The triple helix

John R. Steel
University of California, Berkeley

October 2010

Three staircases

Plan:

I. The interpretability hierarchy.
II. The vision of "ultimate K".
III. The triple helix.
IV. Some locator axioms.
V. Some basic open problems.

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$\text{Arithmetic}_T \subseteq \text{Arithmetic}_U,$$

or

$$\text{Arithmetic}_U \subseteq \text{Arithmetic}_T.$$

The Interpretability Hierarchy

For T a theory, let Arithmetic_T be the set of consequences of T in the language of first order arithmetic.

Phenomenon: If T is a natural extension of ZFC, then there is an extension S axiomatized by large cardinal hypotheses such that

$$\text{Arithmetic}_T = \text{Arithmetic}_S.$$

Moreover, if T and U are natural extensions of ZFC, then

$$\text{Arithmetic}_T \subseteq \text{Arithmetic}_U,$$

or

$$\text{Arithmetic}_U \subseteq \text{Arithmetic}_T.$$

In practice, $\text{Arithmetic}_T \subseteq \text{Arithmetic}_U$ iff PA proves $\text{Con}(U) \Rightarrow \text{Con}(T)$.

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Phenomenon: Let T, U be natural theories such that Arithmetic$_S \subseteq$ Arithmetic$_T$ and Arithmetic$_S \subseteq$ Arithmetic$_U$. Then either

$$\text{1-Arithmetic}_T \subseteq \text{1-Arithmetic}_U,$$

or

$$\text{1-Arithmetic}_U \subseteq \text{1-Arithmetic}_T.$$

Definition

For any theory T in LST, 1-Arithmetic$_T$ is the set of consequences of T of the form "$V_{\omega+1} \models \varphi$".

Let S be the theory ZFC plus "there are infinitely many Woodin cardinals."

Phenomenon: Let T, U be natural theories such that Arithmetic$_S \subseteq$ Arithmetic$_T$ and Arithmetic$_S \subseteq$ Arithmetic$_U$. Then either

$$\text{1-Arithmetic}_T \subseteq \text{1-Arithmetic}_U,$$

or

$$\text{1-Arithmetic}_U \subseteq \text{1-Arithmetic}_T.$$

So at the level of sentences about $V_{\omega+1}$, we know of only **one road upward**. We are led to it many different ways. Strong axioms of infinity are its central markers.
CH is a sentence about $V_{\omega+2}$.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.

Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.

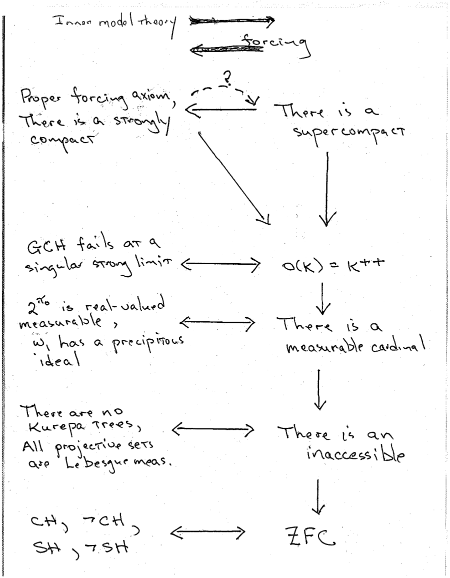
Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty (V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.
4. We shall stick to the initial segment of the interpretability hierarchy where the natural markers are conventional large cardinals, preserved by small forcing.

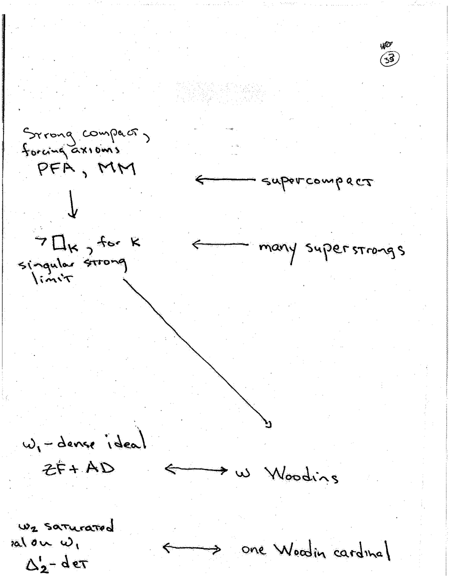
Remarks.

