example by using a property that cannot be true for any set, $x \neq x$, one can easily show using **Axiom 0** and **Axiom 4** that there exists a set with no elements. By **Axiom 1** this set is unique, it is the *emptyset* and is denoted by $\emptyset$.

1.2.2 Finite Set Theory

The formal versions of the axioms on page 3 are the ZFC axioms which is a specific (infinite) theory in the formal first order language for Set Theory–a specific list is given in [12] for the formal language, $\mathcal{L}(\hat{=}, \hat{\in})$, of Set Theory. This theory is too strong for my purposes. The following axioms describe the universe of sets under the assumption that for some finite ordinal α, $V = V_{\alpha+1}$.

Axiom 0 There exists a set.

Axiom 1 (Extensionality) Two sets A and B are equal if and only if they have the same elements.

Axiom 2 (Bounding) There exists a set C such that every set is a subset of C.

Axiom 3 (Union) If A is a set then there exists a set C whose elements are the elements of the elements of A.

Axiom (4a) (Powerset) For all sets A either there exists a set B whose elements are all the subsets of A, or there exists a set C such that every set is a subset of C and such that A is not an element of C.

Axiom (4b) (Powerset) For all sets A, either every set is a subset of A, or there exists a set B such that B is an element of A and such that A does not contain all the subsets of B, or there is a set C whose elements are all the subsets of A.

Axiom 5 (Regularity) If A is a set then either A is empty or there exists an element C of A which is disjoint from A.

Axiom 6 (Comprehension) If A is a set and $P(x)$ formalizes a property of sets then there exists a set C whose elements are the elements of A with this property.

Axiom 7 (Axiom of Finiteness) If A is a nonempty set then there is an element B of A such that for all sets C, if C is an element of A then B is not an element of C.

The two forms of the **Powerset Axiom** are needed to compensate for the lack of the **Pairing Axiom** and the **Bounding Axiom** eliminates the need for the **Axiom of Replacement**. Note that the set specified by the **Bounding Axiom** must be unique (by the **Axiom of Extensionality**). **Axiom 1–Axiom 6** imply that for some ordinal α, $V = V_{\alpha+1}$. By **Axiom 7**, this ordinal is finite and so these axioms actually do imply the assertion, "*For some finite ordinal α, $V = V_{\alpha+1}$*". As a consequence one can show that these axioms also imply the **Axiom of Choice**.

The formal versions of these axioms above define the theory with which I shall be working, it is our base theory and I denote it by ZFC_0.

1.2.3 The formula

I first discuss the standard example of a Gödel sentence modified to our context (in the language $\mathcal{L}(\hat{=}, \hat{\in})$ and relative to the theory ZFC_0). This is the sentence, Ξ_0, which asserts that its negation, $(\neg\Xi_0)$, can be proved from the theory ZFC_0.

By the usual arguments it follows (within our universe of sets) that the theory ZFC_0 does not prove Ξ_0 and ZFC_0 does not prove $(\neg\Xi_0)$; i. e. the sentence Ξ_0 is *independent* of the theory ZFC_0. I give the argument.

For trivial reasons, ZFC_0 cannot prove Ξ_0. This is because

$$(\{\emptyset\}, \in) \vDash \mathrm{ZFC}_0$$

and clearly

$$(\{\emptyset\}, \in) \vDash (\neg\Xi_0)$$

since $\emptyset$ is not a formal proof. Therefore I have only to show that

$$\mathrm{ZFC}_0 \nvDash (\neg\Xi_0).$$

Assume toward a contradiction that

$$\mathrm{ZFC}_0 \vdash (\neg\Xi_0).$$

Then for all sufficiently large finite ordinals, n,

$$(V_n, \in) \vDash \text{“}\mathrm{ZFC}_0 \vdash (\neg\Xi_0)\text{”},$$

and so for all sufficiently large finite ordinals, n, $(V_n, \in) \vDash \Xi_0$. But for all finite ordinals $n > 0$, $(V_n, \in) \vDash \mathrm{ZFC}_0$, and so for all finite ordinals $n > 0$,

$$(V_n, \in) \vDash (\neg\Xi_0)$$

which is a contradiction.

The sentence Ξ_0 is too pathological even for my purposes, a proof of $(\neg\Xi_0)$ cannot belong to a model of ZFC_0 with any extent beyond the proof itself. The sentence I seek is obtained by a simple modification of Ξ_0 which yields the sentence, Ξ.

Informally, the sentence Ξ asserts that there is a proof from ZFC_0 of $(\neg\Xi)$ of length less than 10^{24} and further that V_n exists where $n = |V_{1000}|$. As I have already indicated, the choice of 10^{24} is only for practical reasons. There is no corresponding reason for my particular choice of n, one could quite easily modify the definition by requiring that the choice of n be larger.

The formal specification of Ξ is a completely standard (though tedious) exercise using the modern theory of formal mathematical logic; this involves the formal notion of proof defined so that proofs are finite sequences of natural numbers, etc. [12].

In our universe of sets $(\neg\Xi)$ is true and so there *is* a proof of $(\neg\Xi)$ from ZFC_0. It is not clear just how short such a proof can be. This is a very interesting question. The witness for Armageddon (though with the end of time comfortably distant in the future) is a proof of $(\neg\Xi)$ from ZFC_0 of length less than 10^{24}.

It is important to emphasize that while ZFC_0 is a very weak theory, in attempting to prove $(\neg\Xi)$ from ZFC_0, one is free to augment ZFC_0 with the axiom that V_n exists where $n = |V_{1000}|$. This theory is not weak, particularly as far as the structure of binary sequences of length 10^{24} or even of length $10^{10^{10}}$ is concerned.

The sum total of human experience in mathematics to date (i. e., the number of manuscript pages written to date) is certainly less than 10^{12} pages. The shortest proof from ZFC_0 that no such sequence exists must have length greater than 10^{24}. This is arguably beyond the reach of our current experience but there is an important issue which concerns the *compression* achieved by the informal style in which mathematical arguments are actually written. This is explored a little bit further in [12].

With proper inputs and global determination one could verify with current technology that a given sequence of length at most 10^{24} is a proof of $(\neg\Xi)$ from ZFC_0. But, obviously we do not expect to be able to find a sequence of length less than 10^{24} which is a proof of $(\neg\Xi)$ from ZFC_0. This actually gives a prediction about the physical universe since one can code any candidate for such a sequence by a binary sequence of length at most 10^{26}. The point is that assuming the validity of the quantum view of the world, it is possible to build an actual physical device which must have a nonzero chance of finding such a sequence if such a sequence can exist. The device simply contains (a suitably large number of independent) modules each of which performs an independent series of measurements which in effect *flips a quantum coin*. The point of course is that by quantum law *any outcome is possible*. The prediction is simply that any such device must fail to find a sequence of length less than 10^{24} which is a proof of $(\neg\Xi)$ from ZFC_0. One may object that the belief that any binary sequence of length 10^{26} is *really* a possible outcome of such a device, requires an extraordinary faith in quantum law. But any attempt to build a quantum computer which is useful (for factoring) requires the analogous claim where 10^{26} is replaced by numbers at least as large as 10^5.

This of course requires something like quantum theory. In the universe as described by Newtonian laws, the argument described above does not apply since truly random processes would not exist. One could imagine proving that for a large class of chaotic (but deterministic) processes ("mechanical coin flippers"), no binary sequence of length 10^{24} which actually codes a formal proof, can possibly be generated. In other words, for the non-quantum world, the prediction that no such sequence (as above) can be generated may *not* require that the conception of V_n is meaningful where $n = |V_{1000}|$.

Granting quantum law, and based only on our collective experience in Mathematics to date, how can one account for the prediction (that one *cannot* find a sequence of length less than 10^{24} which is a proof of $(\neg\Xi)$ from ZFC_0) unless one believes that the conception of V_n is meaningful where $n = |V_{1000}|$?

Arguably (given current physical theory) this is *already* a conception of a nonphysical realm.

1.3 Beyond the finite realm

In this section we briefly summarize the basic argument of [15] though here our use of this argument is for a different purpose.

> **Skeptic's Attack:** The mathematical conception of infinity is meaningless and without consequence because the entire conception of the universe of sets is a complete fiction. Further, all the theorems of Set Theory

> are merely finitistic truths, a reflection of the mathematician and not of any genuine mathematical “reality”.

Throughout this section, the “Skeptic” simply refers to the meta-mathematical position which denies any genuine meaning to a conception of uncountable sets. The counter-view is that of the “Set Theorist”.

> **The Set Theorist’s Response:** The development of Set Theory, after Cohen, has led to the realization there is a robust hierarchy of strong axioms of infinity.

