Completeness or Incompleteness of Basic Mathematical Concepts

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0 Introduction

I will address the topic of the workshop through a discussion of basic mathematical structural concepts, what they are, what their role is in mathematics, in what sense they might be complete or incomplete, and what kind of evidence we have or might have for their completeness or incompleteness.

I share with Kurt Gödel and Solomon Feferman the view that mathematical concepts, not mathematical objects, are what mathematics is about.[1] Gödel, in the text of his 1951 Gibbs Lecture, says:

> Therefore a mathematical proposition, although it does not say anything about space-time reality, still may have a very sound objective content, insofar as it says somethng about relations of concepts.[2]

Though probably neither Feferman nor I would put it this way, we both basically agree. There are some differences that I have with each of the them. My view is closer to Gödel's than to Feferman's. Nevertheless, there is a great deal on which I agree with Feferman. In section 2 of his paper for this workshop,[3] he describes his *conceptual structuralism.* Virtually the only thing in section 2 that I disagree with is the one I will mention in the next paragraph.

Feferman takes mathematical concepts to be human creations. He calls them objective, but the objectivity seems ultimately only intersubjective. Gödel thinks that mathematical concepts are genuine objects, part of the basic furniture of the world. For him they are non-spatio-temporal entities. I would say I stand in between the two, but in fact I don't have anything to say about the ontology of mathematical concepts. I do think it is more correct to say that they were discovered than that they were created by us.

Gödel admits two kinds of evidence for truths about mathematical concepts: what are now called *intrinsic* evidence and *extrinsic* evidence. I agree. Feferman

[1] Actually Gödel counts concepts as objects. What I am calling objects he calls "things."

[2] Gödel [13], p. 321.

[3] Feferman [7].

seems to allow only intrinsic evidence, though perhaps he just has very high standards for extrinsic evidence. I am much more with Feferman rather than Gödel on how we get instrinsic evidence. Gödel holds that it is a kind of non-sensory perception. As I will explain, I believe this is a peripheral aspect of Gödel's position. Having placed mathematical concepts in another world, he is impelled to come up with a mechanism for our getting in contact with them. If we ignore this, then everything else he says makes it seem as if our direct knowledge of concepts comes from garden-variety grasping or understanding them.

The question on which Feferman, Gödel, and I most clearly and directly disagree is on the status of the continuum hypothesis. Feferman is sure that CH has no truth-value. Gödel is sure that it has a truth-value. I believe that the question of whether it has a truth-value is open, and I want to understand both possible answers.

The mathematical concepts I will be discussing are concepts of structures. More specifically, I will be concerned with the main concepts that can be taken as what Feferman elsewhere has called *fundamental concepts*, those that we normally construe as concepts of individual structures rather than kinds of structures, and those each of which we can study without a background of knowledge or assumptions about other concepts. I will use the word *basic* rather than "fundamental," mainly because it is a shorter word. I will be talking about the concept of the natural numbers, that of the natural numbers and the sets of natural numbers, etc., the general iterative concept of set, and extensions of this concept.

Some of the main points of my view are the following:

(1) Basic mathematical concepts are not intrinsically different from ordinary concepts. They differ only in that they are sharp enough to support mathematical mathematical study. Basic mathematical concepts, insofar as they are taken as basic rather than as being defined from other mathematical concepts, do not come with anything like certifiably precise characterizations. Gödel, who makes much of the fact that basic mathematical concepts are not definable in any straightforward way, thinks that they have to be objects in something like Frege's third world, and he thinks that our knowledge of them comes from a kind of perception. My views about mathematics have a lot in common with Gödel's, but his reification of mathematical concepts is one of two main points of difference.

(2) The concepts of the natural numbers and of sets are concepts of structures—more accurately, they are concepts of kinds of structures. When I speak of a "structure," I mean some objects and some relations and functions on the objects. Thinking that the concept of set determines what it is for an object to be a set is very common but, I believe, quite wrong. I will thus treat the concept in a

structuralist manner, but nothing important will turn on this.

(3) A fundamental question about a basic concept is that of which statements are *implied by the concept*—would have to be true in any structure that instantiated the concept. I think of this as a kind of necessity, but a logical necessity not a metaphysical one. I take the question to be a meaningful one whether or not there are any structures that instantiate the concept.

(4) The concept of the natural numbers is *first-order complete*: it determines truth values for all sentences of first-order arithmetic. That is, it implies each first-order sentence or its negation. In fact I think that the concept of the natural numbers has a stronger property than first-order completeness. I will discuss this proprty in the next section. It is an open question whether the concept of sets—or even, say, the concept of the sets of sets of natural numbers—is first-order complete.

(5) The concepts of sets and of the natural numbers are both *categorical*: neither has non-isomorphic instantiations. (A more conservative statement would, for the concept of set, replace "categorical" with "categorical except for the length of the rank hierarchy.")