1. Directedness, not linearity, is what is important for a "well-determined" theory of $(V_{\omega+1}, \in)$.
2. The phenomenon extends to sentences of the form "$\exists A \in Hom_\infty(V_{\omega+1}, \in, A) \models \psi$)." These are called $(\Sigma^2_1)^{Hom_\infty}$ *statements*.
3. By Levy-Solovay, it does not extend to arbitrary Σ^2_1 statements like CH, *if* our consistency-strength lower bound is one of the conventional large cardinals. These are preserved by small forcing, so cannot decide CH.
4. We shall stick to the initial segment of the interpretability hierarchy where the natural markers are conventional large cardinals, preserved by small forcing.

Some equiconsistencies



Further up


Strong compact,
forcing axioms
PFA, MM
supercompact
$\neg\Box_\kappa$, for κ
singular strong
limit
many superstrongs
ω_1-dense ideal
ZF+AD
ω Woodins
ω_2 saturated
al on ω_1
Δ^1_2-det
one Woodin cardinal

The multiverse language

What does this picture say about what we should believe, or give preferred development, as a framework theory?

The multiverse language

What does this picture say about what we should believe, or give preferred development, as a framework theory?

Developing one theory develops them all, via the boolean-valued interpretations. At the level of concrete statements, they all agree. The question is how their theoretical superstructures fit together.

The multiverse language

What does this picture say about what we should believe, or give preferred development, as a framework theory?

Developing one theory develops them all, via the boolean-valued interpretations. At the level of concrete statements, they all agree. The question is how their theoretical superstructures fit together.

Calling one theory the "theory of V", and the others theories of generic extensions of V, may have no rational basis. Can we avoid doing that, and yet have a foundation for all mathematics?

The multiverse language

What does this picture say about what we should believe, or give preferred development, as a framework theory?

Developing one theory develops them all, via the boolean-valued interpretations. At the level of concrete statements, they all agree. The question is how their theoretical superstructures fit together.

Calling one theory the "theory of V", and the others theories of generic extensions of V, may have no rational basis. Can we avoid doing that, and yet have a foundation for all mathematics?

A proposed foundation should have a *language*, and *formal theory* stated in that language.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC, and whatever large cardinal axioms you can find.)

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC, and whatever large cardinal axioms you can find.)

(2) (a) Every world is a transitive proper class.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC, and whatever large cardinal axioms you can find.)

(2) (a) Every world is a transitive proper class.
(b) If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC, and whatever large cardinal axioms you can find.)

(2)
   (a) Every world is a transitive proper class.
   (b) If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
   (c) If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world.

We describe a *multiverse language*, and an open-ended *multiverse theory*, in an informal way. It is routine to formalize completely.

Multiverse language: usual syntax of set theory, with two sorts, for the *worlds* and for the *sets*.

Axioms of MV*:*

$(1)_\varphi$ φ^W, for every world W. (For each axiom φ of ZFC, and whatever large cardinal axioms you can find.)

(2) (a) Every world is a transitive proper class.
(b) If W is a world and $\mathbb{P} \in W$ is a poset, then there is a world of the form $W[G]$, where G is $\mathbb{P}$-generic over W.
(c) If U is a world, and $U = W[G]$, where G is $\mathbb{P}$-generic over W, then W is a world.
(d) (Amalgamation.) If U and W are worlds, then there are G, H set generic over them such that $W[G] = U[H]$.

Remarks.

(1) This has nothing to do with Hamkins' multiverse.

Remarks.

(1) This has nothing to do with Hamkins' multiverse.

(2) Woodin's "generic multiverse" is closer, but it does not satisfy amalgamation. (It is too big.)

Remarks.

(1) This has nothing to do with Hamkins' multiverse.

(2) Woodin's "generic multiverse" is closer, but it does not satisfy amalgamation. (It is too big.)

(3) Neither Hamkins' nor Woodin's theories were presented in terms of a formal language, and theory stated in that language.

Remarks.

(1) This has nothing to do with Hamkins' multiverse.