Elaborating further, it has been discovered that in many cases, very different lines of investigation have led to problems whose degree of unsolvability is exactly calibrated by a notion of infinity. Thus the hierarchy of large cardinal axioms emerges an intrinsic, fundamental, conception within Set Theory. To illustrate this I discuss an example from modern Set Theory which concerns infinite games.

Suppose $A \subset \mathcal{P}(\mathbb{N})$ where $\mathcal{P}(\mathbb{N})$ denotes the set of all sets $\sigma \subseteq \mathbb{N}$ and $\mathbb{N}$ is the set of all natural numbers; $\mathbb{N} = \{1, 2, \ldots, k, \ldots\}$.

Associated to the set A is an infinite game involving two players, Player I and Player II. The players alternate declaring at stage k whether $k \in \sigma$ or $k \notin \sigma$:

> Stage 1: Player I declares $1 \in \sigma$ or declares $1 \notin \sigma$;
>
> Stage 2: Player II declares $2 \in \sigma$ or declares $2 \notin \sigma$;
>
> Stage 3: Player I declares $3 \in \sigma$ or declares $3 \notin \sigma$; ...

After infinitely many stages a set $\sigma \subseteq \mathbb{N}$ is specified. Player I wins this run of the game if $\sigma \in A$; otherwise Player II wins. (Note: Player I has control of which odd numbers are in σ, and Player II has control of which even numbers are in σ.).

A *strategy* is simply a function which provides moves for the players given just the current state of the game. More formally a strategy is a function

$$\tau : [\mathbb{N}]^{<\omega} \times \mathbb{N} \to \{0, 1\}$$

where $[\mathbb{N}]^{<\omega}$ denotes the set of all finite subsets of $\mathbb{N}$. At each stage k of the game the relevant player can choose to follow τ by declaring “$k \in \sigma$” if

$$\tau(a, k) = 1$$

and declaring “$k \notin \sigma$” if $\tau(a, k) = 0$, where

$$a = \{i < k \mid \text{“}i \in \sigma\text{” was declared at stage } i\text{”}\}.$$

The strategy τ is a *winning strategy* for Player I if by following the strategy at each stage k where it is Player I’s turn to play (i.e., for all odd k), Player I wins the game *no matter how Player II plays*. Similarly τ is a *winning strategy* for Player II if by following the strategy at each stage k where it is Player II's turn to play (i.e., for all even k), Player II wins the game *no matter how Player I plays*.

The game is *determined* if there is a *winning strategy* for one of the players. Clearly it is impossible for there to be winning strategies for *both* players.

It is easy to specify sets $A \subseteq \mathcal{P}(\mathbb{N})$ for which the corresponding game is determined, however, the problem of specifying a set $A \subseteq \mathcal{P}(\mathbb{N})$ for which the corresponding game is *not* determined, turns out to be quite a bit more difficult. The *Axiom of Determinacy*, AD, is the axiom which asserts that for all sets $A \subseteq \mathcal{P}(\mathbb{N})$, the game given by A, as described above, is determined. This axiom was first proposed by Mycielski and Steinhaus, [9], and contradicts the Axiom of Choice, so the problem here is whether the Axiom of Choice is necessary to construct a set

$$A \subseteq \mathcal{P}(\mathbb{N})$$

for which the corresponding game is not determined. Clearly if the Axiom of Choice is necessary then the existence of such set A is quite a subtle fact.

The unsolvability of this problem is exactly calibrated by large cardinal axioms. The relevant large cardinal notion is that of a *Woodin cardinal* which I shall not define, [5]. The ZF axioms are the ZFC axioms but without the Axiom of Choice. The issue of whether the Axiom of Choice is needed to construct a counterexample to AD is exactly the question of whether the theory, ZF + AD, is formally consistent.

Theorem 2. *The two theories,*

(1) ZF + AD

(2) ZFC + *"There exist infinitely many Woodin cardinals"*

are equiconsistent. □

1.3.1 A prediction and a challenge for the Skeptic

Is the theory, ZF + AD, really formally consistent? The claim that it is consistent is a prediction which can be refuted by finite evidence (a formal contradiction). Taking an admittedly extreme position, I claim in [15] the following.

> *It is only through the calibration by a large cardinal axiom in* **conjunction** *with our understanding of the hierarchy of such axioms as* **true axioms about the universe of sets**, *that this prediction; the formal theory* ZF+AD *is consistent, is justified.*

As a consequence of my belief in this claim, I also made a prediction:

> *In the next 10,000 years there will be no discovery of an inconsistency in this theory.*

This is a specific and unambiguous prediction about the *physical universe* just as is the case for the analogous prediction in the previous section. Further it is a prediction which does *not* arise by a reduction to a previously held truth (as for example is the case for the prediction that no counterexample to Fermat's Last Theorem will be discovered). This is a genuinely new prediction which I make in [15] based on the development of Set Theory over the last 50 years and on my belief that the conception of the transfinite universe of sets is meaningful. I make this prediction independently of all speculation of what computational devices might be developed in the next 10,000 years (or whatever new sources of knowledge might be discovered) which increase the effectiveness of research in Mathematics.

Now the Skeptic might object that this prediction is not interesting or natural because the formal theories are not interesting or natural. But such objections are not allowed in Physics, the ultimate physical theory should explain *all* (physical) aspects of the physical universe, not just those which we regard as natural. How can we apply a lessor standard for the ultimate mathematical theory? Of course, I also predict:

> *There will be no discovery* ***ever*** *of an inconsistency in this theory*;

and this prediction, if true, is arguably a physical law.

> **Skeptic's Retreat:** OK, I accept the challenge noting that I only have to explain the predictions of formal consistency given by the large cardinal axioms. The formal theory of Set Theory as given by the axioms, ZFC, is so "incomplete" that: *Any large cardinal axiom, in the natural formulation of such axioms, is either consistent with the axioms of Set Theory, or there is an elementary proof that the axiom cannot hold.*

To examine the Skeptic's Retreat and to assess how this too might be refuted I need to briefly survey the basic template for large cardinal axioms in Set Theory.

1.3.2 Large cardinal axioms within Set Theory

A set N is *transitive* if every element of N is a subset of N. Transitive sets are fragments of V which are analogous to initial segments. For each ordinal α the set V_α is a transitive set.

The simplest (proper) class is the class of all ordinals. This class is a transitive class where a class $M \subseteq V$ is defined to be a transitive class if every element of M is a subset of M. The basic template for large cardinal axioms is as follows.

> *There is a transitive class M and an elementary embedding*
>
> $$j : V \to M$$
>
> *which is not the identity.*

With the exception of the definition of a *Reinhardt cardinal* which I shall come to below, one can always assume that the classes, M and j, are classes which are logically definable from parameters by formulas of a fixed bounded level of complexity (Σ_2-formulas). Moreover the assertion that j is an elementary embedding—that is the assertion:

- For all formulas $\phi(x)$ and for all sets a,

  $$V \vDash \phi[a]$$

  if and only if $M \vDash \phi[j(a)]$;

—is equivalent to the assertion:

- For all formulas $\phi(x)$, for all ordinals α, and for all sets $a \in V_\alpha$,

  $$V_\alpha \vDash \phi[a]$$

  if and only if $j(V_\alpha) \vDash \phi[j(a)]$.

Therefore this template makes no essential use of the notion of a class. It is simply for convenience that I refer to classes (and this is the usual practice in Set Theory).

Suppose that M is a transitive class and that

$$j : V \to M$$

is an elementary embedding which not the identity. Suppose that $j(\alpha) = \alpha$ for all ordinals α. Then one can show by transfinite induction that for all ordinals α, the embedding, j, is the identity on V_α. Therefore since j is not the identity, there must exist an ordinal α such that $j(\alpha) \neq \alpha$. The least such ordinal is the *critical point* of j. This must be a cardinal. The *critical point* of j is the large cardinal and the existence of the transitive class M and the elementary embedding j are the witnesses for this.

A cardinal κ is a *measurable cardinal* if there exists a transitive class M and an elementary embedding,

$$j : V \to M$$

such that κ is the critical point of j.

It is by requiring M to be *closer* to V that one can define large cardinal axioms far beyond the axiom, "There is a measurable cardinal". In general the closer one requires M to be to V, the stronger the large cardinal axiom. The natural maximum axiom was proposed ($M = V$) by Reinhardt in his Ph.D thesis, see [10]. The associated large cardinal axiom is that of a *Reinhardt cardinal*.

