The Continuum Hypothesis, the generic-multiverse of sets, and the Ω Conjecture

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1 A tale of two problems

The formal independence of Cantor's Continuum Hypothesis from the axioms of Set Theory (ZFC) is an immediate corollary of the following two theorems where the statement of the Cohen's theorem is recast in the more modern formulation of the Boolean valued universe.

Theorem 1 (Gödel, [3]). *Assume $V = L$. Then the Continuum Hypothesis holds.* □

Theorem 2 (Cohen, [1]). *There exists a complete Boolean algebra, $\mathbb{B}$, such that*

$$V^{\mathbb{B}} \models \text{"The Continuum Hypothesis is false".}$$ □

Is this really evidence (as is often cited) that the Continuum Hypothesis has no answer?

Another prominent problem from the early 20th century concerns the *projective sets*, [8]; these are the subsets of $\mathbb{R}^n$ which are generated from the closed sets in finitely many steps taking images by continuous functions, $f : \mathbb{R}^n \to \mathbb{R}^n$, and complements. A function, $f : \mathbb{R} \to \mathbb{R}$, is projective if the graph of f is a projective subset of $\mathbb{R} \times \mathbb{R}$. Let *Projective Uniformization* be the assertion:

> *For each projective set $A \subset \mathbb{R} \times \mathbb{R}$ there exists a projective function, $f : \mathbb{R} \to \mathbb{R}$, such that for all $x \in \mathbb{R}$ if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$.*

The two theorems above concerning the Continuum Hypothesis have versions for Projective Uniformization. Curiously the Boolean algebra for Cohen's theorem is the same in both cases, but in case of the problem of Projective Uniformization an additional hypothesis on V is necessary. While Cohen did not explicitly note the failure of Projective Uniformization, it is arguably implicit in his results. On the other hand Gödel was aware and did note that Projective Uniformization does hold in L; he calculated that if $V = L$ then there is a wellordering of the reals which as a binary relation is a projective subset of the plane.

Theorem 3 (Gödel, [3]). *Assume $V = L$. Then Projective Uniformization holds.* □

Theorem 4 (Cohen, [1]). *Assume $V = L$. There exists a complete Boolean algebra, $\mathbb{B}$, such that*

$$V^{\mathbb{B}} \models \text{“Projective Uniformization does not hold”}. \quad \square$$

Thus one also obtains the formal independence of Projective Uniformization from the axioms of Set Theory. But in this case this is *not* evidence that the problem of Projective Uniformization has no answer. The reason is that there is a fundamental difference in the problem of the Continuum Hypothesis versus the problem of Projective Uniformization–it was realized fairly soon after Cohen's initial results that *large cardinal* axioms could not be used to settle the Continuum Hypothesis, [7].

Theorem 5 (Levy, Solovay [7]). *There exists a complete Boolean algebra, $\mathbb{B}$, such that*

$$V^{\mathbb{B}} \models \text{“The Continuum Hypothesis”}. \quad \square$$

Assuming the consistency of certain large cardinal axioms, the analogous theorem for negation of Projective Uniformization is *false*. The reason is that large cardinal axioms have been shown to imply that Projective Uniformization holds. There are several versions of this theorem and the most recent and essentially optimal version (optimal in its economy of large cardinal axioms) is given by the seminal theorem of Martin and Steel [9], rephrased here to be relevant to this discussion.

Theorem 6 (Martin, Steel). *Assume there are infinitely many Woodin cardinals. Then Projective Uniformization holds.* □

Corollary 7. *Suppose there is a proper class of Woodin cardinals. Then for all complete Boolean algebras, $\mathbb{B}$,*

$$V^{\mathbb{B}} \models \text{“Projective Uniformization”}. \quad \square$$

2 The generic-multiverse of sets

Let the *multiverse* (of sets) refer to the collection of possible universes of sets. The truths of the Set Theory are the sentences which hold in each universe of the multiverse. The multiverse is the *generic-multiverse* if it is generated from each universe of the collection by closing under generic extensions (enlargements) and under generic refinements (inner models of a universe which the given universe is a generic extension of). To illustrate the concept of the generic-multiverse, suppose that M is a countable transitive set with the property that

$$M \models \text{ZFC}.$$

Let $\mathbb{V}_M$ be the smallest set of countable transitive sets such that $M \in \mathbb{V}_M$ and such that for all pairs, (M_1, M_2), of countable transitive sets such that

$$M_1 \models \text{ZFC},$$

and such that M_2 is a generic extension of M_1, if either $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$ then both M_1 and M_2 are in $\mathbb{V}_M$. It is easily verified that for each $N \in \mathbb{V}_M$,

$$\mathbb{V}_N = \mathbb{V}_M,$$

where $\mathbb{V}_N$ is defined using N in place of M. $\mathbb{V}_M$ is the generic-multiverse generated in V from M.

The *generic-multiverse position* is the position that a sentence is true if and only if it holds in each universe of the generic-multiverse generated by V. A priori this conception of truth seems to depend on (and therefore require) a larger universe within which the generic-multiverse generated by V is computed, However this conception of truth can actually be formalized within V without regard to any such larger universe. More precisely, for each sentence ϕ there is a sentence ϕ^*, recursively depending on ϕ, such that ϕ is true in each universe of the generic-multiverse generated by V if and only if ϕ^* is true in V. The recursive transformation which sends ϕ to ϕ^* is explicit (we shall actually specify it) and does not depend on V. Thus with M and $\mathbb{V}_M$ as above, the following are equivalent.

(1) $M \vDash \phi^*$.

(2) $N \vDash \phi$ for all $N \in \mathbb{V}_M$.

(3) $N \vDash \phi^*$ for all $N \in \mathbb{V}_M$.

Since this is an important point in favor of the generic-multiverse position, I give a proof in the appendix. The relevance of this point to the generic-multiverse position is that it shows that as far as assessing truth, the generic-multiverse position is not that sensitive to the meta-universe in which the generic-multiverse is being defined.

The generic-multiverse position has a feature which the multiverse view given by formalism does *not* share: the notion of truth is the same as defined relative to each universe of the multiverse, so from the perspective of evaluating truth *all* the universes of the multiverse are equivalent. This seems another important point in favor of the generic-multiverse position and this point is reinforced by the reduction indicated above of truth in the generic-multiverse to truth in each constituent universe.

In fact the multiverse position given by essentially any generalization of first order logic is *not* the same as defined relative to each universe of the multiverse (the definition of the logic is not absolute to each universe of the multiverse). This only requires that the logic be definable and a more precise version of this claim is given in the following lemma the statement of which requires some notation.

Suppose $\Phi(x)$ is a formula and define T_Φ to be the set of all sentences ψ such that if $\mathcal{M} \vDash \text{ZFC}$ and if $\Phi[\mathcal{M}]$ holds then $\mathcal{M} \vDash \psi$. Thus if there is no such model $\mathcal{M}$, T_Φ is simply the set of all sentences. For each model, $\mathcal{M} \vDash \text{ZFC}$, let $(T_\Phi)^{\mathcal{M}}$ be the set of all sentences ϕ such that

$$\mathcal{M} \vDash \text{“}\phi \in T_\Phi.\text{”}$$

Let $\hat{\Phi}$ be a Gödel sentence, ϕ, which expresses: $\phi \notin T_\Phi$. More precisely, let $\hat{\Phi}$ be a sentence such that for all models $\mathcal{M} \vDash \text{ZFC}$, $\mathcal{M} \vDash \hat{\Phi}$ if and only if $\hat{\Phi} \notin (T_\Phi)^{\mathcal{M}}$.

The following lemma generalizes Gödel's Second Incompleteness Theorem.

Lemma 8. *Suppose that there exists a model, $\mathcal{M} \vDash$ ZFC, such that $\Phi[\mathcal{M}]$ holds. Then there exists a model, $\mathcal{M} \vDash$ ZFC, such that $\Phi[\mathcal{M}]$ holds and such that either:*

(1) $\hat{\Phi} \in T_\Phi$ *and* $\hat{\Phi} \notin (T_\Phi)^{\mathcal{M}}$, *or*

(2) $\hat{\Phi} \notin T_\Phi$ *and* $\hat{\Phi} \in (T_\Phi)^{\mathcal{M}}$. □

The proof of the lemma is immediate from the definitions. In general the conclusion of the lemma best possible. More precisely, assuming there is a model of ZFC there are examples of Φ for which (1) does not hold with $\hat{\Phi}$ replaced by any sentence whatsoever and assuming there is an ω-model of ZFC (less suffices) there are examples of Φ for which (2) does not hold with $\hat{\Phi}$ replaced by any sentence whatsoever.

Suppose that $\Phi(x)$ is trivial, for example suppose that $\Phi(x)$ is the formula, "$x = x$". Then $\hat{\Phi} \in T_\Phi$ if and only if there are no models

$$\mathcal{M} \vDash \text{ZFC}.$$

Therefore an immediate corollary of the lemma is that if there exists a model,

$$\mathcal{M} \vDash \text{ZFC},$$

then there exists a model,

$$\mathcal{N} \vDash \text{ZFC},$$

such that

$$\mathcal{N} \vDash \text{"There is no model of ZFC"}.$$

This of course is the version of Gödel's Second Incompleteness Theorem for the theory, ZFC.

The generic-multiverse position, which is suggested by the results of the previous section, declares that the Continuum Hypothesis is neither true nor false. Assuming that in each universe of the generic-multiverse there is a proper class of Woodin cardinals then the generic-multiverse position declares Projective Uniformization as true.