(2) Woodin's "generic multiverse" is closer, but it does not satisfy amalgamation. (It is too big.)

(3) Neither Hamkins' nor Woodin's theories were presented in terms of a formal language, and theory stated in that language.

(4) The natural way to get a model of MV is as follows. Let M be a transitive model of ZFC, and let G be M-generic for $\mathrm{Col}(\omega, < \mathsf{OR}^M)$. The worlds of the multiverse M^G are all those W such that

$$W[H] = M[G \restriction \alpha],$$

for some H set generic over W, and some $\alpha \in \mathsf{OR}^M$.

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

One can go on to claim this will always be the case, but as Tarski has shown, "all mathematics" is mathematically meaningless!

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

One can go on to claim this will always be the case, but as Tarski has shown, "all mathematics" is mathematically meaningless!

Strong relativist thesis: No individual world is definable in the multiverse language. In particular, there is no inclusion-minimal world.

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

One can go on to claim this will always be the case, but as Tarski has shown, "all mathematics" is mathematically meaningless!

Strong relativist thesis: No individual world is definable in the multiverse language. In particular, there is no inclusion-minimal world.

Weak absolutist thesis: There are individual worlds that are definable in the multiverse language.

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

One can go on to claim this will always be the case, but as Tarski has shown, "all mathematics" is mathematically meaningless!

Strong relativist thesis: No individual world is definable in the multiverse language. In particular, there is no inclusion-minimal world.

Weak absolutist thesis: There are individual worlds that are definable in the multiverse language.

Strong absolutist thesis: "V" makes sense, regardless of whether we can define it in the multiverse language.

Four theses

Weak relativist thesis: All mathematics done to date can be expressed in the multiverse language, and proved in extensions of the multiverse theory by large cardinal hypotheses.

One can go on to claim this will always be the case, but as Tarski has shown, "all mathematics" is mathematically meaningless!

Strong relativist thesis: No individual world is definable in the multiverse language. In particular, there is no inclusion-minimal world.

Weak absolutist thesis: There are individual worlds that are definable in the multiverse language.

Strong absolutist thesis: "V" makes sense, regardless of whether we can define it in the multiverse language.

The speaker believes the weak relativist thesis. He has hopes that the weak absolutist thesis is also true. (They are consistent with each other.) W.H. Woodin has recently suggested a very appealing way in which it might be true.

In this view, the meaningfulness of statements like CH depends there being priveleged worlds in the multiverse, and whether that is the case is yet to be determined.

The multiverse language I have presented is contained in the standard one; the generic relativist's idea is that it contains the meaningful part of the standard language. It is in this respect that that Woodin's "generic multiverse" V_M is not an appropriate model.

The multiverse language I have presented is contained in the standard one; the generic relativist's idea is that it contains the meaningful part of the standard language. It is in this respect that that Woodin's "generic multiverse" V_M is not an appropriate model.

In the multiverse I have described, the theory of the multiverse as a whole is present in the internal theory of each world. This is not the case for V_M. Indeed, it is not at all clear what the theory of a multiverse like V_M would be. Its reals are not closed under Turing join.

The multiverse language I have presented is contained in the standard one; the generic relativist's idea is that it contains the meaningful part of the standard language. It is in this respect that that Woodin's "generic multiverse" V_M is not an appropriate model.

In the multiverse I have described, the theory of the multiverse as a whole is present in the internal theory of each world. This is not the case for V_M. Indeed, it is not at all clear what the theory of a multiverse like V_M would be. Its reals are not closed under Turing join.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the boolean-vaulued interpretations. At the level of concrete statements, they all agree.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the boolean-vaulued interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the boolean-vaulued interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.
3. It might contain a *locator axiom* (saying which staircase you are on), and *height axioms* (saying how far up you are). Its incompleteness might reside solely in the height axioms. So it would decide CH, and much more.

The vision of ultimate K.

What does this picture say about what we should believe, or give preferred development, as a framework theory?