Definition 3. A cardinal κ is a *Reinhardt cardinal* if there is an elementary embedding,

$$j : V \to V$$

such that κ is the critical point of j. □

The definition of a Reinhardt cardinal makes essential use of classes, but the following variation does not and this variation (which is not a standard notion) is only formulated in order to facilitate this discussion. The definition requires a logical notion. Suppose that α and β are ordinals such that $\alpha < \beta$. Then

$$V_\alpha \prec V_\beta$$

if for all formulas, $\phi(x)$, for all $a \in V_\alpha$,

$$V_\alpha \vDash \phi[a]$$

if and only if $V_\beta \vDash \phi[a]$. Thus $V_\alpha \prec V_\beta$ if and only if

$$I : V_\alpha \to V_\beta$$

is an elementary embedding where I is the identity map.

Definition 4. A cardinal κ is a *weak Reinhardt cardinal* if there exist $\gamma > \lambda > \kappa$ such that

(1) $V_\kappa \prec V_\lambda \prec V_\gamma$,

(2) there exists an elementary embedding,

$$j : V_{\lambda+2} \to V_{\lambda+2}$$

such that κ is the critical point of j. □

The definition of a weak Reinhardt cardinal only involves sets. The relationship between Reinhardt cardinals and weak Reinhardt cardinals is unclear but one would naturally conjecture that at least in terms of consistency strength, Reinhardt cardinals are stronger than weak Reinhardt cardinals and hence my choice in terminology. The following theorem is an immediate corollary of the fundamental inconsistency results of Kunen, [6].

Theorem 5 (Kunen). *There are no weak Reinhardt cardinals.* □

The proof is elementary so this does not refute the Skeptic's Retreat. But Kunen's proof makes essential use of the Axiom of Choice. The problem is open without this assumption. Further there is really no known interesting example of a strengthening of the definition of a weak Reinhardt cardinal that yields a large cardinal axiom which can be refuted without using the Axiom of Choice. The difficulty is that without the Axiom of Choice it is extraordinarily difficult to prove anything about sets.

Kunen's proof leaves open the possibility that the following large cardinal axiom might be consistent with the Axiom of Choice. This therefore is essentially the strongest large cardinal axiom not known to be refuted by the Axiom of Choice, see [5] more on this as well as for the actual statement of Kunen's theorem.

Definition 6. A cardinal κ is a *strongly* $(\omega + 1)$*-huge cardinal* if there exist $\gamma > \lambda > \kappa$ such that

(1) $V_\kappa \prec V_\lambda \prec V_\gamma$,

(2) there exists an elementary embedding,

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

such that κ is the critical point of j. □

The issue of whether the existence of a weak Reinhardt cardinal is consistent with the axioms, ZF, is an important issue for the Set Theorist because by the results of [16], the theory

ZF + "There is a weak Reinhardt cardinal"

proves the formal consistency of the theory

ZFC + "There is a proper class of strongly $(\omega + 1)$-huge cardinals".

This number theoretic statement is a theorem of Number Theory. But as indicated above, the notion of a strongly $(\omega + 1)$-huge cardinal is essentially the strongest large cardinal notion which is not known to be refuted by the Axiom of Choice.

Therefore the number theoretic assertion that the theory

ZF + "There is a weak Reinhardt cardinal"

is consistent is a *stronger* assertion than the number theoretic assertion that the theory

ZFC + "There is a proper class of strongly $(\omega + 1)$-huge cardinals"

is consistent. More precisely, the former assertion implies, *but is not implied by*, the latter assertion; unless of course the theory

ZFC + "There is a proper class of strongly $(\omega + 1)$-huge cardinals"

is formally inconsistent. This raises an interesting question.

> *How could the Set Theorist ever be able to argue for the prediction that the existence of weak Reinhardt cardinals is consistent with axioms of Set Theory without the Axiom of Choice?*

Moreover this *one* prediction implies *all* the predictions (of formal consistency) the Set Theorist can currently make based on the *entire* large cardinal hierarchy as presently conceived (in the context of a universe of sets which satisfies the Axiom of Choice). My point is that by appealing to the Skeptic's Retreat, one could reasonably claim that the theory

ZF + "There is a weak Reinhardt cardinal",

is formally consistent and in making this *single* claim one would subsume *all* the claims of consistency that the Set Theorist can make based on our current understanding of the universe of sets (without abandoning the Axiom of Choice).

Before presenting a potential option to deal with this, I describe an analogous option of how the Set Theorist *can* claim that the theory

$$\mathrm{ZF} + \mathrm{AD}$$

is consistent even though as I have indicated AD also refutes the Axiom of Choice. The explanation requires some definitions which I shall require anyway. Gödel defined a very special transitive class $L \subseteq V$ and showed that all the axioms of ZFC hold when interpreted in L. The definition of L does not require the Axiom of Choice and so one obtains the seminal result that if the axioms, ZF, are consistent then so are the axioms ZFC. Gödel also proved that the Continuum Hypothesis holds in L thereby showing that one cannot formally refute the Continum Hypothesis from the axioms ZFC (unless of course these axioms are inconsistent).

The definition of L is simply given by replacing the operation $\mathcal{P}(X)$ in the definition of $V_{\alpha+1}$ by the operation $\mathcal{P}_{\mathrm{Def}}(X)$ which associates to the set X the set of all subsets $Y \subseteq X$ such that Y is logically definable in the structure, $(X, \in)$, from parameters in X. For any infinite set X, $\mathcal{P}_{\mathrm{Def}}(X) \subset \mathcal{P}(X)$ and $\mathcal{P}_{\mathrm{Def}}(X) \neq \mathcal{P}(X)$.

Thus one defines L_α by induction on the ordinal α; setting $L_0 = \emptyset$, setting

$$L_{\alpha+1} = \mathcal{P}_{\mathrm{Def}}(L_\alpha),$$

and taking unions at limit stages. The class L is defined as the class of all sets a such that $a \in L_\alpha$ for some ordinal α. It is perhaps important to note that while there must exist a proper class of ordinals α such that

$$L_\alpha = L \cap V_\alpha,$$

this is not true for all ordinals α.

Relativizing the definition of L to $V_{\omega+1}$ we obtains the class $L(V_{\omega+1})$ which is more customarily denoted by $L(\mathbb{R})$; here one defines,

$$L_0(\mathbb{R}) = V_{\omega+1}$$

and proceeds by induction exactly as above to define $L_\alpha(\mathbb{R})$ for all ordinals α. The class $L(\mathbb{R})$ is the class of all sets a such that $a \in L_\alpha(\mathbb{R})$ for some ordinal α.

Unlike the case for L, one cannot prove that the Axiom of Choice holds in $L(\mathbb{R})$, though one can show that all of the other axioms of ZFC hold in $L(\mathbb{R})$. The following theorem which is related to Theorem 2, not only establishes the consistency of ZF+AD from simply the existence of large cardinals, it also establishes that $L(\mathbb{R}) \models \mathrm{AD}$ (as a new truth about sets)–see [5] for more on the history of this theorem and the attempts to establish that $L(\mathbb{R}) \models \mathrm{AD}$ from large cardinal axioms.

Theorem 7 (Martin, Steel, Woodin). *Suppose there is a proper class of Woodin cardinals. Then*

$$L(\mathbb{R}) \models \mathrm{AD}. \qquad \square$$

For the reasons I have indicated one cannot hope to argue for the consistency of the theory,

$$\text{ZF} + \text{“There is a weak Reinhardt cardinal”},$$

on the basis of *any* large cardinal axiom not known to refute the Axiom of Choice. The experience with the theory, ZF + AD, suggests that as an alternative, one should seek both a generalization of $L(\mathbb{R})$ and some structural principles for this fragment such that the axiom that asserts both the existence of this fragment and that the structural principles hold in this fragment, implies the formal consistency of the axiom which asserts the existence of a weak Reinhardt cardinal, or even better that implies that the latter axiom actually holds in this fragment.

In fact there *are* compelling candidates for generalizations of $L(\mathbb{R})$ and axioms for these fragments generalizing AD. But at present there is simply no plausible candidate for such a generalization of $L(\mathbb{R})$ in which the axiom that there is a weak Reinhardt cardinal can even hold; nor is there a plausible candidate for a fragment together with structural principles for that fragment which would imply the formal consistency of the existence of a weak Reinhardt cardinal. This is explored more fully in [16].

There is another potential option which is suggested by a remarkable theorem of Vopenka. But to explain this further I must give another definition which I shall also require in the subsequent discussion. This is the definition of the class, HOD, which originates in remarks of Gödel at the Princeton Bicentennial Conference in December, 1946. The first detailed reference appears to be [7] (see the review of [7] by G. Kreisel).

Definition 8 (ZF). (1) For each ordinal α, HOD_α is the set of all sets a such that there exists a transitive set $M \subset V_\alpha$ such that $a \in M$ and such for all $b \in M$, b is definable in V_α from ordinal parameters.