Is the generic-multiverse position a reasonable one? The refinements of Cohen's method of *forcing* in the decades since his initial discovery of the method and the resulting plethora of problems shown to be unsolvable, have in a practical sense almost compelled one to adopt the generic-multiverse position. This has been reinforced by some rather unexpected consequences of large cardinal axioms which I shall discuss in the next section. Finally the argument that Cohen's method of forcing establishes that the Continuum Hypothesis has no answer, is implicitly assuming the generic-multiverse conception of truth, a least for sentences about sets of real numbers.

The purpose of this paper is *not* to argue against any possible multiverse position but to more carefully examine the generic-multiverse position within the context of modern Set Theory. I have formalized the generic-multiverse conception of truth simply as a vehicle to explore more fully the claim that Cohen's method of forcing does establish that the Continuum Hypothesis has no answer. In brief I shall argue that modulo the Ω Conjecture, the generic-multiverse position outlined above is not reasonable and a more detailed discussion is given after a brief review of Ω-logic, in the last two sections of this paper. The essence of the argument against the generic-multiverse position is that assuming the Ω Conjecture is true (and that there is a proper class of Woodin cardinals) then this position is simply a brand of formalism that denies the transfinite by a reducing truth about the universe of sets to truth about a simple fragment such as the integers or, in this case, the collection of all subsets of the least Woodin cardinal. The Ω Conjecture is invariant between V and $V^{\mathbb{B}}$ and so the generic-multiverse position must either declare the Ω Conjecture to be true or declare the Ω Conjecture to be false.

It is a fairly common (informal) claim that the quest for truth about the universe of sets is analogous to the quest for truth about the physical universe. However I am claiming an important distinction. While physicists would rejoice in the discovery that the conception of the physical universe reduces to the conception of some simple fragment or model, in my view the set theorist must reject the analogous possibility for truth about the universe of sets. By the very nature of its conception, the set of all truths of the transfinite universe (the universe of sets) cannot be reduced to the set of truths of some explicit fragment of the universe of sets. Taking into account the iterative conception of sets, the set of all truths of an explicit fragment of the universe of sets cannot be reduced to the truths of an explicit *simpler* fragment. The latter is the basic position on which I shall base my arguments.

An assertion is Π_2 if it is of the form,

$$\text{“ For every infinite ordinal } \alpha \text{ , } V_\alpha \vDash \phi \text{ ”,}$$

for some sentence, ϕ. A Π_2 assertion is a *multiverse truth* if the Π_2 assertion holds in each universe of the multiverse.

Let δ_0 denote the least Woodin cardinal (so I now assume there is a proper class of Woodin cardinals so that the existence of δ_0 is invariant across the generic-mulitverse). $H(\delta_0^+)$ denotes the set of all sets X whose transitive closure has cardinality at most δ_0. The multiverse truths of $H(\delta_0^+)$ are those sentences ϕ which hold in the $H(\delta_0^+)$ of each universe of the multiverse.

Note that for each sentence ϕ, it is a Π_2 assertion to say that

$$H(\delta_0^+) \vDash \phi$$

and it is a Π_2 assertion to say that $H(\delta_0^+) \nvDash \phi$. Thus in any one universe of the multiverse, the set of all sentences ϕ such that

$$H(\delta_0^+) \vDash \phi,$$

this is the *theory* of $H(\delta_0^+)$ as computed in that in that universe, is recursive in the set of Π_2 sentences (assertions) which hold in that universe. Further by Tarski's Theorem on the undefinability of truth the latter set cannot be recursive in the former set.

Similarly as computed in any one universe of the multiverse, the theory of any *explicit* fragment of the universe of sets, such as $V_{\omega+\omega}$ or even V_{δ_0} is recursive in the set of Π_2 sentences which hold in that universe and not vice-versa.

These comments suggest the following multiverse laws which I state in reference to an arbitrary multiverse position though in the context that the existence of a Woodin cardinal holds across the muliverse. For the case of the generic-multiverse generated by V, this latter requirement is equivalent to the requirement that there exist a proper class of Woodin cardinals in V.

<u>*First Multiverse Law*</u>

> *The set of Π_2 assertions which are multiverse truths is not recursive in the set of multiverse truths of $H(\delta_0^+)$.* □

The motivation for this multiverse law is that if the set of Π_2 multiverse truths is recursive in the set of multiverse truths of $H(\delta_0^+)$ then as far as evaluating Π_2 assertions is concerned, the multiverse is equivalent to the reduced multiverse of just the fragments $H(\delta_0^+)$ of the universes of the multiverse. This amounts to a rejection of the transfinite beyond $H(\delta_0^+)$ and constitutes in effect the unacceptable brand of formalism alluded to earlier. This claim is reinforced should the multiverse position also violate a second multiverse law which I formulate below.

A set $Y \subset V_\omega$ is definable in $H(\delta_0^+)$ across the multiverse if the set Y is definable in the structure $H(\delta_0^+)$ of each universe of the multiverse (possibly by formulas which depend on the parent universe).

The second multiverse law is a variation of the First Multiverse Law.

Second Multiverse Law

> *The set of* Π_2 *assertions which are multiverse truths, is not definable in* $H(\delta_0^+)$ *across the multiverse.* □

Again, by Tarski's Theorem on the undefinability of truth, this multiverse law is obviously a reasonable one *if* one regards the only possibility for the multiverse to be the universe of sets so that set of multiverse truths of $H(\delta_0^+)$ is simply the set of all sentences which are true in $H(\delta_0^+)$ and the set of Π_2 assertions which are multiverse truths is simply the set of Π_2 assertions which are true in V. More generally the Second Multiverse Law would have to hold if one modified the law to simply require that the set of Π_2 assertions which are multiverse truths, is not uniformly definable in $H(\delta_0^+)$ across the multiverse (i.e. by a single formula).

Assuming both that Ω Conjecture and the existence of a proper class of Woodin cardinals hold in each (or one) universe of the generic-multiverse generated by V, then *both* the First Multiverse Law and the Second Multiverse Law are violated by the generic-multiverse position. This is the basis for the argument I am giving against the generic-multiverse position in this paper. In fact the technical details of how the generic-multiverse position violates these multiverse laws provides an even more compelling argument against the generic-multiverse position since the analysis shows that in addition the generic-multiverse position is truly a form of formalism because of the connections to Ω-logic.

There is a special case which I can present without any additional definitions and which is not contingent on any conjectures.

Theorem 9. *Suppose that* M *is a countable transitive set*

$$M \vDash \text{ZFC} + \textit{“There is a proper class of Woodin cardinals”}$$

and that $M \cap \text{Ord}$ *is as small as possible. Then* $\mathbb{V}_M$ *violates both multiverse laws.* □

3 Ω-logic

A set X is *transitive* if for all $a \in X$, $a \subset X$. It is a consequence of the axioms of Set Theory that every set X is a subset of a transitive set and among these transitive sets there is a least one under containment; this is the *transitive closure* of the set X.

A set X is of hereditary cardinality at most κ if the transitive closure of X has cardinality at most κ. As I indicated above, I denote by $H(\delta_0^+)$ the set of all sets X whose transitive closure has cardinality at most δ_0 where δ_0 is the least Woodin cardinal.

Both the Continuum Hypothesis and Projective Uniformization are *first order* properties of $H(\delta_0^+)$ in the sense that there are sentences ψ_{CH} and ψ_{PU} such that

$$H(\delta_0^+) \vDash \psi_{\mathrm{CH}}$$

if and only if the Continuum Hypothesis holds and

$$H(\delta_0^+) \vDash \psi_{\mathrm{PU}}$$

if and only if Projective Uniformization holds.

Since in the generic-multiverse position, an assertion of the form,

$$H(\delta_0^+) \vDash \phi,$$

is true if and only if the assertion holds in all universes of the generic-multiverse, the generic-multiverse position declares the Continuum Hypothesis to be neither true nor false and declares, granting large cardinals, that Projective Uniformization is true. I note that for essentially all current large cardinal axioms, the existence of a proper class of large cardinals holds in V if and only if it holds in $V^{\mathbb{B}}$ for all complete Boolean algebras, $\mathbb{B}$. In other words, in the generic-multiverse position the existence of a proper class of, say, Woodin cardinals is either true or false since it either holds in every universe of the generic-multiverse or it holds in no universe of the generic-multiverse, [5].

I am going to analyze the generic-multiverse position from the perspective of Ω-logic which I first briefly review.

Definition 10. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then

$$T \vDash_\Omega \phi$$

if for all complete Boolean algebras, $\mathbb{B}$, for all ordinals, α, if

$$V_\alpha^{\mathbb{B}} \vDash T$$

then $V_\alpha^{\mathbb{B}} \vDash \phi$. □

If there is a proper class of Woodin cardinals then the relation $T \vDash_\Omega \phi$, is generically absolute. This fact which arguably was a completely unanticipated consequence of large cardinals, makes Ω-logic interesting from a meta-mathematical point of view. For example the set

$$\mathcal{V}_\Omega = \{\phi \mid \emptyset \vDash_\Omega \phi\}$$

is generically absolute in the sense that for a given sentence, ϕ, the question whether or not ϕ is logically Ω-valid, i.e. whether or not $\phi \in \mathcal{V}_\Omega$, is absolute between V and all of its generic extensions. In particular the method of (set) forcing *cannot* be used to show the formal independence of assertions of the form $\emptyset \vDash_\Omega \phi$.