(6) It is irrelevant to pure mathematics whether either of these concepts is instantiated.

(7) We do not at present know that that the concept of set—or even just the concept of the sets of sets of natural numbers—is instantiated. I do not have an opinion as to whether it is known that the concept of natural number is instantiated. But I have no real quarrel with those who say it is. I will explain this in the next section.

A central point of my view is that mathematical objects (e.g., numbers and sets) are not what mathematics is about, that the truth or falsity of mathematical statements does not depend on mathematical objects or even with whether they exist. There are a number of familiar difficulties with *object-based* accounts of mathematics and mathematical truth. I will present an additional difficulty: Object-based accounts of truth could not accommodate interdeterminate truth-values, and indeterminate truth-values in set theory are a genuine epistemic possibility.

One of the points of disagreement between my views and Gödel's is the role of mathematical objects in mathematics. Gödel believed that they play an important role. For one thing, he considered mathematical concepts to be a species of mathematical objects. Since he characterized mathematical truth in terms of relations of concepts, his view has to count as object-based. But the role of he ascribes to non-concept mathemetical objects such as numbers and sets is limited. It is, I believe, more limited than what I said it was in [17]. I will discuss Gödel's views about the role of such objects in the next section.

1 Mathematical Objects

Most philosophical accounts of mathematics are *object-based*. They take the subject matter of mathematics to be mathematical objects. They characterize mathematical truth in terms of structures composed of objects.

What seems to me the strongest argument in favor of object-based accounts is that they—or, at least, some of them—allow one to take mathematical discourse at face value. Euclid's theorem that there are infinitely many prime numbers is, on face value, about a domain of objects, the positive integers. What makes it true is, on face value, that infinitely many of these objects have a certain property.

Of course, many object-based accounts accounts involve taking mathematical discourse at something other than face value. Some structuralist accounts are examples. But such accounts take one aspect of mathematical language at face value: its existential import. The statement that there are infinitely many prime numbers seems to assert the existence of something, and pretty much all object-based accounts construe such statements as genuinely an assertion of existence.

There are major problems that object-based accounts must face. There is Benacerraf's [1] problem of how we can know truths about objects with which we seemingly do not interact. There is the problem of how we know even that these objects exist. There is the problem of just what objects such things as numbers and sets are.

Dealing with thes problems has led philosophers of mathematics to come up with what seem to me strange sounding notions about mathematical objects. Here are some of them:

- The natural numbers are (or are being) created by us.
- Mathematical objects are "thin" objects.
- Specifying the internal identity conditions for a supposed kind of mathematical objects can be sufficient for determining what these objects are.
- Mathematical objects are "logical" objects, and this guarantees their existence.

For me the problem with assuming as a matter of course that the existence of mathematical objects instantiating our mathematical concepts is that such assumptions are not innocent. They can have consequences that we have good reasons to doubt. Assume, for example, that we know that the concept of the sets of sets of natural numbers is instantiated. I will argue later that the this concept is categorical. (This is an old argument, due to Zermelo.) But instantiation plus categoricity imply that the concept is first-order complete, it would seem—and I believe. Hence we know that CH, which is a first-order statement about the concept in question, has a definite truth-value. Do we really know that it has a truth-value? I don't think so.

There is a propery of concepts short of being instantiated that has all the important consequences of instantion. Say that a concept is *fully determinate* it is determined, in full detail, what a structure instantiating it would be like. Most of us—including me—think of the concept of the natural numbers a having this property. Full determinateness follows, I believe, from categoricity plus instantiation. Full determinateness is what we want our basic concepts to have. It does all the important work of instantiation. I do not see any reason that full determinateness implies instantiation. However, all my worries about too easily assuming instantiation apply just as well to too easily assuming full determinateness. Hence I don't mind if someone asserts that any fully determinate concept is instantiated.

Gödel on mathematical objects

Gödel classified objects into two sorts, *things* and *concepts*. The role he ascribes to mathematical concepts is a central one. Mathematical propositions are about the relation of concepts. A true mathematical proposition is *analytic*, true in virtue of meaning. In one place[4] he says that the meaning in question is the meaning of the concepts occurring in the propositon. In another place[5]—thinking of propostions as linguistic—he says that it is the meaning of the terms occurring in the proposition, where the meaning of he terms is the concepts they denote.

What role Gödel ascribes to mathematical things, e.g., to sets and numbers, is less clear.

On the one hand, he says of classes (i.e., sets) and concepts:

> It seems to me that the assumption of such objects is quite as legitimate as the assumption of physical bodies and there is quite as much

[4]Gödel [10], p. 139

[5]Gödel [13], p. 346

> reason to believe in their existence. They are in the same sense necessary to obtain a satisfactory system of mathematics as physical bodies are necessary for a satisfactory theory of our sense perceptions and in both cases it is impossible to interpret the propositions one wants to assert about these entities as propostions about the "data", i.e., in the latter case the actually occurring sense perceptions.