1. Developing one theory develops them all, via the boolean-vaulued interpretations. At the level of concrete statements, they all agree.
2. But there might be a best framework theory, or equivalence class of them under inter-translations.
3. It might contain a *locator axiom* (saying which staircase you are on), and *height axioms* (saying how far up you are). Its incompleteness might reside solely in the height axioms. So it would decide CH, and much more.
4. The best location is the center! It is easier to leave a canonical inner model by forcing than to get back into one by core model theory.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

    (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

   (b) nothing appears until you are fully ready to understand it,

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:
   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,
   (b) nothing appears until you are fully ready to understand it,
   (c) when you are ready, it does appear! (Unlike L.)

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,

   (b) nothing appears until you are fully ready to understand it,

   (c) when you are ready, it does appear! (Unlike L.)

   (d) In particular, you see natural models for all other natural theories. For example, you will find plenty of ways to enter PFA- worlds with the same theory of the concrete as your own.

(5) Nothing rivals fine structure theory as a global theory. There is no candidate locator axiom which produces a detailed, global theory of the universe of sets, except those asserting that V is some kind of canonical inner model.

(6) If you can make precise sense of "$V =$ ultimate K", and it is compatible with all consistency strengths up to rank-to-rank embeddings, then it has many virtues as a locator:

   (a) everything there is is arranged in a finely-grained, harmoniously ordered hierarchy,
   (b) nothing appears until you are fully ready to understand it,
   (c) when you are ready, it does appear! (Unlike L.)
   (d) In particular, you see natural models for all other natural theories. For example, you will find plenty of ways to enter PFA- worlds with the same theory of the concrete as your own.

(7) Parallel: if we were to show $0^\sharp$ does not exist, then $V = L$ would become a natural locator axiom.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.
3. You can't develop the theory of one hierarchy without developing the theory of all three. You can't prove consistency strength lower bounds without constructing all three types of model simultaneously.

The triple helix

The initial segment of the consistency-strength hierarchy we understand well enough to prove nontrivial equiconsistencies is $\leq$ a Woodin limit of Woodin cardinals. (Yes, that's all!) In this region:

1. At the center are 3 intertwined hierarchies of sets. Staircase 1 is the Wadge hierarchy in a canonical model of AD containing all reals. Staircases 2 and 3 are the fine-structural hierarchies of two inter-related types of canonical inner model for ZFC.
2. Models in each hierarchy see the others at levels $\leq$ their own.
3. You can't develop the theory of one hierarchy without developing the theory of all three. You can't prove consistency strength lower bounds without constructing all three types of model simultaneously.
4. One of the 3 types of models may be "preferred".

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function $x \mapsto \langle (M_n^x, i_{n,m}^x) \mid n, m < \omega \rangle$ on ω^ω such that for all x, $M_0^x = V$, each M_n^x is closed under κ-sequences, and

$$x \in A \Leftrightarrow \lim_n M_n^x \text{ is wellfounded.}$$

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is Hom_∞ iff for any κ, there is a continuous function $x \mapsto \langle (M_n^x, i_{n,m}^x) \mid n, m < \omega \rangle$ on ω^ω such that for all x, $M_0^x = V$, each M_n^x is closed under κ-sequences, and

$$x \in A \Leftrightarrow \lim_n M_n^x \text{ is wellfounded.}$$

The concept comes from Martin 1968. Hom_∞ sets are determined. The definition seems to capture what it is about sets of reals that makes them "well-behaved".

Staircase 1

Definition

A set $A \subseteq \omega^\omega$ is *Hom*$_\infty$ iff for any κ, there is a continuous function $x \mapsto \langle (M_n^x, i_{n,m}^x) \mid n, m < \omega \rangle$ on ω^ω such that for all x, $M_0^x = V$, each M_n^x is closed under κ-sequences, and

$$x \in A \Leftrightarrow \lim_n M_n^x \text{ is wellfounded.}$$

The concept comes from Martin 1968. *Hom*$_\infty$ sets are determined. The definition seems to capture what it is about sets of reals that makes them "well-behaved".

Theorem (Martin, S., Woodin)

If there are arbitrarily large Woodin cardinals, then for any pointclass Γ *properly contained in Hom*$_\infty$*, every set of reals in* $L(\Gamma, \mathbb{R})$ *is in Hom*$_\infty$*, and thus* $L(\Gamma, \mathbb{R}) \models \mathrm{AD}$.

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ statements are absolute for set forcing.