Theorem 11. *Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Suppose that there exists a proper class of Woodin cardinals. Then for all complete Boolean algebras, $\mathbb{B}$,*

$$V^{\mathbb{B}} \vDash \text{“}T \vDash_\Omega \phi\text{”}$$

if and only if $T \vDash_\Omega \phi$. □

There are a variety of technical theorems which show that one cannot hope to prove the generic invariance of Ω-logic from any large cardinal hypothesis weaker than the existence of a proper class of Woodin cardinals–for example if $V = L$ then definition of $\mathcal{V}_\Omega$ is not absolute between V and $V^{\mathbb{B}}$, for *any* nontrivial complete Boolean algebra, $\mathbb{B}$, of cardinality c.

It follows easily from the definition of Ω-logic, that for any Π_2-sentence, ϕ,

$$\emptyset \models_\Omega \phi$$

if and only if for all complete Boolean algebras, $\mathbb{B}$,

$$V^{\mathbb{B}} \models \phi.$$

Therefore by the theorem above, assuming there is a proper class of Woodin cardinals, for each sentence, ψ, the assertion

$$\textit{For all complete Boolean algebras, } \mathbb{B},\ V^{\mathbb{B}} \models \text{“}H(\delta_0^+) \models \psi\text{”}$$

is itself absolute between V and $V^{\mathbb{B}}$ for all complete Boolean algebras $\mathbb{B}$. This remarkable consequence of the existence of a proper class of Woodin cardinals actually seems to be evidence for the generic-multiverse position. In particular this shows that the generic-multiverse position, at least for assessing Π_2 assertions, and so for assessing all assertions of the form,

$$H(\delta_0^+) \models \phi,$$

is equivalent to the position that a Π_2 assertion is true if and only if it holds in $V^{\mathbb{B}}$ for all complete Boolean algebras $\mathbb{B}$. Notice that if $\mathbb{R} \not\subset L$ and if V is a generic extension of L then this equivalence is *false*. In this situation the Π_2 sentence which expresses $\mathbb{R} \not\subset L$ holds in $V^{\mathbb{B}}$ for all complete Boolean algebras, $\mathbb{B}$, but this sentence fails to hold across the generic-multiverse generated by V (since L belongs to this multiverse).

To summarize, suppose that there exists a proper class of Woodin cardinals in each universe of the generic-multiverse (or equivalently that there is a proper class of Woodin cardinals in at least one universe of the generic-multiverse). Then for each Π_2 sentence ϕ; the following are equivalent:

(1) ϕ is holds across the generic-multiverse;

(2) “$\emptyset \models_\Omega \phi$” holds across the generic-multiverse;

(3) “$\emptyset \models_\Omega \phi$” holds in at least one universe of the generic-multiverse.

Therefore to evaluate the generic-multiverse position one must understand the logical relation, $T \models_\Omega \phi$. In particular a natural question arises: is there a corresponding proof relation?

4 The Ω Conjecture

I define the proof relation, $T \vdash_\Omega \phi$. This requires a preliminary notion that a set of reals be *universally Baire*, [2]. In fact I shall define $T \vdash_\Omega \phi$, assuming the existence of a proper class of Woodin cardinals and exploiting the fact that there are a number of (equivalent) definitions. Without the assumption that there is a proper class of Woodin cardinals, the definition is a bit more technical, [13]. Recall that if S is a compact Hausdorff space then a set $X \subseteq S$ has the *property of Baire* in the space S if there exists an open set $O \subseteq S$ such that symmetric difference,

$$X \triangle O,$$

is meager in S (contained in a countable union of closed sets with empty interior).

Definition 12. A set $A \subset \mathbb{R}$ is *universally Baire* if for all compact Hausdorff spaces, S, and for all continuous functions,

$$F : S \to \mathbb{R},$$

the preimage of A by F has the property of Baire in the space S. □

Suppose that $A \subseteq \mathbb{R}$ is universally Baire. Suppose that M is a countable transitive model of ZFC. Then M is *strongly A-closed* if for all countable transitive sets N such that N is a generic extension of M,

$$A \cap N \in N.$$

Definition 13. Suppose there is a proper class of Woodin cardinals. Suppose that T is a countable theory in the language of Set Theory, and ϕ is a sentence. Then $T \vdash_\Omega \phi$ if there exists a set $A \subset \mathbb{R}$ such that:

(1) A is universally Baire,

(2) for all countable transitive models, M, if M is strongly A-closed and $T \in M$, then

$$M \vDash \text{“}T \vDash_\Omega \phi\text{”}.$$ □

Assuming there is a proper class of Woodin cardinals, the relation, $T \vdash_\Omega \phi$, is generically absolute. Moreover *Soundness* holds as well.

Theorem 14. *Assume there is a proper class of Woodin cardinals. Then for all* (T, ϕ) *and for all complete Boolean algebras,* $\mathbb{B}$,

$$T \vdash_\Omega \phi \quad \textit{if and only if} \quad V^{\mathbb{B}} \vDash \text{“}T \vdash_\Omega \phi\text{”}.$$ □

Theorem 15 (Soundness). *Assume there is a proper class of Woodin cardinals. If* $T \vdash_\Omega \phi$ *then* $T \vDash_\Omega \phi$. □

I now come to the Ω Conjecture which in essence is simply the conjecture that the Gödel Completeness Theorem holds for Ω-logic; see [13] for a more detailed discussion.

Definition 16 (**Ω Conjecture**). Suppose that there exists a proper class of Woodin cardinals. Then for all sentences ϕ, $\emptyset \vDash_\Omega \phi$ if and only if $\emptyset \vdash_\Omega \phi$. □

5 The complexity of Ω-logic

Let (as defined on page 7) $\mathcal{V}_\Omega$ be the set of sentences ϕ such that

$$\emptyset \models_\Omega \phi,$$

and let $\mathcal{V}_\Omega(H(\delta_0^+))$ be the set of sentences, ϕ, such that

$$\text{ZFC} \models_\Omega \text{“}H(\delta_0^+) \models \phi\text{”}.$$

Assuming there is a proper class of Woodin cardinals then the set of generic-multiverse truths which are Π_2 assertions is of the same Turing complexity as $\mathcal{V}_\Omega$ (i.e., each set is recursive in the other). Further (assuming there is a proper class of Woodin cardinals) the set, $\mathcal{V}_\Omega(H(\delta_0^+))$, is precisely the set of generic-multiverse truths of $H(\delta_0^+)$. Thus the requirement that the generic-multiverse position satisfies the First Multiverse Law, as discussed on page 11, reduces to the requirement that $\mathcal{V}_\Omega$ not be recursive in the set $\mathcal{V}_\Omega(H(\delta_0^+))$.

The following theorem is a corollary of the basic analysis of Ω-logic in the context that there is a proper class of Woodin cardinals, in fact one obtains the stronger conclusion that set $\mathcal{V}_\Omega$ is recursive in the set $\mathcal{V}_\Omega(H(\omega_2))$ (which is the set of sentences, ϕ, such that ZFC $\models_\Omega$ “$H(\omega_2) \models \phi$”).

Theorem 17. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the set $\mathcal{V}_\Omega$ is recursive in the set $\mathcal{V}_\Omega(H(\delta_0^+))$.* □

Therefore, assuming the existence of a proper class of Woodin cardinals and that the Ω Conjecture both hold across the generic-multiverse generated by V, the generic-multiverse position violates the First Multiverse Law. What about the Second Multiverse Law (on page 11)? This requires understanding the complexity of the set $\mathcal{V}_\Omega$. From the definition of $\mathcal{V}_\Omega$ it is evident that this set is definable in V by a Π_2 formula: if $V = L$ then this set is recursively equivalent to the set of all Π_2 sentences which are true in V. However in the context of large cardinal axioms the complexity of $\mathcal{V}_\Omega$ is more subtle.

Theorem 18. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the set $\mathcal{V}_\Omega$ is definable in $H(\delta_0^+)$.* □

Therefore if the Ω Conjecture holds and there is a proper class of Woodin cardinals then the generic-multiverse position that the only Π_2 assertions which are true are those which are true in each universe of the generic-multiverse also violates the Second Multiverse Law–for this set of assertions is itself definable in $H(\delta_0^+)$ across the generic-multiverse.

In summary, assuming the existence of a proper class of Woodin cardinals and that the Ω Conjecture both hold across the generic-multiverse generated by V, then both the First Multiverse Law and the Second Multiverse Law are violated by the generic-multiverse view of truth.

In particular, assuming the existence of a proper class of Woodin cardinals and that the Ω Conjecture both hold across the generic-multiverse generated by V, then as far evaluating truth across the generic-multiverse for Π_2 assertions, the generic-multiverse is equivalent to the reduced multiverse given by the structures $H(\delta_0^+)$ of the universes in the generic-multiverse. Why not just adopt formalism where the multiverse is the collection of all possible universes constrained only by the formal axioms, ZFC, so that truth is reduced to truth within V_ω (i.e. to truth within the integers)?

As I have indicated, the actual argument against the generic-multiverse position is a much more compelling one. This is because the reduction of truth for Π_2 assertions to truth about $H(\delta_0^+)$ is much stronger than is abstractly indicated by the mere violation of the two multiverse laws. It seems incoherent to me to have a conception of the transfinite which reduces to simply a conception of $H(\delta_0^+)$ which in essence is just the truncation of the universe of sets to the level of the least Woodin cardinal.

6 The Weak Multiverse Laws and $H(c^+)$

The standard structure for *Third Order Number Theory* is the structure, $\langle \mathcal{P}(\mathbb{R}), \mathbb{R}, \cdot, +, \in \rangle$. This structure is logically equivalent to $H(c^+)$ which is the set of all sets X whose transitive closure has cardinality at most $c = 2^{\aleph_0}$.