(2) HOD is the class of all sets a such that $a \in \text{HOD}_\alpha$ for some α. □

I caution that just as is the case for L_α, in general

$$\text{HOD}_\alpha \neq \text{HOD} \cap V_\alpha,$$

though for a proper class of ordinals α it is true that $\text{HOD}_\alpha = \text{HOD} \cap V_\alpha$.

The class HOD is quite interesting for a number of reasons, one of which is illustrated by the following observation of Gödel which as indicated is stated within just the theory ZF, in other words without assuming the Axiom of Choice.

Theorem 9 (ZF). $\text{HOD} \vDash \text{ZFC}$. □

This theorem gives a completely different approach to showing that if the theory, ZF, is formally consistent then so is the theory, ZFC.

One difficulty with HOD is that the definition of HOD is not absolute, for example in general HOD is not even the same as defined within HOD. As a consequence almost any set theoretic question one might naturally ask about HOD, is formally unsolvable. Two immediate such questions are whether V = HOD and more simply, whether HOD contains all the real numbers. Both of these questions are formally unsolvable but are of evident importance because they specifically address the complexity of the Axiom of Choice. If V = HOD then there is no mystery as to why the Axiom of Choice holds but of course one is left with the problem of explaining why V = HOD.

I end this section with the remarkable theorem of Vopenka alluded to above. The statement involves Cohen's method of *forcing* adapted to produce extensions in which the Axiom of Choice can fail, these are called *symmetric generic extensions*. For each ordinal α, there is a minimum extension of the class HOD which contains both HOD and V_α and in which the axioms ZF hold. This minimum extension is denoted by HOD(V_α).

Theorem 10 (ZF; Vopenka). *For all ordinals α,* HOD(V_α) *is a symmetric generic extension of* HOD. □

The alternative conception of truth for Set Theory which is suggested by this theorem and which could provide a basis for the claim that weak Reinhardt cardinals are consistent is the subject of the next section.

1.4 The generic-multiverse of sets

The challenge presented in the previous section—the challenge to account for the prediction that the existence of a weak Reinhardt cardinal is formally consistent with ZF axioms—suggests that one should consider a multiverse conception of the universe of sets. The point of course is that while the existence of a weak Reinhardt cardinal is not possible, granting the Axiom of Choice, this does not rule out that there may be (symmetric) generic extensions of V in which there are weak Reinhardt cardinals. Such a multiverse approach to the conception of the universe of sets would also mitigate the difficulties associated to the formal unsolvability of fundamental problems such as that of the Continum Hypothesis, and this latter feature is the primary motivation for such an approach. This section is based on [13].

Let the *multiverse* (of sets) refer to the collection of possible universes of sets. The truths of the Set Theory according to the multiverse conception of truth are the sentences which hold in each universe of the multiverse. Cohen's method of *forcing* which is the fundamental technique for constructing non-trivial extensions of a given (countable) model of ZFC suggests a natural candidate for a multiverse; the *generic-multiverse* is generated from each universe of the collection by closing under generic extensions (enlargements) and under generic refinements (inner models of a universe which the given universe is a generic extension of). To illustrate the concept of the generic-multiverse, suppose that M is a countable transitive set with the property that

$$M \vDash \text{ZFC}.$$

Let $\mathbb{V}_M$ be the smallest set of countable transitive sets such that $M \in \mathbb{V}_M$ and such that for all pairs, (M_1, M_2), of countable transitive sets such that

$$M_1 \vDash \text{ZFC},$$

and such that M_2 is a generic extension of M_1, if either $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$ then both M_1 and M_2 are in $\mathbb{V}_M$. It is easily verified that for each $N \in \mathbb{V}_M$,

$$\mathbb{V}_N = \mathbb{V}_M,$$

where $\mathbb{V}_N$ is defined using N in place of M. $\mathbb{V}_M$ is the generic-multiverse generated in V from M.

The *generic-multiverse conception of truth* is the position that a sentence is true if and only if it holds in each universe of the generic-multiverse generated by V. This can be formalized within V in the sense that for each sentence ϕ there is a sentence ϕ^*, recursively depending on ϕ, such that ϕ is true in each universe of the generic-multiverse generated by V if and only if ϕ^* is true in V. The sentence ϕ^* is explicit given ϕ and does not depend on V. For exmple, given *any* countable transitive set, M, such that $M \vDash \text{ZFC}$,

$$M \vDash \phi^*$$

if and only if $N \vDash \phi$ for all $N \in \mathbb{V}_M$ (the proof is given in [14]). This is an important point in favor of the generic-multiverse position since it shows that as far as assessing truth is concerned, the generic-multiverse position is not that sensitive to the meta-universe in which the generic-multiverse is being defined.

Is the generic-multiverse position a reasonable one? The refinements of Cohen's method of *forcing* in the decades since his initial discovery of the method and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position. This has been reinforced by some rather unexpected consequences of large cardinal axioms which I shall discuss later in this section.

The purpose of this section is *not* to argue against *any* possible multiverse position but to more carefully examine the generic-multiverse position within the context of modern Set Theory. In brief I shall argue that modulo the Ω Conjecture (which I shall define in the next section), the generic-multiverse position outlined above is not plausible. The essence of the argument against the generic-multiverse position is that assuming the Ω Conjecture is true (and that there is a proper class of Woodin cardinals) then this position is simply a brand of formalism that denies the transfinite by a reducing truth about the universe of sets to truth about a simple fragment such as the integers or, in this case, the sets of real numbers. The Ω Conjecture is invariant between V and any generic extension of V and so the generic-multiverse position must either declare the Ω Conjecture to be true or declare the Ω Conjecture to be false.

It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However I am claiming an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, the set theorist rejects this possibility. I claim that by the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. Taking into account the iterative conception of sets, the set of all truths of an explicit fragment of the universe of sets cannot be reduced to the truths of an explicit *simpler* fragment. The latter is the basic position on which I shall base my arguments.

An assertion is Π_2 if it is of the form,

$$\text{“ For every infinite ordinal } \alpha\text{ , } V_\alpha \vDash \phi \text{ ”,}$$

for some sentence, ϕ. A Π_2 assertion is a *multiverse truth* if the Π_2 assertion holds in each universe of the multiverse. A key point:

Remark 11. Arguably, the generic-multiverse view of truth is only viable for Π_2-sentences and not in general even for Σ_2-sentences (these are sentences expressible as the negation of a Π_2-sentence). This is because of the restriction to *set forcing* in the definition of the generic-multiverse. Therefore one can quite reasonably question whether the generic-multiverse view can possibly account for the predictions of consistency given by large cardinal axioms. At present there is no reasonable candidate for the definition of an expanded version of the generic-multiverse which allows *class forcing* extensions and yet which preserves the existence of large cardinals across the multiverse. □

In the context where there is a Woodin cardinal, let us use “δ_0” to denote the least Woodin cardinal. So I am just fixing a notation just as “ω_1” is fixed as the notation for the least uncountable ordinal. Both ω_1 and δ_0 can change in passing from one universe of sets to an extension of that universe.

The assertion,

$$\text{“}\delta \text{ is a Woodin cardinal”}$$

is equivalent to the assertion,

$$V_{\delta+1} \vDash \text{“}\delta \text{ is a Woodin cardinal”}$$

and so $\delta = \delta_0$ if and only if

$$V_{\delta+1} \vDash \text{“}\delta = \delta_0\text{”.}$$

Therefore assuming there is a Woodin cardinal, for each sentence ϕ, it is a Π_2 assertion to say that

$$V_{\delta_0+1} \vDash \phi$$

and it is a Π_2 assertion to say that $V_{\delta_0+1} \nvDash \phi$. Thus in any one universe of the multiverse, the set of all sentences ϕ such that $V_{\delta_0+1} \vDash \phi$—that is, the *theory* of V_{δ_0+1} as computed in that in that universe—is recursive in the set of Π_2 sentences (assertions) which hold in that universe. Further by Tarski's Theorem on the undefinability of truth the latter set cannot be recursive in the former set.

These comments suggest the following multiverse laws which I state in reference to an arbitrary multiverse position (though assuming that the existence of a Woodin cardinal holds throughout the multiverse).

First Multiverse Law

The set of Π_2 *assertions which are multiverse truths is not recursive in the set of multiverse truths of* V_{δ_0+1}. □

The motivation for this multiverse law is that if the set of Π_2 multiverse truths is recursive in the set of multiverse truths of V_{δ_0+1} then as far as evaluating Π_2 assertions is concerned, the multiverse is equivalent to the reduced multiverse of just the fragments V_{δ_0+1} of the universes of the multiverse. This amounts to a rejection of the transfinite beyond V_{δ_0+1} and constitutes in effect the unacceptable brand of formalism alluded to earlier. This claim would be reinforced should the multiverse position also violate a second multiverse law which I now formulate.