On the other hand, his account of mathematical truth makes it puzzling what role sets and other mathematical "things" are supposed to play. The revised and expanded version of his paper on the Continuum Hypothesis has passages that look relevant to this puzzle. Here is one of them.

> But, despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception.[6]

It would be natural to suppose that Gödel is talking about both perception of concepts and perception of sets. Nevertheless nothing he says in the paper (or elsewhere, so far as I know) suggests that perception of sets could yield significant mathematical knowledge. The ZFC axioms' forcing themselves on us is surely intended as evidence that we perceive the *concept* of set. When elsewhere in the paper, he discusses actual or imagined new axioms, the source of our certain knowledge of their truth is always characterized as the concept of set and other concepts. He talks of large cardinal axioms that are suggested by the "very concept of set" on which the ZFC axioms are based; of new axioms that "only unfold the content of the concept of set"; of new axioms that "are implied by the general concept of set"; of the possibility that new axioms will be found via "more profound understanding of the concepts underlying logic and mathematics."[7] There is nothing to suggest that perception of *sets* could help in finding new axioms or played a role in finding the old ones.

A second relevant-looking passage is the following.

> For if the meanings of the primitive terms of set theory as explained on page 262 and in footnote 14 are accepted as sound, it follows

[6]Gödel [11], p. 268.

[7]See pp. 260-261.

> that the set-theoretical concepts and theorems describe some well-determined reality, in which Cantor's conjecture must be either true or false. Hence its undecidability from the axioms being assumed today can only mean that these axioms do not contain a complete description of that reality.[8]

What he explained on page 262 (of Benacerraf and Putnam [2]) and in footnote 14 was the iterative concept of set. The quoted passage thus seems to be saying that if that concept of set is sound then it is instantiated by some structure and, moreover, the instantiation is unique.[9] In what sense the instantiation is supposed to be unique is not clear. No doubt he at least intends uniqueness up to isomorphism.

The argument given in the quoted passage seems to be the only argument given in the Continuum Hypothesis papers for why the CH must have a truth-value. The corresponding passage[10] in the original version of the paper has what is probably supposed to be the same argument. Instead of the assumption that the explained meanings of the primitive terms of set theory are sound, there is the assumption that "the concepts and axioms" have a "well-defined meaning."[11]

Some naturally arising questions about the argument are:

(1) Is soundness of meaning the same as well-definedness of meaning? I.e., are the two versions of the argument the same? A related question is: Why did Gödel replace the first version by the second?

(2) Do these assumptions imply *by definition* the existence of an instantiation (the well-determined reality)? I.e., is existence of a unique instantiation part of what is being assumed?

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[8]Gödel [11], p. 260.

[9]There is a way of reading this passage on which no uniqueness is asserted and there is no implication that the Continuum Hypothesis has a definite (instantiation-independent) truth-value. But this is not the reading Gödel intends. He intends that the concept of set determines a unique instantiantion—at least, one that is unique enough determine a truth-value for the CH.

[10]Gödel [12], p.181.

[11]The real reason for the qualification "seems to be" is not this earlier version of the argument. It is another passage in the revised version that, on one reading, gives a different argument for there being a truth-value for the CH. On page 268 of Gödel [11], he says, "The mere psychological fact of the existence of an intuition which is sufficiently clear to produce the axioms of set theory and an open series of extensions of them suffices to give meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis." If there being "meaning to the question of the truth or falsity of propositions like Cantor's continuum hypothesis" is understood as implying that such propositions have truth-values, then the argument seems very weak, so charity suggests that Gödel intended something weaker. See Parsons [18] for a discussion of the passage.

(3) Does Gödel have reasons for thinking that the assumptions are true? Evidently he does think they are true.

Whatever the answers to these individual questions are, the important question is: *Do we have good reasons for believing that the concept of set has a unique instantiation?* As I have already indicated, I think that the answer is yes for uniqueness and no for existence, and I will say why later in this paper.

2 The Concept of the Natural Numbers

Despite the title of this section, I will mainly discuss not the concept of the natural numbers but the more general concept of an ω-sequence. The concept of the natural numbers is often taken to be the concept of a single structure, a concept that determines not just the isomorphism type of a structure but also the objects that form the structure's domain. Whether the concept of the natural numbers determines what object, e.g., the number 3 is has long been debated. I don't believe the concept does this. Nor do I—as an agnostic about the existence of mathematical objects—believe that the concept of the natural numbers determines what 3 has to be if it exists. But probably the concept does determine some properties of the numbers. Perhaps, for example, it is part of the concept that numbers have to be abstract objects and that being cardinalities has to be part of their essences. By talking mainly about the blatantly structuralist concept of an ω-sequence, I will avoid these issues.