The multiverse truths of $H(c^+)$ are those sentences ϕ which hold in the $H(c^+)$ of each universe of the multiverse. The following are the natural formally weaker versions of the two multiverse laws which are obtained by simply replacing $H(\delta_0^+)$ by $H(c^+)$. For the Second Multiverse Law, this gives the weakest version which is not provable for the generic-multiverse assuming the existence of a proper class of Woodin cardinals; i.e., replacing $H(c^+)$ by $H(c)$ yield a second multiverse law which is provable for the generic-multiverse assuming the existence of a proper class of Woodin cardinals.

Weak First Multiverse Law

The set of Π_2 assertions which are multiverse truths is not recursive in the set of multiverse truths of $H(c^+)$. □

Weak Second Multiverse Law

The set of Π_2 assertions which are multiverse truths, is not definable in $H(c^+)$ across the multiverse. □

The following theorem shows that assuming there exist a proper class of Woodin cardinals and that the Ω Conjecture holds, then the generic-multiverse position violates the Weak First Multiverse Law.

Theorem 19. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then the set $\mathcal{V}_\Omega$ is recursive in the set $\mathcal{V}_\Omega(H(c^+))$.* □

Theorem 9 also holds for the weak multiverse laws.

Theorem 20. *Suppose that M is a countable transitive set*

$$M \vDash \text{ZFC} + \text{“There is a proper class of Woodin cardinals”}$$

and that $M \cap \text{Ord}$ is as small as possible. Then $\mathbb{V}_M$ violates both weak multiverse laws. □

The issue of whether the generic-multiverse position violates the Weak Second Multiverse Law is much more subtle and the issue is whether assuming the Ω Conjecture one can show that $\mathcal{V}_\Omega$ is definable in $H(c^+)$ across the generic-multiverse. I originally thought I could prove this but the proof was based on an implicit restriction regarding the universally Baire sets, see [14] for more details.

The definability of $\mathcal{V}_\Omega$ in $H(c^+)$ is a consequence of the Ω Conjecture augmented by the following conjecture. The statement involves both AD^+ which is a technical variant of AD, the *Axiom of Determinacy*, and the notion that a set $A \subseteq \mathbb{R}$ be ω_1-universally Baire. The definition of AD^+ and a brief survey of some of the basic aspects of the theory of AD^+ are given in [14].

Suppose that δ is an infinite cardinal. A set $A \subset \mathbb{R}$ is δ-universally Baire if for all compact Hausdorff spaces Ω and for all continuous functions

$$\pi : \Omega \to \mathbb{R}$$

if the topology of Ω has a basis of cardinality κ for some $\kappa \leq \delta$, then the preimage of A by π has the property of Baire in Ω. Clearly A is universally Baire if and only if A is δ-universally Baire for all δ.

Definition 21 (AD^+ **Conjecture**). Suppose that $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ each satisfy AD^+. Suppose that every set

$$X \in (L(A, \mathbb{R}) \cup L(B, \mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is ω_1-universally Baire. Then either

$$(\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})}$$

or

$$(\underset{\sim}{\Delta}^2_1)^{L(B,\mathbb{R})} \subseteq (\underset{\sim}{\Delta}^2_1)^{L(A,\mathbb{R})}$$ □

There is a stronger version of this conjecture.

Definition 22 (**Strong** AD^+ **Conjecture**). Suppose that $L(A, \mathbb{R})$ and $L(B, \mathbb{R})$ each satisfy AD^+. Suppose that every set

$$X \in (L(A, \mathbb{R}) \cup L(B, \mathbb{R})) \cap \mathcal{P}(\mathbb{R})$$

is ω_1-universally Baire. Then either $A \in L(B, \mathbb{R})$ or $B \in L(A, \mathbb{R})$.

Assuming the AD^+ Conjecture one obtains a significant improvement on the calculation of the complexity of $\mathcal{V}_\Omega$ and here, as opposed to the previous theorem, the distinction between $H(c^+)$ and $H(\omega_2)$ is critical.

Theorem 23 (AD^+ *Conjecture*). *Assume there is a proper class of Woodin cardinals. Then the set*

$$\mathcal{V}_\Omega = \{\phi \mid \emptyset \models_\Omega \phi\}$$

is definable in $H(c^+)$. □

Thus assuming the AD^+ Conjecture holds across the generic-multiverse then the generic-multiverse position violates the Weak Second Multiverse Law assuming of course that both the existence of a proper class of Woodin cardinals and the Ω Conjecture hold in V. Quite a number of combinatorial propositions are known to imply the AD^+ Conjecture [11], and the results of [14] offer evidence that the AD^+ Conjecture is true.

For my basic argument against the generic-multiverse position (assuming the Ω Conjecture), arguably the distinction between $H(c^+)$ and $H(\delta_0^+)$ is not relevant. It is however crucial for arguments such as those given in [12] that the Continuum Hypothesis is false. Those arguments are now contingent on *both* the Ω Conjecture and the AD^+ Conjecture. Nevertheless the results of [14] strongly suggest that these conjectures are both true. However the results of [14] also suggest that Continuum Hypothesis is true. This is not a contradiction. The basic argument against the Continuum Hypothesis in [12] is based on optimizing the theory of $H(\omega_2)$ and this approach cannot extend to even $H(c^+)$. While this approach is perhaps compelling from the perspective of just $H(\omega_2)$, it now seems likely that it will not be so compelling from the perspective of V.

7 Conclusions

If the Ω Conjecture is true then one cannot reasonably claim that the only true Π_2 assertions are those which are true across the generic-multiverse. Of course I am assuming large cardinals exist, in particular I am assuming that in some (and hence all) universes of the generic-multiverse, there is a proper class of Woodin cardinals.

For the skeptic who acknowledges this and yet claims that the problem of the Continuum Hypothesis is meaningless the challenge is the following:

> *Exhibit a* Π_2 *assertion which is true and which is not true across the generic-multiverse.*

The notion, $T \vdash_\Omega \phi$, has many features of the classical notion, $T \vdash \phi$. These features include a reasonable definition for the length of proof and so one can construct Gödel and Rosser sentences within Ω-logic. Moreover if the Ω Conjecture holds then $\mathcal{V}_\Omega$ is definable in $H(\delta_0^+)$. Thus if the Ω Conjecture holds then meeting the challenge posed above would seem to be entirely straightforward, being analogous to the challenge of exhibiting a sentence ψ for which the assertion

$$V_\omega \vDash \psi$$

is both true and not provable (in first order logic). In fact it would seem one could do much better and exhibit an assertion of the form

$$H(\delta_0^+) \vDash \phi$$

which is true but not true across the generic-multiverse.

However there is a feature of Ω logic which is not shared by classical logic. While it is true that if the Ω Conjecture holds then $\mathcal{V}_\Omega$ is definable in $H(\delta_0^+)$, the definition is *not uniform*. More precisely the actual definition of $\mathcal{V}_\Omega$ within $H(\delta_0^+)$ *depends* on the universe V. A strong form of this claim is given by the following theorem.

Theorem 24. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then for each formula $\Psi(x)$ there exists a complete Boolean algebra $\mathbb{B}$ such that*

$$V^{\mathbb{B}} \models \text{“}\mathcal{V}_\Omega \neq \left\{\phi \mid H(\delta_0^+) \models \Psi[\phi]\right\}\text{”}. \qquad \square$$

Thus, assuming there is a proper class of Woodin cardinals and that the Ω Conjecture holds, one *cannot* use either Gödel or Rosser style sentences to produce an assertion of the form

$$H(\delta_0^+) \models \psi$$

which is true and not Ω-valid; where a Π_2 assertion is Ω-*valid* if it holds in $V^{\mathbb{B}}$ for all complete Boolean algebras, $\mathbb{B}$. Any such construction would have to be based on a specific choice of the definition of $\mathcal{V}_\Omega$ within $H(\delta_0^+)$; and there is no possible choice which one can make. For exactly the same reasons, one cannot specify a single sentence ψ for which the assertion;

$$H(\delta_0^+) \models \psi$$

expresses, say, the Ω-consistency of ZFC in each universe of the generic-multiverse.

A similar argument applies to attempting to meet the challenge stated above–producing a Π_2 assertion which is true but not Ω-valid–because of the following stronger version of the previous theorem.

Theorem 25. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then for each Σ_2 formula $\Psi(x)$ there exists a complete Boolean algebra $\mathbb{B}$ such that*

$$V^{\mathbb{B}} \models \text{“}\mathcal{V}_\Omega \neq \{\phi \mid \Psi[\phi] \text{ holds }\}\text{”}. \qquad \square$$

Of course one could just conclude from all of this that the Ω Conjecture is *false*; see [13] for a more detailed discussion of this; declaring the Ω Conjecture to be meaningless is *not* an option since the Ω Conjecture is either true in all the universes of the generic-multiverse or false in all the universes of the generic-multiverse.

Note that even if the Ω Conjecture is false this does not necessarily resolve the objections to generic-multiverse position. For this end one would need something much stronger than the simple failure of the Ω Conjecture; one would need at the very least either that $\mathcal{V}_\Omega$ is not recursive in $\mathcal{V}_\Omega(H(\delta_0^+))$ or that $\mathcal{V}_\Omega$ is not definable in $H(\delta_0^+)$ across the generic-multiverse. If the Ω Conjecture is false then the problem of trying to understand the complexity of $\mathcal{V}_\Omega$ looks extremely difficult. For example, is it consistent for there to be a proper class of Woodin cardinals and for $\mathcal{V}_\Omega$ to be recursively equivalent to the set of all Π_2-sentences which hold in V (as happens if $V = L$) and so be as complicated as its natural definition suggests?

