STRICT PREDICATIVITY[1]

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The most basic notion of impredicativity applies to specifications or definitions of sets or classes. If a set b is specified as $\{x: A(x)\}$ for some predicate A, then the specification is impredicative if A contains quantifiers such that the set b falls in the range of these quantifiers. Already in the early history of the notion, it was extended to other cases, such as propositions in Russell's discussions of the liar paradox. Mathematics will be predicative if it avoids impredicative definitions.

The logicist reduction of the concept of natural number met a difficulty on this point, since the definition of 'natural number' already given in the work of Frege and Dedekind is impredicative. More recently, it has been argued by Michael Dummett, the author, and Edward Nelson that more informal explanations of the concept of natural number are impredicative as well. That has the consequence that impredicativity is more pervasive in mathematics, and appears at lower levels, than the earlier debates about the issue generally presupposed.

The appearance to the contrary resulted historically from the fact that many opponents of impredicative methods, in particular Poincaré and Weyl, were prepared to assume the natural numbers in their work. In this they were followed by later analysts of predicativity, in particular Kreisel, Schütte, and Feferman. The result was that the working conception of predicativity was of predicativity given the natural numbers. Thus Feferman characterized "the predicative conception" as holding that "only the natural numbers can be

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regarded as 'given' to us. … In contrast, sets are created by man to act as convenient abstractions (*façons de parler*) from particular constructions or definitions."[2] There is clearly a conceptual distinction between predicativity assuming the natural numbers, in practice allowing as predicative theories with induction on all predicates of a relevant formal language, and predicativity without that assumption, where the possibility is at least open that the concept of natural number, or some part even of elementary arithmetic, should be impredicative. It is the latter conception of predicativity that I call strict predicativity.

That there is such a conceptual distinction should not be controversial, and that it is the former, non-strict conception that the work on the analysis of predicative definability and predicative provability in the 1950s and 1960s had in view was in effect quite clear to the participants, as the above quotation from Feferman illustrates. It was already made quite clear by Hermann Weyl in arguing in *Das Kontinuum* that the concept of natural number is "extensionally definite", so that quantification over numbers is unproblematic even in the context of specification of sets, while the concept "property of natural numbers", from which he took that of set of natural numbers to be derivative, is not. It is clearly the conception that has played the more prominent role in the foundations of mathematics. Nonetheless, strict predicativity would be the relevant conception where predicativity was at issue in logicist work, such as Russell's unsuccessful attempt in the second edition of *Principia* to show that the natural numbers could be defined in ramified type theory without reducibility.

The concept of strict predicativity raises two questions: (1) Does the conceptual distinction make a real difference in what mathematical methods

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[2] "Systems of predicative analysis," pp. 1-2.

count as predicative; that is, are there mathematical theories that are predicative in the more usual sense but are not reducible to theories that are strictly predicative? (2) Assuming that there is such a difference, is there a body of arithmetic that is strictly predicative, and what are its limits? Both these questions suggest a logical research program, but one objective of this program would have to be to make the notion of strict predicativity more precise, as was done by earlier work for the notion of predicativity given the natural numbers. The main purpose of this talk is to commend such a research program and to discuss some technical work, none of it my own, that has contributed to it.

The informal arguments for the impredicativity of induction suggest an affirmative answer to the first question. Suppose for example that we characterize the natural numbers by rules, first of all introduction rules stating that 0 is a number and that if x is a number so is Sx. Then, by way of interpreting an extremal clause that states that nothing is a number except by these rules, we add an elimination rule saying that for any well-defined predicate $A(x)$, if $A(0)$ holds and $A(Sx)$ holds whenever $A(x)$ holds, then $A(x)$ holds for all numbers. We take the notion of "predicate" to be quite open-ended, indefinitely extensible in Dummett's phrase. It therefore includes the number predicate itself and others built up by logical operations with the number predicate as one of the constituents. That makes the explanation impredicative; furthermore, the impredicativity appears to make a difference in what can be proved, since inductions involving predicates containing the number predicate occur early on in the development of number theory.

This observation is not a decisive argument for an affirmative answer to (1), since other explanations of the number concept are possible, and even given this one it would need to be argued that impredicative instances of induction are indispensable. More clarity on the question can be obtained by looking at logical results motivated by question (2), as we shall see shortly. A step toward a negative answer to (1) would be taken if we could show that there is a theory of arithmetic that passes muster as strictly predicative that is, let's say, at least as strong as PA. The most plausible candidates for such a theory known to me are the theories EFS and EFSC of Solomon Feferman and Geoffrey Hellman. EFS is a theory of finite sets; EFSC adds first-order definable classes. Feferman and Hellman don't pose the questions in quite the way I have above, so that they don't clearly claim that EFS and EFSC are strictly predicative. The question whether they are, however, is clearly relevant to our question (1).