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ *statements are absolute for set forcing.*

In practice, generic absoluteness of a class of statements can be proved by reducing them to $(\Sigma^2_1)^{Hom_\infty}$ statements. (You may need more than arbitrarily large Woodin cardinals to do that!)

Theorem (Woodin)

If there are arbitrarily large Woodin cardinals, then $(\Sigma^2_1)^{Hom_\infty}$ statements are absolute for set forcing.

In practice, generic absoluteness of a class of statements can be proved by reducing them to $(\Sigma^2_1)^{Hom_\infty}$ statements. (You may need more than arbitrarily large Woodin cardinals to do that!)
There is an approximation to being Hom_∞ which can be used in constructing the sets in staircase 1 in universes where we have no measurable cardinals.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding. The extenders in a coherent sequence appear in order of their strength, without leaving gaps.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."
- Every real in an iterable extender model is ordinal definable in an absolute way.

Staircase 2

Definition

A *pure extender model* is the constructible closure of a *coherent sequence of extenders*.

- An *extender* is a system of ultrafilters coding an elementary embedding.The extenders in a coherent sequence appear in order of their strength, without leaving gaps.
- What makes a countable pure extender model M canonical is the existence of a Hom_∞ iteration strategy for M. This is a $(\Sigma^2_1)^{Hom_\infty}$ statement about M.It is reasonable to hope that all $(\Sigma^2_1)^{Hom_\infty}$ statements ψ can be reduced to statements of the form " there is a Hom_∞ iteration strategy for some $M \models f(\psi)$."
- Every real in an iterable extender model is ordinal definable in an absolute way.
- At the moment, we can only construct iteration strategies for M a bit past Woodin limits of Woodins. The fine structure theory for iterable M works up through superstrong cardinals.

Staircase 3

Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Staircase 3

Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Psuedo-definition: A *hod mouse* is a model constructed from a coherent sequence of extenders, together with an iteration strategy for the model.

Staircase 3

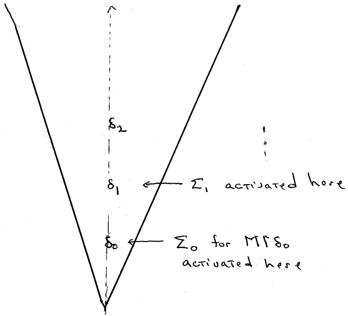
Leaning heaviliy on work of Woodin:

Theorem

No iterable pure extender model with a Woodin cardinal satisfies "I am iterable".

Psuedo-definition: A *hod mouse* is a model constructed from a coherent sequence of extenders, together with an iteration strategy for the model.

The concept was made precise by Woodin for hod mice having countably many Woodin cardinals. Grigor Sargsyan developed it further, up to measurable cardinals which are limits of Woodins. Steel and Sargsyan have gone somewhat beyond that.



A hod mouse M

δ_ξ = ξ^{th} Woodin of M

Σ_ξ = iteration strategy for $M|\delta_\xi$

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathrm{AD}_{\mathbb{R}} +$ *"θ is measurable" containing all the reals; then HOD is a hod mouse.*

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathrm{AD}_{\mathbb{R}}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathrm{AD}_{\mathbb{R}}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathsf{AD}_{\mathbb{R}}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathsf{AD}_{\mathbb{R}}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Mouse Set Conjecture: Assume AD^+ and there is no iterable pure extender model with a superstrong; then HOD is a pure extender model below its least Woodin cardinal.

Some connections

Theorem (Sargsyan 2008)

Assume AD^+ *and there is no model of* $\mathsf{AD}_{\mathbb{R}}$ + "θ *is measurable" containing all the reals; then HOD is a hod mouse.*

For determinacy models M, HOD^M knows the theory of M. E.g.:

Theorem (Woodin late 80's)

Assume $\mathsf{AD}_{\mathbb{R}}$ + "θ *is regular"; then* V *is elementarily embeddable into a symmetric extension of HOD.*

Mouse Set Conjecture: Assume AD^+ and there is no iterable pure extender model with a superstrong; then HOD is a pure extender model below its least Woodin cardinal.
Sargsyan proved this for determinacy models in the region to which his work applied.