The skeptic might try a different approach even granting the Ω Conjecture and that there exists proper class of Woodin cardinals, by proposing that the generic-multiverse is *too small*. In particular why should a Π_2 assertion which is Ω-valid in *one* universe of the multiverse be true in *every* universe of the multiverse?

The counter to this approach is simply that, since the Ω Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse, if “$\emptyset \models_\Omega \phi$” holds in one universe of the multiverse then “$\emptyset \vdash_\Omega \phi$” holds in that universe. But then "$\emptyset \vdash_\Omega \phi$" must hold across the multiverse and so by the Soundness Theorem for Ω-logic, the sentence ϕ must hold in every universe of the multiverse. The argument that "$\emptyset \vdash_\Omega \phi$" must hold across the multiverse is made by appealing to the intricate connections between proofs in Ω-logic and notions of large cardinals; so one is really arguing that however the multiverse is defined, each universe of the multiverse is *as transcendent* as every other universe of the multiverse. Of course the Ω Conjecture is essential for this argument though interestingly the use of the Ω Conjecture is quite different here.

The Ω Conjecture is consistent (with a proper class of Woodin cardinals), more precisely if the theory

$$\text{ZFC} + \text{"There is a proper class of Woodin cardinals"}$$

is consistent then so is this theory together with the assertion that the Ω Conjecture is true. It is at present not known if the Ω Conjecture is consistently false.

The *Inner Model Program* is the attempt to generalize the definition of Gödel's constructible universe, L, to define (canonical) transitive inner models of the universe of sets for *large cardinal axioms*. Fairly general requirements on the structure of the inner model imply that the Ω Conjecture must hold in the inner model, further very recent results indicate that if this program can succeed at the level of supercompact cardinals then no large cardinal hypothesis whatsoever can refute the Ω Conjecture. Such an analysis would in turn strongly suggest that the Ω Conjecture is true; [14]. In summary there is (at present) a plausible framework for actually proving the Ω Conjecture–of course evidence is not a proof and this framework could collapse under an onslaught of theorems which reveal the true nature of sets.

But suppose that the Ω Conjecture is in fact provable (in classical logic, from the axioms for Set Theory). After all, as indicated above, all the evidence to date points to this possibility. What would this say about truth within Set Theory? It certainly would say that there is no evidence for the claim that the Continuum Hypothesis has no answer.

If the Ω Conjecture holds then there *must* be a Π_2 assertion which is true and which is not Ω-valid–in fact for essentially the same reasons there must be an assertion of the form,

$$H(\delta_0^+) \models \phi$$

for some sentence ϕ, which is true but not Ω-valid.

The latter class of Π_2 assertions, Φ, have the feature that they can be equivalently formulated as Σ_2 assertions–in the sense that there is a Σ_2 sentence, Φ', with the property that in each universe of the generic-multiverse generated by V, Φ holds if and only if Φ' holds. Any such Π_2 assertion, Φ, is qualitatively just like both the *Continuum Hypothesis* and its negation–assuming there is a proper class of Woodin cardinals there are complete Boolean algebras, $\mathbb{B}$, in which the assertion holds as interpreted in $V^{\mathbb{B}}$ and there are complete Boolean algebras, $\mathbb{B}$, in which the assertion fails as interpreted in $V^{\mathbb{B}}$. So if there is such a sentence Φ which is true then why could this not also be the case for Continuum Hypothesis (or its negation)?

I just do not see how one can maintain a position that there is any meaning to a conception of the transfinite universe beyond formalism, and yet be unwilling to acknowledge that there is some statement about $H(\delta_0^+)$ which is both true and not Ω-valid (unless the Ω Conjecture is false).

But again the skeptic can reasonably object: OK, even if the Continuum Hypothesis has an answer, is there any evidence whatsoever that we can or will ever determine what that answer is? The sympathetic skeptic might soften the position implicit in this question and simply claim that we are as far from finding and understanding the answer to the problem of the Continuum Hypothesis as the mathematicians studying the projective sets in the early 20th century were from finding and understanding the answer to the problem of Projective Uniformization.

Perhaps the generic-multiverse position can be (non-trivially) resurrected by adding single sentence to the axioms, ZFC, and still assuming both the Ω Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse. Adding a sentence, Ψ, to the axioms, ZFC, one would restrict the multiverse to a collection of universes where Ψ holds. Applying this restriction to the generic-multiverse gives the corresponding restricted generic-multiverse leading to a *revised generic-multiverse position*. This can be formalized within in any universe of the restricted generic-multiverse; Lemma 33, just as in the case for the generic-multiverse position. Since by adding a Π_2 sentence to the axioms one can preserve the truth of any given Σ_2 sentence, preserve essentially all large cardinals, and force the corresponding restricted generic-multiverse to contain only one universe, I shall restrict consideration to the case where the additional axiom is a Σ_2 sentence.

Suppose Ψ is a sentence and let $\mathcal{V}_\Omega^\Psi$ be the set of sentences, ϕ, such that

$$\{\Psi\} \models_\Omega \phi,$$

and let $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$ be the set of sentences, ϕ, such that

$$\text{ZFC} \cup \{\Psi\} \models_\Omega \text{“}H(\delta_0^+) \models \phi\text{”}.$$

Now if Ψ is a Σ_2 sentence then $\mathcal{V}_\Omega^\Psi$ is of the same Turing complexity as the set of Π_2 assertions which hold across the restricted generic-multiverse (where now Ψ is required to hold in each universe of the multiverse) and $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$ is the corresponding set of multiverse truths of $H(\delta_0^+)$.

Clearly $\mathcal{V}_\Omega^\Psi$ is recursive in $\mathcal{V}_\Omega$ and so by Theorem 18, the revised generic-multiverse position will still violate the Second Multiverse Law. What about the First Multiverse Law?

Theorem 26. *Assume there is a proper class of Woodin cardinals and that the Ω Conjecture holds. Then for each sentence Ψ, the set $\mathcal{V}_\Omega^\Psi$ is recursive in the set $\mathcal{V}_\Omega^\Psi(H(\delta_0^+))$.* $\square$

Thus assuming both the Ω Conjecture and the existence of a proper class of Woodin cardinals both hold across the multiverse then for all enlargements of ZFC by adding a single Σ_2 sentence to the axioms, ZFC, the revised generic-multiverse position still violates both the First Multiverse Law and the Second Multiverse Law. Assuming in addition that the AD^+ Conjecture holds across the generic-multiverse this violation extends to both the Weak First Multiverse Law and the Weak Second Multiverse Law. Though here as above, the distinction between $H(\omega_2)$ and $H(c^+)$ is critical.

I am an optimist, perhaps even a transfinitist. There is in my view no reason at all, beyond a lack of faith, for believing that there is no extension of the axioms ZFC, by one axiom, a posteriori true, which settles all instances of the Generalized Continuum Hypothesis and more generally which yields a theory of the universe of sets which is as "complete" as the theory of Gödel's constructible universe, L, which is given by the axioms ZFC. (Or as complete as the theory of the integers that is given by the axioms for that structure.) The new axiom of any such extension cannot be Ω-valid since in particular it must settle the Continuum Hypothesis.

Rephrasing my position slightly, there is in my view no credible evidence at present refuting the existence of a single additional axiom to ZFC which is consistent with large cardinal axioms and which in a practical sense provides a complete description of $H(\delta_0^+)$ or even of V_κ where κ is any cardinal which is definable within the universe of sets as the least cardinal with a Σ_2-property; where the gold standard for practical completeness is the theory of L as given by the ZFC axioms.

Until recently I have always viewed such a possibility as very implausible at best. The change in my view is motivated by the results of [14] which provide some evidence for the existence of such an axiom. This is not to say that in the final analysis I will not revert.

Why not the axiom "$V = L$"? The difficulty is that this is a limiting axiom for it refutes large cardinal axioms. The results of [14] suggest the possibility that if there is a *supercompact cardinal* then there is a generalization of L which is both *close* to V and which inherits large cardinals from V exactly as L inherits large cardinals from V if $0^\#$ does not exist. Should this turn out to be true, it would be remarkable. The axiom that V is such an inner model would have all the advantages of the axiom "$V = L$" without limiting V as far as large cardinals are concerned. In particular the often cited arguments against the axiom "$V = L$" *would not apply to this new axiom*.

A far stronger view than that outlined above and which I also currently hold because of the suggestive results of [14], is that there *must* be such an axiom and in understanding it we will understand why it is essentially unique and therefore true. Further this new axiom will in a transparent fashion both settle the classical questions of combinatorial set theory where to date independence has been the rule and explain the large cardinal hierarchy. There is already a specific candidate for this axiom [14] though it is not only too early to argue that this axiom is true, it is too early to be sure that this axiom is even consistent with all large cardinal axioms. The issue at present is that there are actually *two* families of enlargements of L and it is not yet clear whether *both* can be *transcendent* relative to large cardinal axioms, [14].

Even if this does happen (we find and understand this missing axiom), the specter of independence remains–it is just that now the vulnerability of Set Theory to the occurrence of independence becomes the same as that of Number Theory. In other words, we would have come to a conception of the transfinite universe which is as clear and unambiguous as our conception of the fragment V_ω, the universe of the finite integers. To me this a noteworthy goal to aspire to.