The notions of finite set and of natural number are obviously intimately related. In particular, one might explain the former notion by rules, along the same lines as such an explanation of the natural numbers. The introduction rules would say that is a set; if a is a set, then the result $a \oplus x$ of adding an object x to a as an element is also a set. Nothing is a (finite) set except by virtue of these rules. The extremal clause is cashed in by Zermelo's induction principle: If $A(\varnothing)$ and $A(a \oplus x)$ for any set a for which $A(a)$ and any object x, then $A(a)$ holds for all finite sets.[3] If the "objects" include sets, we obtain a theory of hereditarily finite sets; if they do not (if we have a two-sorted theory), we have a theory of finite sets of individuals.

The notion of strict predicativity should not allow the assumption of finite sets if explained in something like this way, since a case for its impredicativity could be made along the same lines. The most a development of arithmetic in a theory of the type just sketched, with axioms corresponding to the introduction rules, Zermelo induction, and extensionality, could show is that arithmetic is

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[3] Of course $a \oplus x$ is just $a \cup \{x\}$, but I have in mind a theory in which $\oplus$ is primitive. Then $\{x\}$ is obviously $\varnothing \oplus x$. Union can be introduced by primitive recursion.

predicative relative to the notion of finite set. EFS differs from such a theory by replacing Zermelo induction by the schema of separation; it is a two-sorted theory with pairing on individuals, so that an infinity of individuals is provided for. It is then possible to exhibit a structure for which the axioms of first-order arithmetic can be proved. In EFSC, one can prove that a progression exists and is unique up to isomorphism.

There is, however, a serious objection to the claim of this development to be strictly predicative: to derive instances of the induction schema involving quantification over numbers, one needs to apply the separation schema to formulae involving quantification over all sets in the domain; thus one has impredicativity of a rather traditional kind. So I don't think EFS or EFSC is strictly predicative, and the prospects of a strictly predicative theory that would interpret PA do not look bright. However, a definitely affirmative answer to (1) would require more exact criteria of strict predicativity than I have put forth thus far or than I think we have at present.

Let us turn now to (2). Edward Nelson's *Predicative Arithmetic* is probably the first sustained attempt to develop arithmetic on a strictly predicative basis, and it offers a possible criterion of strict predicativity. Nelson's way of dealing with the problem of the impredicativity of induction is to begin with the theory Q, first-order arithmetic without induction but with addition and multiplication and an axiom stating that every non-zero number has a predecessor. Though extremely weak in its ability to prove generalizations, this theory is rather strong from the point of view of interpretability. Nelson's strategy is to develop arithmetic as far as possible in theories that are interpretable in Q. We can take such interpretability as a proposed criterion of strict predicativity. Nelson then showed that what is called $I\Delta_0$, the subsystem of classical first-order arithmetic PA in which induction is restricted to formulae containing only bounded quantifiers, is locally interpretable in Q, and subsequently it was shown that even extensions of $I\Delta_0$ are globally interpretable in Q. That amounts to saying that we could write down in the language of PA a predicate N_p (to be read: x is a pseudo-number) so that all theorems of $I\Delta_0$ are provable in Q when the quantifiers are restricted to N_p.[4] The extension consists of adding axioms stating that certain functions are total, where, however, the functions grow more slowly than exponentiation.[5]

Thus on the suggested criterion, an arithmetic whose recursive functions are by a reasonable standard feasibly computable is strictly predicative. Nelson doubted that interpretability in Q would be preserved if exponentiation is added, thus going beyond the feasible. Subsequently it was shown that $I\Delta_0 + exp$ is not interpretable in Q, confirming Nelson's doubt. Thus Nelson's criterion puts the boundary of strict predicativity somewhere between feasible arithmetic and arithmetic with exponentiation.

A competing apparently more direct analysis, however, leads to the conclusion that the total character of exponentiation can be proved in a strictly predicative way. This analysis pursues the natural idea that one should see how much arithmetic one can do by starting with something like Russell's ramified type theory without the axiom of reducibility. Admitting 0 and S with the elementary Peano axioms amounts to assuming a version of the type-theoretic axiom of infinity. We can then use the Frege-Dedekind definition of the natural numbers, with the second-order quantifier assigned a specific order. If we write N^{k+1} when the quantifier is of order k, then the defining formula of N^k is of order

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[4] Local interpretability amounts to the claim that one can do this for any finite subset of the axioms of $I\Delta_0$.