A set $Y \subset V_\omega$ is definable in V_{δ_0+1} across the multiverse if the set Y is definable in the structure V_{δ_0+1} of each universe of the multiverse (possibly by formulas which depend on the parent universe). The second multiverse law is a variation of the First Multiverse Law.

Second Multiverse Law

The set of Π_2 *assertions which are multiverse truths, is not definable in* V_{δ_0+1} *across the multiverse.* □

Again, by Tarski's Theorem on the undefinability of truth, this multiverse law is obviously a reasonable one *if* one regards the only possibility for the multiverse to be the universe of sets so that set of multiverse truths of V_{δ_0+1} is simply the set of all sentences which are true in V_{δ_0+1} and the set of Π_2 assertions which are multiverse truths is simply the set of Π_2 assertions which are true in V. Likewise the Second Multiverse Law would have to hold if one modified the law to simply require that the set of Π_2 assertions which are multiverse truths, is not uniformly definable in V_{δ_0+1} across the multiverse (i.e., by a single formula).

Assuming both that Ω Conjecture and the existence of a proper class of Woodin cardinals hold in each (or one) universe of the generic-multiverse generated by V, then *both* the First Multiverse Law and the Second Multiverse Law are violated by the generic-multiverse position. This is the basis for the argument I am giving against the generic-multiverse position in this paper. In fact the technical details of how the generic-multiverse position violates these multiverse laws provides an even more compelling argument against the generic-multiverse position since the analysis shows that in addition the generic-multiverse position is truly a form of formalism because of the connections to Ω-logic. The argument also shows that the violation of the First Multiverse Law is explicit; i.e. assuming the Ω Conjecture, there is an explicit recursive reduction of the set of Π_2 assertions which are generic-multiverse truths is to the set of generic-multiverse truths of V_{δ_0+1}.

There is a special case which I can present without any additional definitions and which is not contingent on any conjectures.

Theorem 12. *Suppose that M is a countable transitive set*

$$M \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$$

and that $M \cap \text{Ord}$ is as small as possible. Then $\mathbb{V}_M$ violates both multiverse laws. □

1.4.1 Ω-logic

The generic-multiverse conception of truth declares the Continuum Hypothesis to be neither true nor false and declares, granting large cardinals, that assertion,

$$L(\mathbb{R}) \vDash \text{AD},$$

to be true (see Theorem 7). I note that for essentially all current large cardinal axioms, the existence of a proper class of large cardinals holds in V if and only if it holds in $V^{\mathbb{B}}$ for all complete Boolean algebras, $\mathbb{B}$. In other words, in the generic-multiverse position the existence of a proper class of, say, Woodin cardinals is either true or false since it either holds in every universe of the generic-multiverse or it holds in no universe of the generic-multiverse, [4].

I am going to analyze the generic-multiverse position from the perspective of Ω-logic which I first briefly review. I will use the standard modern notation for Cohen's method of forcing; potential extensions of the universe, V, are given by complete Boolean algebras $\mathbb{B}$, $V^{\mathbb{B}}$ denotes the corresponding boolean valued extension and for each ordinal α, $V_\alpha^{\mathbb{B}}$ denotes V_α as defined in that extension.

Definition 13. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then

$$T \vDash_\Omega \phi$$

if for all complete Boolean algebras, $\mathbb{B}$, for all ordinals, α, if $V_\alpha^{\mathbb{B}} \vDash T$ then $V_\alpha^{\mathbb{B}} \vDash \phi$. □

If there is a proper class of Woodin cardinals then the relation $T \models_\Omega \phi$, is generically absolute. This fact which arguably was a completely unanticipated consequence of large cardinals, makes Ω-logic interesting from a meta-mathematical point of view. For example the set

$$\mathcal{V}_\Omega = \{\phi \mid \emptyset \models_\Omega \phi\}$$

is generically absolute in the sense that for a given sentence, ϕ, the question whether or not ϕ is logically Ω-valid; i.e., whether or not $\phi \in \mathcal{V}_\Omega$, is absolute between V and all of its generic extensions. In particular the method of forcing *cannot* be used to show the formal independence of assertions of the form $\emptyset \models_\Omega \phi$.

Theorem 14. *Suppose that T is a countable theory in the language of Set Theory, ϕ is a sentence and that there exists a proper class of Woodin cardinals. Then for all complete Boolean algebras, $\mathbb{B}$, $V^{\mathbb{B}} \models$ "$T \models_\Omega \phi$" if and only if $T \models_\Omega \phi$.* □

There are a variety of technical theorems which show that one cannot hope to prove the generic invariance of Ω-logic from any large cardinal hypothesis weaker than the existence of a proper class of Woodin cardinals—for example if $V = L$ then definition of $\mathcal{V}_\Omega$ is not absolute between V and $V^{\mathbb{B}}$, for *any* non-atomic complete Boolean algebra, $\mathbb{B}$.

It follows easily from the definition of Ω-logic, that for any Π_2-sentence, ϕ,

$$\emptyset \models_\Omega \phi$$

if and only if for all complete Boolean algebras, $\mathbb{B}$,

$$V^{\mathbb{B}} \models \phi.$$

Therefore by the theorem above, assuming there is a proper class of Woodin cardinals, for each sentence, ψ, the assertion

For all complete Boolean algebras, $\mathbb{B}$, $V^{\mathbb{B}} \models$ "$V_{\delta_0+1} \models \psi$"

is itself absolute between V and $V^{\mathbb{B}}$ for all complete Boolean algebras $\mathbb{B}$. This remarkable consequence of the existence of a proper class of Woodin cardinals actually seems to be evidence for the generic-multiverse position. In particular this shows that the generic-multiverse position, at least for assessing Π_2 assertions, and so for assessing all assertions of the form,

$$V_{\delta_0+1} \models \phi,$$

is equivalent to the position that a Π_2 assertion is true if and only if it holds in $V^{\mathbb{B}}$ for all complete Boolean algebras $\mathbb{B}$. Notice that if $\mathbb{R} \not\subset L$ and if V is a generic extension of L then this equivalence is *false*. In this situation the Π_2 sentence which expresses $\mathbb{R} \not\subset L$ holds in $V^{\mathbb{B}}$ for all complete Boolean algebras, $\mathbb{B}$, but this sentence fails to hold across the generic-multiverse generated by V (since L belongs to this multiverse).

To summarize, suppose that there exists a proper class of Woodin cardinals in each universe of the generic-multiverse (or equivalently that there is a proper class of Woodin cardinals in at least one universe of the generic-multiverse). Then for each Π_2 sentence ϕ; the following are equivalent:

(1) ϕ is holds across the generic-multiverse;

(2) "$\emptyset \models_\Omega \phi$" holds across the generic-multiverse;

(3) "$\emptyset \models_\Omega \phi$" holds in at least one universe of the generic-multiverse.

For any Σ_2-sentence ϕ, the assertion, "$\emptyset \not\models_\Omega (\neg\phi)$", is by the definitions equivalent to the assertion that for some complete Boolean algebra, $\mathbb{B}$, $V^{\mathbb{B}} \models \phi$. Therefore assuming that there exists a proper class of Woodin cardinals, for any Σ_2-sentence, if in one universe of the generic-multiverse generated by V, the sentence ϕ is true, then in *every* universe of the generic-multiverse generated by V, the sentence ϕ can be forced to be true by passing to a generic extension of that universe. This same remarkable fact applies to symmetric forcing extensions as well.

Therefore assuming there is a proper class of Woodin cardinals, it seems that the generic-multiverse view of truth can account for the prediction that weak Reinhardt cardinals are consistent with ZF; the relevant Σ_2-sentence is the sentence which asserts that there exists a complete Boolean algebra $\mathbb{B}$ and there exists a term $\tau \in V^{\mathbb{B}}$ for a transitive set such that (with Boolean value 1),

$$V(\tau) \models \text{ZF} + \text{“\textit{There is a weak Reinhardt cardinal}”}.$$

This sentence if true in one universe of the generic-multiverse generated by V must be true in *every* universe of the generic-multiverse generated by V. Further by Theorem 10, if there is a weak Reinhardt cardinal then in the generic-multiverse generated by HOD this Σ_2-sentence is actually declared as true.

But on close inspection one realizes this is not really a justification at all. The sentence above is *meaningful* in the generic-multiverse view of truth but there is no explanation of why it is true. This is exactly as is the case for the sentence which asserts that the formal theory,

$$\text{ZF} + \text{“\textit{There is a weak Reinhardt cardinal.}”},$$

is consistent.

To more fully evaluate the generic-multiverse position one must understand the logical relation, $T \models_\Omega \phi$. In particular a natural question arises: is there a corresponding proof relation?