The core model induction

One proves consistency strength lower bounds for theories like PFA by climbing all three staircases together. A sample theorem:

The core model induction

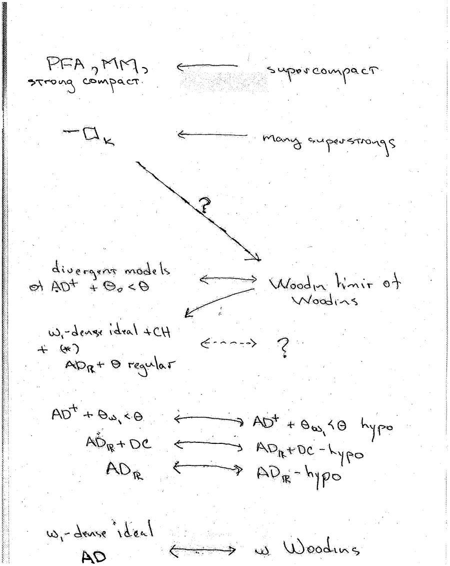
One proves consistency strength lower bounds for theories like PFA by climbing all three staircases together. A sample theorem:

Theorem (Woodin 90's, Sargsyan 2008)

The following are equiconsistent

(1) ZFC + *"there is an ω_1-dense ideal on ω_1* + CH + $(*)$,

(2) ZF + $\mathsf{AD}_{\mathbb{R}}$ + *"Θ is regular"*.



PFA, MM, strong compact.
supercompact
$\neg\square_\kappa$
many superstrongs
?
divergent models of $AD^+ + \Theta_0 < \Theta$
Woodin limit of Woodins
ω_1-dense ideal + CH + (*)
$AD_{\mathbb{R}} + \Theta$ regular
?
$AD^+ + \Theta_{\omega_1} < \Theta$
$AD^+ + \Theta_{\omega_1} < \Theta$ hypo
$AD_{\mathbb{R}} + DC$
$AD_{\mathbb{R}} + DC$ - hypo
$AD_{\mathbb{R}}$
$AD_{\mathbb{R}}$ - hypo
ω_1-dense ideal
AD
ω Woodins

Some locator axioms

($A1$) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathsf{HOD}^M|\theta.$$

Some locator axioms

($A1$) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

Some locator axioms

$(A1)$ There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

(A2) implies there are no strong cardinals, but may be compatible with all the local forms of large cardinal axioms. (Having a strong cardinal is like having a largest rank.)

Some locator axioms

($A1$) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

(A2) implies there are no strong cardinals, but may be compatible with all the local forms of large cardinal axioms. (Having a strong cardinal is like having a largest rank.)

Both (A1) and (A2) say that V looks like the HOD of an AD model.

Some locator axioms

(*A*1) There is a strong cardinal, and arbitrarily large Woodin cardinals, and for κ the least strong cardinal, and M the *derived model* of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

$(A2)_n$ There are arbitrarily large Woodins, and $V_\kappa \prec_{\Sigma_n} V$, such that for M the derived model of V at κ, there is an elementary

$$j\colon V_\kappa \to \mathrm{HOD}^M|\theta.$$

(A2) implies there are no strong cardinals, but may be compatible with all the local forms of large cardinal axioms. (Having a strong cardinal is like having a largest rank.)
Both (A1) and (A2) say that V looks like the HOD of an AD model.Woodin has shown that (A2) is true in the HOD of a model of $\mathrm{AD}_{\mathbb{R}}$ + "θ is regular.

$(A3)_n$ For arbitrarily large α, V is Σ_n equivalent to the α-complete backgrounded pure extender model.

$(A3)_n$ For arbitrarily large α, V is Σ_n equivalent to the α-complete backgrounded pure extender model.

$(A4)$ AD^+.

Some basic open problems

By now, philosophy has outrun the math. Some problems:

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?Con(supercompacts)?

Some basic open problems

By now, philosophy has outrun the math. Some problems:

(1) How iterable is V? Are there iterable pure extender models having superstrongs? Measurable Woodin cardinals? Supercompacts?

(2) Are there hod mice with Woodin limits of Woodin cardinals?

(3) Is the Mouse Set Conjecture true?

(4) Does Con(PFA) imply Con(Woodin limit of Woodins)?Con(supercompacts)?

(5) And ...

What's up there?