8 Appendix

The purpose of this appendix is to show how the generic-multiverse position can be formalized within V. In particular I shall prove that for each sentence ϕ there is a sentence ϕ^*, recursively depending on ϕ, such that for each countable transitive set M such that $M \vDash \text{ZFC}$, the following are equivalent:

(1) $M \vDash \phi^*$;

(2) For each $N \in \mathbb{V}_M$, $N \vDash \phi$;

where as defined on page 2, $\mathbb{V}_M$ is the generic-multiverse generated (in V) by M. In fact this is a straightforward corollary of Lemma 27 and Lemma 28 below noting that to verify that $N \vDash \phi$ for each $N \in \mathbb{V}_M$ one need only verify $N \vDash \phi$ for each transitive set N such that N can be generated from M in only 3 steps (actually 2 steps as noted by Hamkins) of taking generic enlargements or generic refinements (as opposed to finitely many steps). However the proof I shall give easily adapts to prove the corresponding result for *any* restricted generic-multiverse position obtained by limiting the generic extensions to those given by a definable class of partial orders, Lemma 33. In this general case, the reduction to models generated in only 3 steps (or any fixed finite number of steps) from the initial model M is not always possible.

Let $\text{ZC}^{(\text{VN})}$ denote the axioms ZC together with the axiom which asserts that for all ordinals α, V_α exists.

I fix some notation generalizing the definition of $\mathbb{V}_M$ to the case that M is a countable transitive set such that

$$M \vDash \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}.$$

Note that

$$V_{\omega+\omega} \vDash \text{ZC}^{(\text{VN})},$$

but for all (ordinals) $\lambda > \omega$,

$$V_\lambda \vDash \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement},$$

if and only if $|V_\lambda| = \lambda$.

Let $\mathbb{V}_M$ be the generic-multiverse generated by M. This is the smallest collection of transitive sets such that $M \in \mathbb{V}_M$ and such that for all pairs, (M_1, M_2), of countable transitive sets if

(1) $M_1 \vDash \text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}$,

(2) M_2 is a generic extension of M_1,

(3) either $M_1 \in \mathbb{V}_M$ or $M_2 \in \mathbb{V}_M$,

then both M_1 and M_2 are in $\mathbb{V}_M$.

The only change here from the definition of $\mathbb{V}_M$ on page 2, is that here I am not requiring that $M \vDash \text{ZFC}$; but just that M be a model of the weaker set of axioms,

$$\text{ZC}^{(\text{VN})} + \Sigma_1\text{-Replacement}.$$

The following lemma has an interesting corollary. Suppose M is a transitive set, $M \models \mathrm{ZFC}$, and that $M[G]$ is a generic extension of M. Then M is definable in $M[G]$ from parameters. This in turn implies that the property of being a generic extension is first-order. The lemma is motivated by [4] and both a version of this lemma and the application indicated above are due independently to Laver, [6]; also see Reitz [10] for further developments.

Lemma 27. *Suppose that M, M' are transitive sets,*

$$\mathbb{P} \in M \cap M'$$

and $G \subset \mathbb{P}$. Suppose:

(i) $M \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement*;*

(ii) $M' \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement*;*

(iii) $\mathcal{P}(\mathbb{P}) \cap M = \mathcal{P}(\mathbb{P}) \cap M'$*;*

(iv) *G is M-generic for $\mathbb{P}$;*

(v) $M[G] = M'[G]$.

Then $M = M'$.

Proof. Suppose toward a contradiction that $M \neq M'$. Then since

$$M \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

it follows that $M \cap \mathcal{P}(\mathrm{Ord}) \neq M' \cap \mathcal{P}(\mathrm{Ord})$.

Let $\gamma \in \mathrm{Ord} \cap M$ be least such that

$$\mathcal{P}(\gamma) \cap M \neq \mathcal{P}(\gamma) \cap M'.$$

Clearly γ is a cardinal in both M and M'. Let $\kappa = |\mathbb{P}|^M$. Thus $(\kappa^+)^M$ is a cardinal in $M[G]$ and since,

$$\mathcal{P}(\mathbb{P}) \cap M = \mathcal{P}(\mathbb{P}) \cap M',$$

it follows that $\gamma > \kappa$. By interchanging M and M' if necessary we can suppose that there exists a set

$$A \in \mathcal{P}(\gamma) \cap M' \backslash M.$$

Thus by choice of γ, for all $\alpha < \gamma$, $A \cap \alpha \in M$. Further since $A \in M[G]$, it follows that

$$(\mathrm{cof}(\gamma))^M \leq \kappa.$$

Therefore γ is not a regular cardinal in M and so $\gamma > (\kappa^+)^M$. Let

$$\delta = (\kappa^+)^M = (\kappa^+)^{M'} = (\kappa^+)^{M[G]}.$$

The key point is that in $M[G]$ the following hold.

(1.1) $\mathcal{P}(\gamma) \cap M'$ is closed in under strictly increasing unions of length δ.

(1.2) For every set $a \subset \gamma$ with $|a|^{M[G]} \leq \delta$, there exists $b \subset \gamma$ such that $a \subseteq b$, $|b|^{M[G]} \leq \delta$, and such that $b \in \mathcal{P}(\gamma) \cap M'$.

(1.3) For each $\xi < \gamma$, $\mathcal{P}(\xi) \cap M' = \mathcal{P}(\xi) \cap M$.

(1.4) $A \in \mathcal{P}(\gamma) \cap M' \backslash M$.

Let $\tau \in M$ be a term for $\mathcal{P}(\gamma) \cap M'$. Let $p \in G$ be a condition which forces that (1.1)–(1.3) hold for $I_G(\tau)$, where $I_G(\tau)$ is the interpretation of τ in $M[G]$. Let $\sigma \in M$ be a term for A. By shrinking p if necessary we can suppose that p forces both $I_G(\sigma) \in I_G(\tau)$ and that $I_G(\sigma) \notin M$.

We now work in M. Let

$$X = \{Z \subset \gamma \mid \kappa \subset Z, |Z| \leq \delta\}$$

and let S be the set of all $Z \in X$, such that p forces $Z \in I_G(\tau)$. Since p forces that (1.1)–(1.2) hold for $I_G(\tau)$, it follows that S is stationary as a subset of $\{Z \subset \gamma \mid |Z| \leq \delta\}$. To see this let

$$H : \gamma^{<\omega} \to \gamma.$$

We must find $Z \in S$ such that $H[Z^{<\omega}] \subset Z$. Let

$$\langle (Z_\alpha, \tau_\alpha) : \alpha < \delta \rangle$$

be a sequence such that for all $\alpha < \beta < \delta$,

(2.1) $p \Vdash \tau_\alpha \in \tau$,

(2.2) $p \Vdash \tau_\alpha \subseteq Z_\beta$,

(2.3) $p \Vdash Z_\alpha \subseteq \tau_\beta$,

(2.4) $|Z_\alpha| \leq \delta$ and $H[Z_\alpha^{<\omega}] \subset Z_\alpha$.

Such a sequence is easily constructed by induction. Notice that for all $\alpha < \beta$, $p \Vdash \tau_\alpha \subseteq \tau_\beta$. Thus

$$p \Vdash \cup \{\tau_\alpha \mid \alpha < \delta\} \in \tau.$$

But letting $Z = \cup \{Z_\alpha \mid \alpha < \delta\}$,

$$p \Vdash \cup \{\tau_\alpha \mid \alpha < \delta\} = Z,$$

and so $Z \in S$. Clearly $H[Z^{<\omega}] \subset Z$ and this shows that S is stationary.

For each $Z \in S$, there exist $p_Z \leq p$ and $A_Z \subseteq Z$ such that p_Z forces that

$$I_G(\sigma) \cap Z = A_Z.$$

This is because p forces that (1.3)–(1.4) holds for $I_G(\tau)$. But $|\mathbb{P}|^M = \kappa$ and $\kappa \subset Z$ for each $Z \in S$. Therefore there exists $q \leq p$ such that

$$S_q = \{Z \in S \mid p_Z = q\}$$

is stationary as a subset of $\{Z \subset \gamma \mid |Z| \leq \delta\}$. For each $Z_1, Z_2 \in S_q$, it follows that

$$A_{Z_1} \cap Z_1 \cap Z_2 = A_{Z_2} \cap Z_1 \cap Z_2.$$

Thus there exists a set $A_q \subset \gamma$ such that for all $Z \in S_q$, $A_q \cap Z = A_Z$. But then q forces $I_G(\sigma) = A_q$ which contradicts the choice of σ and p.

Therefore $M \cap \mathcal{P}(\mathrm{Ord}) = M' \cap \mathcal{P}(\mathrm{Ord})$ and so $M = M'$. □

The next lemma is a corollary of Lemma 27.

Lemma 28. *Suppose that N is transitive, the set*

$$\left\{\xi \in N \cap \mathrm{Ord} \mid N \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}\right\},$$

is cofinal in $N \cap \mathrm{Ord}$, $\alpha \in N \cap \mathrm{Ord}$, and that $N_\alpha \prec_{\Sigma_2} N$.

Suppose that $M \in N$ is a transitive set and:

(i) $M \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement*;*

(ii) $\mathbb{P} \in M$*,* $G \subseteq \mathbb{P}$*, and* G *is* M*-generic for* $\mathbb{P}$*;*

(iii) $G \in N$*;*

(iv) $M[G] = N \cap V_\alpha$.

Then there exists a transitive set $M^* \subset N$ such that

(1) $M^* \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement,

(2) $M^* \cap V_\alpha = M$,

(3) $N = M^*[G]$.

Proof. Let I be the set of $\xi \in N \cap \mathrm{Ord}$ such that

$$N \cap V_\xi \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}$$

and such that $\mathbb{P} \in N \cap V_\xi$.

Thus $\alpha \in I$ and both the hypothesis on N,

(1.1) $I \cap \alpha$ is cofinal in α,

(1.2) I is cofinal in $N \cap \mathrm{Ord}$.