1.4.2 The Ω Conjecture

I define the proof relation, $T \vdash_\Omega \phi$. This requires a preliminary notion that a set of reals be *universally Baire*, [1]. In fact I shall define $T \vdash_\Omega \phi$, assuming the existence of a proper class of Woodin cardinals and exploiting the fact that there are a number of (equivalent) definitions. Without the assumption that there is a proper class of Woodin cardinals, the definition is a bit more technical, [13]. Recall that if S is a compact Hausdorff space then a set $X \subseteq S$ has the *property of Baire* in the space S if there exists an open set $O \subseteq S$ such that symmetric difference,

$$X \triangle O,$$

is meager in S (contained in a countable union of closed sets with empty interior).

Definition 15. A set $A \subset \mathbb{R}$ is *universally Baire* if for all compact Hausdorff spaces, S, and for all continuous functions,

$$F : S \to \mathbb{R},$$

the preimage of A by F has the property of Baire in the space S. □

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Suppose that M is a countable transitive model of ZFC. Then M is *strongly A-closed* if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

Definition 16. Suppose there is a proper class of Woodin cardinals. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then $T \vdash_\Omega \phi$ if there exists a set $A \subset \mathbb{R}$ such that:

(1) A is universally Baire,

(2) for all countable transitive models, M, if M is strongly A-closed and $T \in M$, then

$$M \vDash \text{“}T \vDash_\Omega \phi\text{”}.$$ □

Assuming there is a proper class of Woodin cardinals, the relation, $T \vdash_\Omega \phi$, is generically absolute. Moreover *Soundness* holds as well.

Theorem 17. *Assume there is a proper class of Woodin cardinals. Then for all* (T, ϕ) *and for all complete Boolean algebras,* $\mathbb{B}$*,* $T \vdash_\Omega \phi$ *if and only if* $V^{\mathbb{B}} \vDash$ “$T \vdash_\Omega \phi$”. □

Theorem 18 (Soundness). *Assume that there is a proper class of Woodin cardinals and that* $T \vdash_\Omega \phi$*. Then* $T \vDash_\Omega \phi$. □

I now come to the Ω Conjecture which in essence is simply the conjecture that the Gödel Completeness Theorem holds for Ω-logic; see [13] for a more detailed discussion.

Definition 19 (**Ω Conjecture**). Suppose that there exists a proper class of Woodin cardinals. Then for all sentences ϕ, $\emptyset \vDash_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$. □

Assuming the Ω Conjecture one can analyze the generic-multiverse view of truth by computing the logical complexity of Ω-logic. The key issue of course is whether the generic-multiverse view of truth satisfies the two multiverse laws. This is the subject of the next section. We end this section with a curious connection between the Ω Conjecture, HOD, and the universally Baire sets. This requires a definition.

Definition 20. A set $A \subseteq \mathbb{R}$ is OD if there exists an ordinal α and a formula ϕ such that

$$A = \{x \in \mathbb{R} \mid V_\alpha \vDash \phi[x]\}. \qquad \square$$

Theorem 21. *Suppose that there is a proper class of Woodin cardinals and that for every set $A \subseteq \mathbb{R}$, if A is* OD *then A is universally Baire. Then*

$$\text{HOD} \vDash \text{“}\Omega \text{ Conjecture”} \qquad \square$$

1.4.3 The complexity of Ω-logic

Let (as defined on page 21) $\mathcal{V}_\Omega$ be the set of sentences ϕ such that

$$\emptyset \vDash_\Omega \phi,$$

and (assuming there is a proper class of Woodin cardinals) let $\mathcal{V}_\Omega(V_{\delta_0+1})$ be the set of sentences, ϕ, such that

$$\text{ZFC} \vDash_\Omega \text{“}V_{\delta_0+1} \vDash \phi\text{”}.$$

Assuming there is a proper class of Woodin cardinals then the set of generic-multiverse truths which are Π_2 assertions is of the same Turing complexity as $\mathcal{V}_\Omega$ (i.e., each set is recursive in the other). Further (assuming there is a proper class of Woodin cardinals) the set, $\mathcal{V}_\Omega(V_{\delta_0+1})$, is precisely the set of generic-multiverse truths of V_{δ_0+1}. Thus the requirement that the generic-multiverse position satisfies the First Multiverse Law, as discussed on page 19, reduces to the requirement that $\mathcal{V}_\Omega$ not be recursive in the set $\mathcal{V}_\Omega(V_{\delta_0+1})$.

The following theorem is a corollary of the basic analysis of Ω-logic in the context that there is a proper class of Woodin cardinals.

Theorem 22. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the set $\mathcal{V}_\Omega$ is recursive in the set $\mathcal{V}_\Omega(V_{\delta_0+1})$.* $\square$

Therefore, assuming the existence of a proper class of Woodin cardinals and that the Ω Conjecture both hold across the generic-multiverse generated by V, the generic-multiverse position violates the First Multiverse Law. What about the Second Multiverse Law (on page 19)? This requires understanding the complexity of the set $\mathcal{V}_\Omega$. From the definition of $\mathcal{V}_\Omega$ it is evident that this set is definable in V by a Π_2 formula: if $V = L$ then this set is recursively equivalent to the set of all Π_2 sentences which are true in V. However in the context of large cardinal axioms the complexity of $\mathcal{V}_\Omega$ is more subtle.

Theorem 23. *Assume there is a proper class of Woodin cardinals and that the* Ω *Conjecture holds. Then the set* $\mathcal{V}_\Omega$ *is definable in* V_{δ_0+1}. □

Therefore if the Ω Conjecture holds and there is a proper class of Woodin cardinals then the generic-multiverse position that the only Π_2 assertions which are true are those which are true in each universe of the generic-multiverse also violates the Second Multiverse Law—for this set of assertions is itself definable in V_{δ_0+1} across the generic-multiverse. I make one final comment here. The *weak multiverse laws* are the versions of the two multiverse laws I have defined where V_{δ_0+1} is replaced by $V_{\omega+2}$. Assuming the Ω Conjecture and that there is a proper class of Woodin cardinals then the generic-multiverse position actually violates the Weak First Multiverse Law, and augmented by a second conjecture, it also violates the Weak Second Multiverse Law, more details can be found in [14]. The example of Theorem 12 violates both weak multiverse laws.

1.5 The infinite realm

If the Ω Conjecture is true then what is the only plausible alternative to the conception of the universe of sets is arguably ruled out. The point is that any multiverse conception based on a (reasonable) multiverse not smaller than the generic-multiverse also violates the two multiverse laws. Here a multiverse, $\mathbb{V}^*_M$, associated to a countable transitive model M is *not smaller* than the generic-multiverse if for each $N \in \mathbb{V}^*_M$, $\mathbb{V}_N \subseteq \mathbb{V}^*_M$. Similarly, $\mathbb{V}^*_M$ is *smaller* than the generic multiverse (generated by M) if $\mathbb{V}^*_M \subseteq \mathbb{V}_M$. The following theorem gives a more precise version of this claim.

Theorem 24. *Suppose that* M *is a countable transitive set*

$$M \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals.”},$$

and that

$$M \vDash \text{“The } \Omega \text{ Conjecture.”}$$

Suppose $\mathbb{V}^*_M$ *is a multiverse generated by* M *which is not smaller than the generic-multiverse and which contains only transitive sets* N *such that*

$$N \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals.”},$$

and such that

$$N \vDash \text{“The } \Omega \text{ Conjecture.”}$$

Then the multiverse view of truth given by $\mathbb{V}^*_M$ *violates both multiverse laws.* □

Finally, there is no compelling candidate for a multiverse view of truth based on a multiverse generated by V which is smaller than the generic-multiverse; other than the multiverse which just contains V. But there certainly are candidates. For example, restricting the generic-multiverse to the multiverse generated by only allowing forcing notions which are *homogeneous* could (as far as I know) give a multiverse view which does not violate the multiverse laws. Further given one motivation I cited for a generic-multiverse view (based on Theorem 10), such a restriction is a natural one.

At this point, I just do not see any argument for such a restricted multiverse view. The fundamental problem with a multiverse view based on a multiverse which is smaller than the generic-multiverse is that there must exist Π_2 sentences which are declared to be true and which are not true in the generic-multiverse view. At present there is simply no natural candidate for such a collection of Π_2 sentences which is not simply the set of all Π_2 sentences true in V.