[5] Hájek and Pudlák, p. 367.

k. Then the project would be to show that for some k, if we define the numbers by N^k, then we will be able to develop some arithmetic in our ramified higher-order theory. In fact, for the work reported below, ramified second-order logic is sufficient.

It was already shown by Skolem, in *Abstract Set Theory* , that we can develop the arithmetic of addition, multiplication, and order in this way. This result was extended to bounded induction by Allen Hazen, and recent work of John Burgess and Hazen has shown that exponentiation can be added; for this method the obstacles found by Nelson can be overcome. Thus the system $I\Delta_0$ + *exp*, which already allows as total a function that is not feasibly computable, is predicative by the Russellian criterion.[6] It follows that the Russellian criterion is more generous than Nelson's. I am inclined to conclude that Nelson's criterion is too narrow. In fact it seems to have been motivated as much by considerations of feasibility as of predicativity.

Is it possible to go further with the approach of Burgess and Hazen? In the same direct way, probably not. The consistency of the Russellian system with which they work can be proved in primitive recursive arithmetic; it follows that it is not possible to extend their result to primitive recursion in general. Even for iterated exponentiation, they obtain only the weak result that if one uses all finite levels of the ramified hierarchy, one can interpret arithmetic with iterated exponentiation and open induction.

That might suggest that Kalmár elementary arithmetic represents what one can obtain on a strictly predicative basis. However, we could draw such a conclusion at this point only on the basis of a too easy identification of ramified logic with finite levels as expressing what strict predicativity allows. A

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[6] Burgess and Hazen, "Predicative logic and formal arithmetic."

possibility that might be explored is to admit transfinite levels, as was done in the study of predicativity given the natural numbers. There would be no difficulty in working with an ordering < of the natural numbers that is quite simple, say Δ_0, to represent the ordinal levels. Then we would index levels by numbers preceding in the ordering a certain number *n*, where transfinite induction up to *n* has been proved by means already recognized as strictly predicative. General statements of transfinite induction would themselves have to be indexed by a level, but it might be possible to lift such results to higher levels. At all events, transfinite levels could be used only with some condition of autonomy, admitting a level only if transfinite induction up to it has been proved by means already recognized as strictly predicative.

Such a development of the concept of strict predicativity is so far unexplored. I would commend it to proof theorists as a possible line of research. What limits of strictly predicative proof it would in the end give rise to is at present unknown. Results of Feferman and others about "reflective closure" and "unfolding" might, however, suggest conjectures. It is not clear to me that more than primitive recursive arithmetic, or its conservative extension by admitting classical quantificational logic, would turn out to be strictly predicative on the most generous criterion of this kind, and I would be very surprised if more than Peano arithmetic turned out to be so.

A concern might be felt about what is presupposed by the methods of an analysis of this kind. Probably a characterization of strictly predicative provability can be given only from a point of view that is not strictly predicative, in parallel to the situation with predicativity given the natural numbers. A worry might arise right at the beginning, however, because we are talking about formal systems such as Q or ramified type theory, and it may be thought that some impredicative notion, such as natural number or finite set or sequence, is needed in order even to understand what they are, and therefore to understand what it is of which we are asking whether it is strictly predicative. On the whole this worry would be defused by observing that it is possible to describe these theories by very elementary means, and in showing a simply described theory such as $I\Delta_0$ + *exp* to be strictly predicative what we need to do is to make positive claims of provability, where we can actually describe the procedure for constructing the proof. We can in such cases be pretty sure there is not some hidden impredicativity. That is to say, positive claims of strict predicativity are unlikely to face difficulties of this kind. Definite negative claims will in any case have to wait for more work on the criteria of strict predicativity, and probably they will be possible only from a point of view that is not strictly predicative.

Another problem that we have finessed is the following: The theories of Feferman and Hellman, Q, and the ramified logic of Burgess and Hazen have the common feature that they are theories with classical logic, but we have not inquired what the domain of individuals is, and each has axioms that force the domain to be infinite. Apart from a mathematical conception such as that of number or finite set, what could entitle us to assume a theory with such a domain? This is a question that does not arise for predicativity given the natural numbers, because the natural numbers would either directly provide the domain required or offer the resources for constructing it. Moreover some critiques of impredicativity, in particular Weyl's, get some of their force from the contrast between the claim of a clear conception of the natural numbers and of the lack of such a conception of sets or properties of natural numbers.

I'm not sure of the best response to this difficulty. A general idea for such a response is that the predicativist is operating with quantifiers whose range he has not yet determined and will not determine finally. Both Nelson's technique and the work with ramified logic involve using the intuitive idea of characterizing the numbers by induction to get subdomains that are more determinate. Once one has done this, one can operate within the subdomain.

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