Let I' be the set of $\xi \in I$ such that there exists a transitive set $M' \in N$ such that

(2.1) $M' \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement,

(2.2) $\mathbb{P} \in M'$,

(2.3) $\mathcal{P}(\mathbb{P}) \cap M' = \mathcal{P}(\mathbb{P}) \cap M$,

(2.4) $N \cap V_\xi = M'[G]$.

Clearly $\alpha \in I'$ and $I \cap \alpha \subset I'$. Therefore since

$$N \cap V_\alpha \prec_{\Sigma_2} N,$$

it follows $I \subset I'$. Thus $I = I'$.

For each $\xi \in I$ let $M_\xi \in N$ be such that

(3.1) $M_\xi \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement,

(3.2) $\mathbb{P} \in M_\xi$,

(3.3) $\mathcal{P}(\mathbb{P}) \cap M_\xi = \mathcal{P}(\mathbb{P}) \cap M$,

(3.4) $N \cap V_\xi = M_\xi[G]$.

By Lemma 27, for each $\xi_1 < \xi_2$ in I,

$$M_{\xi_1} = M_{\xi_2} \cap V_{\xi_1}.$$

Let $M^* = \cup\left\{M_\xi \mid \xi \in I\right\}$. Thus

(4.1) $M = M^* \cap V_\alpha$,

(4.2) $N = M^*[G]$,

(4.3) for each $\xi \in I$, $M^* \cap V_\xi \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement.

Finally by (4.3),

$$M^* \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

since I is cofinal in $M^* \cap \mathrm{Ord}$. ◻

Lemma 29. *Suppose that $M \subset N$ are transitive sets,*

$$\mathbb{P} \in M$$

and $G \subset \mathbb{P}$ satisfy:

(i) $M \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement*;*

(ii) $N \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement*;*

(iii) *G is M-generic;*

(iv) $N = M[G]$.

Suppose $k \geq 1$, $\gamma \in N \cap \mathrm{Ord}$, $\mathbb{P} \in N \cap V_\gamma$, and that the set

$$\left\{\xi \in N \cap \mathrm{Ord} \mid N \cap V_\xi \prec_{\Sigma_k} N\right\}$$

is cofinal in $N \cap \mathrm{Ord}$.

Then

$$M \cap V_\gamma \prec_{\Sigma_k} M$$

if and only if $N \cap V_\gamma \prec_{\Sigma_k} N$.

Proof. The case $k = 1$ is immediate since for $\gamma > \omega$,

$$N \cap V_\gamma \prec_{\Sigma_1} N$$

if and only if $|N \cap V_\gamma|^N = \gamma$ and

$$M \cap V_\gamma \prec_{\Sigma_1} M$$

if and only if $|M \cap V_\gamma|^M = \gamma$.

We now suppose that $k > 1$. If

$$M \cap V_\gamma \prec_{\Sigma_k} M$$

then it follows easily that

$$N \cap V_\gamma \prec_{\Sigma_k} N.$$

This is because $N = M[G]$ and $\mathbb{P} \in M \cap V_\gamma$.

Finally suppose that

$$N \cap V_\gamma \prec_{\Sigma_k} N.$$

Let I_1^M be the set of $\xi \in M \cap \mathrm{Ord}$ such that

$$M \cap V_\xi \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}$$

and such that $\mathbb{P} \in M \cap V_\xi$. Clearly

$$I_1^M = \left\{\xi \in M \cap \mathrm{Ord} \mid M \cap V_\xi \prec_{\Sigma_1} M\right\} \cap \left\{\xi \in M \cap \mathrm{Ord} \mid \mathbb{P} \in M \cap V_\xi\right\},$$

and I_1^M is also the set of $\xi \in N \cap \mathrm{Ord}$ such that

$$N \cap V_\xi \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}$$

and such that $\mathbb{P} \in N \cap V_\xi$.

By Lemma 27 and Lemma 28, it follows that the set

$$\left\{(\xi, M \cap V_\xi) \mid \xi \in I_1^M\right\}$$

is Π_1-definable in N from the parameter,

$$(\mathbb{P}, G, \mathcal{P}(\mathbb{P}) \cap M).$$

Since

$$M \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

for each $\xi \in I_1^M$, $M \cap V_\xi \prec_{\Sigma_1} M$.

For each $1 \leq n \leq k$ let I_n^M be the set of $\xi \in I_1^M$ such that

$$M \cap V_\xi \prec_{\Sigma_n} M.$$

We prove by induction on n:

(1.1) $I_n^M = \left\{\xi \in I_1^M \mid N \cap V_\xi \prec_{\Sigma_n} N\right\}$;

(1.2) $\left\{(\xi, M \cap V_\xi) \mid \xi \in I_n^M\right\}$ is Π_n-definable in N from the parameter,

$$(\mathbb{P}, G, \mathcal{P}(\mathbb{P}) \cap M).$$

We have just proved (1.1)–(1.2) for $n = 1$, so we suppose that $1 < n < k$ and (1.1)–(1.2) hold for n. We first prove (1.1) holds for $n + 1$ and for this it suffices to simply show that

$$\left\{\xi \in I \mid N \cap V_\xi \prec_{\Sigma_{n+1}} N\right\} \subseteq I_{n+1}^M.$$

Suppose that $\xi \in I_1^M$ and

$$N \cap V_\xi \prec_{\Sigma_{n+1}} N.$$

We must prove that $M \cap V_\xi \prec_{\Sigma_{n+1}} M$. Since $n < k$ we have that

$$\left\{\eta \in N \cap \mathrm{Ord} \mid N \cap V_\eta \prec_{\Sigma_n} N\right\}$$

is cofinal in $N \cap \mathrm{Ord}$. Therefore by the induction hypothesis,

$$M \cap V_\xi \prec_{\Sigma_{n+1}} M$$

if and only if

$$\left(M \cap V_\xi, \left\{M \cap V_{\xi'} \mid \xi' \in I_n^M \cap \xi\right\}\right) \prec_{\Sigma_1} \left(M, \left\{M \cap V_{\xi'} \mid \xi' \in I_n^M\right\}\right).$$

Since (1.2) holds for n and since

$$N \cap V_\xi \prec_{\Sigma_{n+1}} N,$$

it follows that

$$\left(N \cap V_\xi, \left\{M \cap V_{\xi'} \mid \xi' \in I_n^M \cap \xi\right\}\right) \prec_{\Sigma_1} \left(N, \left\{M \cap V_{\xi'} \mid \xi' \in I_n^M\right\}\right),$$

and so $M \cap V_\xi \prec_{\Sigma_{n+1}} M$.

Finally we prove that (1.2) holds for $n+1$. Note that $M \cap V_\xi \prec_{\Sigma_{n+1}} M$ if and only if for all $a \in M \cap V_\xi$ and for all formulas $\phi(x)$ if

$$\left\{(\xi', M \cap V_{\xi'}) \mid \xi' \in I_n^M \text{ and } M \cap V_{\xi'} \models \phi[a]\right\} \neq \emptyset$$

then

$$\left\{(\xi', M \cap V_{\xi'}) \mid \xi' \in I_n^M \text{ and } M \cap V_{\xi'} \models \phi[a]\right\} \cap V_\xi \neq \emptyset.$$

This implies that (1.2) holds for $n+1$ since (1.2) holds for n.

Thus (1.1) and (1.2) hold for all $n \leq k$ and in particular (1.1) holds for $n = k$. Therefore

$$M \cap V_\gamma \prec_{\Sigma_k} M,$$

and this completes the proof of the lemma. □

Lemma 30. *Suppose that N_1 is a countable transitive set,*

$$N_1 \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

and that $k \geq 1$. Let

$$I = \left\{\gamma \in N_1 \cap \mathrm{Ord} \mid N_1 \cap V_\gamma \prec_{\Sigma_k} N_1\right\}$$

and suppose that I is cofinal in $N_1 \cap$ Ord.

(1) *Suppose $\gamma \in I$. Then for each $N \in \mathbb{V}_{N_1 \cap V_\gamma}$ there exists $N^* \in \mathbb{V}_{N_1}$ such that $N = N^* \cap V_\gamma$ and*

$$N \prec_{\Sigma_k} N^*.$$

(2) *Suppose that $N \in \mathbb{V}_{N_1}$. Then there exists $\xi \in N_1 \cap$ Ord such that for all $\gamma \in I \backslash \xi$,*

$$N \cap V_\gamma \in \mathbb{V}_{N \cap V_\gamma}$$

and $N \cap V_\gamma \prec_{\Sigma_k} N$.

Proof. We first prove (1). Fix $\gamma \in I$, let $N_2 = N_1 \cap V_\gamma$ and fix $N \in \mathbb{V}_{N_2}$. Then there exists a finite sequence,

$$\langle (\mathbb{P}_i, G_i, M_i) : i \leq m \rangle$$

such that

(1.1) $M_0 = N_2$ and $M_m = N$,

(1.2) for all $i < m$, $M_i \in \mathbb{V}_{N_2}$,

(1.3) for all $i + 1 \leq m$ either

a) $\mathbb{P}_i \in M_i$, $G_i \subset \mathbb{P}_i$ is M_i-generic and $M_{i+1} = M_i[G_i]$, or

b) $\mathbb{P}_i \in M_{i+1}$, $G_i \subset \mathbb{P}_i$ is M_{i+1}-generic and $M_i = M_{i+1}[G_i]$.

We prove by induction on $i \leq m$ that there exists $M_i^* \in \mathbb{V}_{N_1}$ such that

(2.1) $M_i^* \cap V_\gamma = M_i$,

(2.2) $M_i \prec_{\Sigma_k} M_i^*$.