The determined advocate for the generic-multiverse conception of truth might simply accept the failure of the multiverse laws as a deep fact about truth in Set Theory, and seek salvation in the multiverse truths which are beyond Π_2 sentences. But then this advocate must either explain the restriction to set forcing extensions in the definition of the generic-multiverse or specify exactly how the multiverse is to be defined. Restricting to set generic extensions resurrects the specter of unsolvable problems. For example consider the question of whether there exists a cardinal λ such that the Generalized Continuum Hypothesis holds for all cardinals above λ. The answer to this question is invariant across the generic-multiverse. Alternatively, specifying the class forcing extensions which are to be allowed seems utterly hopeless at the present time if the corresponding multiverse conception of truth is to declare the question above to be meaningless *and* yet preserve large axioms as meaningful. Further there is the issue of whether truth in this expanded generic-multiverse is reducible to truth in the universes of that multiverse as is the case of the generic-multiverse.

Of course one could just conclude from all of this that the Ω Conjecture is *false* and predict that the solution to specifying the true generic-multiverse will be revealed by the nature of the failure of the Ω Conjecture. But the Ω Conjecture is invariant across the generic-multiverse. Thus it is not unreasonable to expect both that the Ω Conjecture has an answer and further if that answer is that it is false, then the Ω Conjecture be *refuted* from some large cardinal hypothesis. Many of the meta-mathematical consequences of the Ω Conjecture follow from the nontrivial Ω-satisfiability of the Ω Conjecture; this is the assertion that for some universe V^* of the generic-multiverse generated by V, there exists an ordinal α such that

$$V^*_\alpha \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$$

and such that

$$V^*_\alpha \vDash \text{“The } \Omega \text{ Conjecture”}.$$

This assertion is itself a Σ_2-assertion and so assuming there is a proper class of Woodin cardinals, this assertion must also be invariant across the generic-multiverse generated by V. While the claim that if the Ω Conjecture is false, then the Ω Conjecture must be refuted from some large cardinal hypothesis, is debatable, the corresponding claim for the nontrivial Ω-satisfiability of the Ω Conjecture (in the sense just defined) is much harder to argue against. The point here is that while there are many examples of sentences which are provably absolute for set forcing and which cannot be decided by any large cardinal axiom, there are no known examples where the sentence is Σ_2. In fact if the Ω Conjecture is true then there really can be no such example. Finally it seems unlikely that there is a large cardinal axiom which proves the nontrivial Ω-satisfiability of the Ω Conjecture and yet there be no large cardinal axiom which proves the Ω Conjecture.

1.5.1 Probing the universe of sets; the Inner Model Program

The *Inner Model Program* is the detailed study of large cardinal axioms. The first construction of an inner model is due to Gödel, [2] and [3]. This construction founded the Inner Model Program, the transitive class constructed is denoted by L and I have given the definition. The question of whether $V = L$ is an important one for Set Theory. The answer has profound implications for the conception of the universe of sets.

Theorem 25 (Scott, [11]). *Suppose there is a measurable cardinal. Then $V \neq L$.* □

The *Axiom of Constructibility* is the axiom which asserts, "$V = L$"; more precisely this is the axiom which asserts that for each set a there exists an ordinal α such that $a \in L_\alpha$. Scott's theorem provided the first indication that the Axiom of Constructibility is independent of the ZFC axioms. At the time there was no compelling reason to believe that the existence of a measurable cardinal was consistent with the ZFC axioms, so one could not make the claim that Scott's theorem established the formal independence of the Axiom of Constructibility from the ZFC axioms. Of course it is an immediate corollary of Cohen's results that the Axiom of Constructibilit is formally independent of the ZFC axioms. The modern significance of Scott's theorem is more profound; I would argue that Scott's theorem establishes that the Axiom of Constructibility is *false*. This claim (that $V \neq L$) is not universally accepted but in my view no one has come up with a credible argument against this claim.

The Inner Model Program seeks generalizations of L for the large cardinal axioms, in brief it seeks generalizations of the Axiom of Constructibility which are compatible with large cardinal axioms (such as the axioms for measurable cardinals and beyond). It has been a very successful program and its successes have led to the realization that the large cardinal hierarchy is a very "robust" notion. The results which have been obtained provide some of our deepest glimpses into the universe of sets. Despite the rather formidable merits as indicated above, there is a fundamental difficulty with the prospect of using the Inner Model Program to counter the Skeptic's Retreat. The problem is in the basic methodology of the Inner Model Program. But to explain this I must give a (brief) description of the (technical) template for inner models.

The inner models which are the goal and focus of the Inner Model Program are defined layer by layer working up through the hierarchy of large cardinal axioms, which in turn is naturally revealed by the construction of these inner models. Each layer provides the foundation for the next and L is the first layer.

Roughly (and in practice) in constructing the inner model for a specific large cardinal axiom, one obtains an exhaustive analysis of all weaker large cardinal axioms. There can be surprises here in that seemingly different notions of large cardinals can coincide in the inner model. Finally as one ascends through the hierarchy of large cardinal axioms, the construction generally becomes more and more difficult.

However there is a fundamental problem with appealing to the Inner Model Program to counter the Skeptic's Retreat. Suppose (for example) that a hypothetical large cardinal axiom "Φ" provides a counterexample to the Skeptic's Retreat and this is accomplished by the Inner Model Program. *To use the Inner Model Program to refute the existence of an "Φ-cardinal" one first must be able to successfully construct the inner models for all smaller large cardinals and this hierarchy would be fully revealed by the construction.*

Perhaps this could happen, but it can only happen *once*. This is the problem. Having refuted the existence of an "Φ-cardinal" how could one then refute the existence of any *smaller* large cardinals, for one would have solved the inner model problem for these smaller large cardinals. The fundamental problem is that *the Inner Model Program seems inherently unable, by virtue of its inductive nature, to provide a framework for an evolving understanding of the boundary between the possible and the impossible (large cardinal axioms).*

Thus it would seem that the Skeptic's Retreat is in fact a powerful counter-attack. But there is something wrong here and the answer lies in understanding large cardinal axioms which are much stronger than those within reach of the current hierarchy of inner models.

1.5.2 Supercompact cardinals and beyond

Paraphrasing a standard definition: an *extender* is an elementary embedding

$$E : V_{\alpha+1} \cap M \to V_{\beta+1} \cap M$$

where M is a transitive class such that $M \vDash \mathrm{ZFC}$. Necessarily $\alpha \leq \beta$ and $E(\alpha) = \beta$. It is not difficult to show that if for all ordinals $\xi \leq \alpha$, $E(\xi) = \xi$ then for all $a \in V_{\alpha+1}$, $E(a) = a$. Thus if E is non-trivial there must exist an ordinal $\gamma \leq \alpha$ such that $E(\gamma) \neq \gamma$ and the least such ordinal γ is the *critical point* of E and is denoted $\mathrm{CRT}(E)$.

Extenders are the building blocks for the Inner Model Program which seeks enlargements of L which are transitive classes N such that N contains *enough* extenders to witness that the targeted large cardinal axiom holds in N. The complication is in specifying just which extenders are to be included in N.

The definition of a supercompact cardinal is due to Reinhardt and Solovay—see [5] for more on the history of the axiom. Below is a reformulation of the definition due to Magidor in terms of extenders.

Definition 26. A cardinal δ is a *supercompact cardinal* if for each ordinal $\beta > \delta$ there exists an extender

$$E : V_{\alpha+1} \to V_{\beta+1}$$

such that $E(\kappa) = \delta$ where $\kappa = \mathrm{CRT}(E)$. □

Slightly stronger is the notion that δ is an *extendible cardinal*: for all $\alpha > \delta$ there exists an extender,

$$E : V_{\alpha+1} \to V_{\beta+1}$$

such that $\mathrm{CRT}(E) = \delta$.

As I have already indicated, the strongest large cardinal axioms not known to be inconsistent with the Axiom of Choice are the family of axioms asserting the existence of strongly $(\omega + 1)$-huge cardinals. These axioms have seemed so far beyond any conceivable inner model theory that they simply are not understood.

The possibilities for an inner model theory at the level of supercompact cardinals and beyond, has been essentially a complete mystery until recently. The reason lies in the nature of extenders. Again for expository purposes, let me define an extender

$$E : V_{\alpha+1} \to V_{\beta+1}$$

to be a *suitable* extender if $E(\mathrm{CRT}(E)) > \alpha$. Thus an extender

$$E : V_{\alpha+1} \cap M \to V_{\beta+1} \cap M$$

is a suitable extender if it is not *too long* and if $V_{\beta+1} \subset M$. For example suppose that κ is a strongly $(\omega + 1)$-huge cardinal as defined in Definition 6 on page 13. Then there exists there exist $\gamma > \lambda > \kappa$ such that

(1) $V_\kappa \prec V_\lambda \prec V_\gamma$,

(2) there exists an elementary embedding,

$$j : V_{\lambda+1} \to V_{\lambda+1}$$

such that κ is the critical point of j.

Thus j is an extender but j is *not* a suitable extender. In particular the existence of a strongly $(\omega + 1)$-huge cardinal *cannot* be witnessed by a suitable extender.

The Inner Model Program at the level of supercompact cardinals and beyond seeks enlargements N of L such that there are enough extenders E such that $E|N \in N$ to witness the targeted large cardinal axiom holds in N. For the weaker large cardinal axioms this has been an extremely successful program. For example, Mitchell-Steel [8], have defined enlargements at the level of Woodin cardinals. In fact they define the basic form of such enlargements of L up to level of *superstrong cardinals* which are just below the level of supercompact cardinals. The Mitchell-Steel models are constructed from sequences of extenders. The basic methodology is to construct N from a sequence of extenders which includes enough extenders to directly witness that the targeted large cardinal axiom holds in N.

For the construction of the Mitchell-Steel models there is a fundamental requirement that the extenders on sequence from which the enlargement of L is constructed be derived from extenders

$$E : V_{\alpha+1} \cap M \to V_{\beta+1} \cap M$$

such that $V_{\alpha+1} \subset M$. Following this basic methodology the enlargements of L at the level of supercompact cardinals and beyond must be constructed from extender sequences which now include extenders which are *restrictions* of extenders of the form,

$$E : V_{\alpha+1} \to V_{\beta+1},$$

and Steel has shown that the basic methodology of analyzing extender models encounters serious obstructions once there are such extenders on the sequence, particularly if the extenders are not suitable.

But by some fairly recent theorems something completely unexpected and remarkable happens. Suppose that N is a transitive class, for some cardinal δ,

$$N \vDash \text{“}\delta \text{ is a supercompact cardinal”},$$

and that this is witnessed by class of all $E|N$ such that $E|N \in N$ and such that E is a suitable extender. Then the transitive class N is close to V and N inherits essentially all large cardinals from V. The amazing thing is that this must happen *no matter how N is constructed*. This would seem to undermine my earlier claim that inner models should be constructed from extender sequences which contain enough extenders to witness that the targeted large cardinal axiom holds in the inner model. It does not and the reason is that by simply requiring that $E|N \in N$ for enough suitable extenders from V to witness that the large cardinal axiom, “There is a supercompact cardinal”, holds in N, one (and this is the surprise) necessarily must have $E|N \in N$ for a *much* larger class of extenders, $E : V_{\alpha+1} \to V_{\beta+1}$. So the *principle* that there are enough extenders in N to witness the targeted large cardinal axiom holds in N *is preserved* (as it must be). The *change*, in the case that N is constructed from a sequence of extenders which includes restrictions of suitable extenders, is that these extenders do not have to be on the sequence from which N is constructed. In particular in this case, large cardinal axioms can be witnessed to hold in N by "phantom" extenders, these are extenders of N which are not on the sequence, which *cannot* be witnessed to hold by *any* extender on the sequence. This includes large cardinal axioms at the level of strongly $(\omega + 1)$-huge cardinals. As a consequence of this, one can completely avoid the cited obstacles because: *One does not need to have the kinds of extenders on the sequence which give rise to the obstacles.* Specifically, one can restrict consideration to extender sequences of just extenders derived from suitable extenders and this is a paradigm shift in the whole conception of inner models.

The analysis yields still more. Suppose that there is a positive solution (in ZFC) to the inner model problem for just one supercompact cardinal. Then as a corollary one would would obtain a proof of the following conjecture.

Conjecture (ZF) *There are no weak Reinhardt cardinals.* □

Suppose this conjecture is actually true and it is proved according to the scenario that I have just described. This would in a convincing fashion refute the Skeptic's Retreat providing for the *first time* an example of a natural large cardinal axiom proved to be inconsistent as a result of a deep structural analysis.

In fact, it is possible to isolate a specific conjecture which must be true if there is a positive solution to the inner model problem for one supercompact cardinal and which itself suffices for this inconsistency result. This conjecture (which is the HOD Conjecture of [16]) concerns HOD and we refer the interested (and dedicated) reader to [16] for details. Actually a corollary of the HOD Conjecture suffices to prove the inconsistency conjectured above and this corollary we can easily state. First, a cardinal κ is a *regular cardinal* if every subset $X \subset \kappa$ with $|X| < \kappa$, is bounded in κ. Thus ω is a regular cardinal as is ω_1 (assuming the Axiom of Choice). The corollary of the HOD Conjecture is the following: There is a proper class of regular cardinals which are not measurable cardinals in HOD.

I mention this for two reasons. First, this is a specific and precise conjecture which does involve the Inner Model Program at all and so offers an independent route to proving the conjectured inconsistency above. Second, it identifies specific combinatorial consequences of having a successful solution to the inner model problem for one supercompact cardinal, and so provides a potential basis for establishing that there is no solution to the inner model problem for one supercompact cardinal.

The extension of the Inner Model Program to level of one supercompact cardinal (the definition of "ultimate L") will come with a price. The successful extension of the Inner Model Program to a large cardinal axiom can no longer serve as the basis for the claim of the formal consistency of that axiom. The reason is that as the previous discussion indicates, in extending the Inner Model Program to the level of one supercompact cardinal, one will have extended the Inner Model Program to essentially all known large cardinals. The ramifications are discussed at length in [16]. In brief further progress in understanding (and even discovering) large cardinal axioms would have to depend on *structural* considerations of "ultimate L".

1.6 Conclusions

The development of the mathematical theory of infinity has led to a number of specific predictions. These predictions assert that certain technical axioms concerning the existence of large cardinals, are not formally inconsistent with the axioms of set theory. As I have indicated these predications are actually predications about the physical universe. To date there is no known (and credible) explanation for these predictions except that they are true because the corresponding axioms are true in the universe of sets. As the arguments of the first section indicate, these same issues arise even for the conception of large finite sets.

As discussed in the second section, there is a serious challenge to this claim, even ignoring the often cited challenge; the ubiquity of unsolvable problems in Set Theory. The challenge arises from the fact that here are formal axioms of infinity which are arguably a serious foundational issue for Set Theory for two reasons. First, these axioms are known to refute the Axiom of Choice and second, these axioms are known to be "stronger" than essentially all the notions of infinity believed to be formally consistent with the Axiom of Choice. Here the metric for strength is simply the inference relation for the corresponding predictions (of formal consistency). The issues raised by this are twofold. First (regarding the debate between the Set Theorist and the Skeptic), there is no need to explain the success of a single prediction, it is a succession of ever stronger successful predictions which demands explanation. But this one prediction of consistency subsumes all the predictions made to date and so there is no series of predictions which requires explanation. Second, for the Set Theorist to account for this one prediction it would seem that a different conception of the Universe of Sets is required.

The conception of a Universe of Sets in which the Axiom of Choice fails creates more difficulties than it solves and so this does not seem to be a viable option. However any large cardinal axiom (which is expressible by a Σ_2-sentence) which can hold in a universe of sets satisfying all of the axioms except for the Axiom of Choice, can hold in a generic extension of a universe of sets which does satisfy the Axiom of Choice. Therefore this challenge, and the challenge posed by formally unsolvable problems such as that of the Continuum Hypothesis, might both be addressed (but perhaps not completely in a satisfactory manner) by adopting the conception of a multiverse of sets. Here the Ω Conjecture emerges as a key conjecture. If this conjecture is true then what is arguably the only candidate for a multiverse view for the infinite realm which can address these challenges, also fails to be a viable alternative (accepting the requirement that the multiverse laws of Section 3 be satisfied). Therefore if the multiverse view is correct, the Ω Conjecture must be false.

The attempt to understand how the Ω Conjecture might be refuted leads directly to the Inner Model Program. The Inner Model Program is the attempt to generalize the definition of L to yield transitive classes M in which large cardinal axioms hold. If the Inner Model Program as described in the fourth section can be extended to the level of a single supercompact cardinal then no known large cardinal axiom can refute the Ω Conjecture. Further one would also obtain as corollary the verification of a series of conjectures. These conjectures imply that the large cardinal axioms—such as the axiom which asserts the existence of a weak Reinhardt cardinal—which pose such a challenge to the conception of the universe of sets, are formally inconsistent. These inconsistency results would be the first examples of inconsistency results for large cardinal axioms obtained only through a very detailed analysis.

Finally the extension of the Inner Model Program to the level of one supercompact cardinal will yield examples (where *none* are currently known) of a *single* formal axiom which is compatible with all the known large cardinal axioms and which provides an axiomatic foundation for Set Theory which is immune to independence by Cohen's method. This axiom will not be unique but there is the very real possibility that among these axioms, there is an optimal one (from structural and philosophical considerations). In which case we will have returned, against all odds or reasonable expectation, to the view of truth for Set Theory which was present at the time when the investigation of Set Theory began.

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