If $i = 0$ then $M_0^* = N_1$ and (2.1)–(2.2) are immediate.

Suppose M_i^* exists satisfying (2.1)–(2.2) and that $i + 1 \leq m$. There are two cases. The first case is that $M_{i+1} = M_i[G_i]$. Then set $M_{i+1}^* = M_i^*[G_i]$. Thus $M_{i+1}^* \in \mathbb{V}_{N_1}$ since $M_i^* \in \mathbb{V}_{N_1}$ and by Lemma 29

$$M_{i+1} \prec_{\Sigma_k} M_{i+1}^*.$$

The second case is that $M_i = M_{i+1}[G_i]$. By Lemma 28 there exists M_{i+1}^* such that $M_i^* = M_{i+1}^*[G_i]$ and such that $M_{i+1} = M_{i+1}^* \cap V_\gamma$. By Lemma 29 and the induction hypothesis,

$$M_{i+1} \prec_{\Sigma_k} M_{i+1}^*.$$

This proves that for all $i \leq m$, M_i^* exists satisfying (2.1)–(2.2). Set $N^* = M_m^*$. This proves (1). The proof of (2) is similar. □

Lemma 31. *Suppose M is a countable transitive set,*

$$M \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

$\gamma \in M \cap \mathrm{Ord}$ *and that*

$$M_\gamma \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}.$$

Suppose $G \subset \mathrm{Coll}(\omega, \gamma)$ is M-generic. Then for each sentence ϕ the following are equivalent.

(1) *For all $N \in \mathbb{V}_{M\cap V_\gamma}$, $N \models \phi$.*

(2) $M[G] \models$ "For all $N \in \mathbb{V}_{M\cap V_\gamma}$, $N \models \phi$".

Proof. By absoluteness,

$$M[G] \cap V_{\omega+1} \prec_{\Sigma_1} V_{\omega+1}$$

and

$$\mathbb{V}_{M\cap V_\gamma} \cap M[G] = \left(\mathbb{V}_{M\cap V_\gamma}\right)^{M[G]}.$$

The lemma follows. □

Suppose that ϕ is a sentence and let n be the length of ϕ. Let ϕ^* be a sentence which expresses:

> *Suppose γ is an ordinal such that*
>
> $$V_\gamma \prec_{\Sigma_{n+1}} V$$
>
> *and that $X \prec V_\gamma$ is a countable elementary substructure. Let M_X be the transitive collapse of X. Then*
>
> $$N \models \phi$$
>
> *for each $N \in \mathbb{V}_{M_X}$.*

The next lemma shows that ϕ^* is as required.

Lemma 32. *Suppose that M is a countable transitive set such that*

$$M \models \mathrm{ZFC}.$$

Then the following are equivalent.

(1) $M \models \phi^*$.

(2) *For each $N \in \mathbb{V}_M$, $N \models \phi$.*

Proof. Let γ be an ordinal of M such that

$$M \cap V_\gamma \prec_{\Sigma_{n+1}} M$$

and let

$$I = \left\{\xi \in M \cap \mathrm{Ord} \mid M \cap V_\xi \prec_{\Sigma_n} M\right\}.$$

We first show that (1) implies (2). Suppose $\xi \in I \cap \gamma$. We claim that for all $N \in \mathbb{V}_{M \cap V_\xi}$, $N \models \phi$. To see this let

$$X \prec M \cap V_\gamma$$

be an elementary substructure such that $\xi \in X$, $X \in M$, and such that $|X|^M = \omega$. Let M_X be the transitive collapse of X and let ξ_X be the image of ξ under the collapsing map. Thus

$$M_X \models \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement},$$

and $M_X \cap V_{\xi_X} \prec_{\Sigma_n} M_X$.

By (1), $M \models \phi^*$. Therefore by absoluteness and since

$$(\mathbb{V}_{M_X})^M = \mathbb{V}_{M_X} \cap M,$$

for all $N \in \mathbb{V}_{M_X}$, $N \models \phi$. Therefore if $G \subset \mathrm{Coll}(\omega, \xi)$ is M_X-generic then for all $N \in \mathbb{V}_{M_X} \cap M_X[G]$, $N \models \phi$. Since $X \prec M \cap V_\gamma$ it follows that if $G \subset \mathrm{Coll}(\omega, \xi)$ is M-generic then for all $N \in \mathbb{V}_{M \cap V_\xi} \cap M[G]$, $N \models \phi$ and so by Lemma 31, for all $N \in \mathbb{V}_{M \cap V_\xi}$, $N \models \phi$.

In summary we have proved that for each $\xi \in I \cap \gamma$, for all $N \in \mathbb{V}_{M \cap V_\xi}$, $N \models \phi$. By induction on k it follows that for each $k \geq 1$, the

$$\left\{M \cap V_\xi \mid M \cap V_\xi \prec_{\Sigma_k} M\right\}$$

is Π_k-definable in M. Therefore by Lemma 31, and since

$$M_\gamma \prec_{\Sigma_{n+1}} M,$$

it follows that for each $\xi \in I$, for all $N \in \mathbb{V}_{M \cap V_\xi}$, $N \models \phi$.

Finally suppose that $N \in \mathbb{V}_M$. Then by Lemma 30, there exists $\xi \in I$ such that $N \cap V_\xi \in \mathbb{V}_{M \cap V_\xi}$ and such that

$$N \cap V_\xi \prec_{\Sigma_n} N.$$

Since ϕ has length at most n, $N \models \phi$ if and only if

$$N \cap V_\xi \models \phi.$$

Therefore $N \models \phi$. This proves (2).

We now suppose (2) holds and prove (1). Let

$$X \prec M \cap V_\gamma$$

be an elementary substructure such that $X \in M$ and such that $|X|^M = \omega$. Let M_X be the transitive collapse of X. We must show that for all $N \in \mathbb{V}_{M_X} \cap M$, $N \vDash \phi$.

Let

$$I_X = \left\{\xi \in M_X \cap \mathrm{Ord} \mid M_X \cap V_\xi \prec_{\Sigma_n} M_X\right\}.$$

Thus I_X is the image of $I \cap X$ under the collapsing map.

By (2) and Lemma 30(1), for each $\xi \in I$, for each $N \in \mathbb{V}_{M \cap V_\xi}$, $N \vDash \phi$. Therefore by Lemma 31, for each $\xi \in I_X$, for each $N \in \mathbb{V}_{M_X \cap V_\xi}$, $N \vDash \phi$.

Suppose $N \in \mathbb{V}_{M_X} \cap M$. Then by Lemma 30(2), there exists $\gamma_0 \in I_X$ such that $N \cap V_{\gamma_0} \in \mathbb{V}_{M_X \cap V_{\gamma_0}}$ and such that

$$N \cap V_{\gamma_0} \prec_{\Sigma_n} N.$$

Since $N \cap V_{\gamma_0} \in \mathbb{V}_{M_X \cap V_{\gamma_0}}$, $N \cap V_{\gamma_0} \vDash \phi$ and therefore since $N \cap V_{\gamma_0} \prec_{\Sigma_n} N$, $N \vDash \phi$.

Therefore , for all $N \in \mathbb{V}_{M_X} \cap M$, $N \vDash \phi$. This proves (1). $\square$

I finish by discussing the generalization of Lemma 32 to any restricted generic-multiverse of the following form.

Suppose $\psi(x)$ is a formula in the language of set theory and that M is a countable transitive set such that

$$M \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1\text{-Replacement}.$$

Let $\mathbb{V}_M^{(\psi)} \subseteq \mathbb{V}_M$ be the be smallest set such that such that $M \in \mathbb{V}_M^{(\psi)}$ and such that for all pairs, (M_1, M_2), of countable transitive sets if

(1) $M_1 \vDash \mathrm{ZC}^{(\mathrm{VN})} + \Sigma_1$-Replacement,

(2) $M_2 = M_1[G]$ for some set G such that G is M_1-generic for some partial order $\mathbb{P}$ such that $M_1 \vDash \psi[\mathbb{P}]$,

(3) either $M_1 \in \mathbb{V}_M^{(\psi)}$ or $M_2 \in \mathbb{V}_M^{(\psi)}$,

then both M_1 and M_2 are in $\mathbb{V}_M^{(\psi)}$.

For example one can choose ψ such that

$$\mathbb{V}_M^{(\psi)} = \left\{N \in \mathbb{V}_M \mid (\omega_1)^M = (\omega_1)^N\right\}$$

or such that

$$\mathbb{V}_M^{(\psi)} = \left\{N \in \mathbb{V}_M \mid (\text{cardinals})^M = (\text{cardinals})^N\right\}.$$

Suppose that ϕ is a sentence and let $n = \mathrm{length}(\phi) + \mathrm{length}(\psi)$. Let $\phi^{(\psi)}$ be a sentence which expresses:

> *Suppose γ is an ordinal such that*
>
> $$V_\gamma \prec_{\Sigma_{n+1}} V$$
>
> *and that $X \prec V_\gamma$ is a countable elementary substructure. Let M_X be the transitive collapse of X. Then*
>
> $$N \vDash \phi$$
>
> *for each $N \in \mathbb{V}_{M_X}^{(\psi)}$.*

The proof of Lemma 32 easily adapts to prove the following generalization.

Lemma 33. *Suppose that M is a countable transitive set such that*

$$M \vDash \mathrm{ZFC}.$$

Then the following are equivalent.

(1) $M \vDash \phi^{(\psi)}$.

(2) *For each $N \in \mathbb{V}_M^{(\psi)}$, $N \vDash \phi$.*

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