The concept of an ω-sequence—or that of the natural number sequence—may be taken not as *basic* but as *defined*, usually as defined from the concept of set. Throughout this section, I will take the concept as basic. An instantiation of the concept will consist of some objects and a function, the successor function. One can, if one wishes, think of the successor function as merely a relation, not as an additional object. As I will explain shortly, ordering and basic arithmetical functions are determined by terms the successor function, and so we might as well think of them as belonging to the instantiation proper.

There are various ways in which we can explain to one another the concept of the sequence of all the natural numbers or, more generally, the concept of an ω-sequence. Often these explanations involve metaphors: counting forever; an endless row of telephone poles (or cellphone towers); etc. If we want to avoid metaphor, we can talk of an unending sequence or of an infinite sequence. If we wish not to pack to pack so much into the word "sequence," then we can say that

that an ω-sequence consists of some objects ordered so that there is no last one and so that each of them has only finitely many predecessors. This explanation makes the word "finite" do the main work. We can shift the main work from one word to another, but somewhere we will use a word that we do not explicitly define or define only in terms of other words in the circle. One might worry—and in the past many did worry—that all these concepts are incoherent or at least vague and perhaps non-objective.

The fact that we can latch onto and communicate to one another concepts that we cannot precisely define is not easily explained. It surely has a lot to do with the way we are wired.

Is the concept of an ω-sequence a clear and precise one? In particular, is it clear and precise enough to determine a truth-value for every every sentence expressible in the language of first-order arithmetic? (To be specific, let's declare this to be the first-order language with S, $+$ and $\cdot$.) As I indicated earlier, I will call the concept *first-order complete* if the answer is yes—if it does determine truth-values for all these statements. First-order completeness does not mean that the answers to all arithmetical questions are knowable by us. In terminology (of Gödel) that I introduced earlier, it means that an answer is implied by the concept. I take this as equivalent to saying that there is an answer that would have to be correct for any structure that instantiated the concept. Here the modal "would have to be" should be taken as something like logical.

I will use the phrase "first-order complete" in in a similar way in discussing other concepts E.g., by the question of whether the concept of sets is *first-order complete* I mean the question of whether that concept determines truth-values for all sentences of the usual first-order language of set theory.

Of course, what one means by "first-order completeness" of a concept depends on what functions and relations one includes. Since I am (mostly) taking the concept of ω-sequence to be a concept of structures with only one unary operation, it would perhaps seem more correct to define first-order completeness for that concept in terms of the language with only with only S as the only non-logical symbol. But order, addition, and multiplication are recursively definable from successor, so it makes sense to include them. Indeed, it makes sense to take them to be part of the concept. I won't worry about whether doing so would yield a different, or just an equivalent, concept.

The question of the first-order completeness of a concept may not be a clear and precise one. If one is unsure about the answer in the ω-sequence case, one may worry even about whether the notion of a first order formula is clear and precise.

I suspect that most mathematicians believe that the concept of an ω-sequence is first-order complete. I believe that it is. But I suspect that most mathematicians believe—as I do—that the concept is clear and precise in a stronger way. It may be impossible to give a clear description of this stronger way, But I will try. Say that the concept of an ω-sequence is *fully determinate* if it fully determines what any instantiation would be like. In the language of a structuralist, one might try to describe this as saying that the concept fully fully determines a single structure. I don't take full determinateness as implying that there are such objects as structures, but I don't mind too much if it is taken in that way. I don't even mind if one says that the full determinateness of the concept of an ω-sequence means that such a sequence exists or even that the natural numbers exist. My objection to assuming to assuming that instantiantions of, e.g., the concept of set exist is entirely based on uncertainty about whether the concept is fully determinate.

I am now going to discuss some questions about the ω-sequence concept that are related to first-order completeness and full determinateness but are—I believe, importantly different questions.

One question that is not the same as the full determinatenesss question or the first-order completeness question for a concept is the question of whether the concept is a genuine mathematical concept. The concept of an ω-sequence is the paradigm of a fundamental mathematical concept. It supports rich and intricate mathematics. It is also fully determinate, but that is an additional fact about it. There could be genuine mathematical concept that was not fully determinate or even first-order complete. Many think that the concept of set is such a concept.

Another question that is different from those of full determinateness and first-order full determinateness is the question of categoricity: Are any two structures instantiating the concept isomorphic? Obviously a concept can be clear, precise, and first-order complete without being categorical. The the concept of a dense linear ordering without endpoints is an example. But I also think it possible that a concept be categorical without being first-order complete. The concept of an ω-sequence is not an example, but I do contend that (a) we know the concept of an ω-sequence to be categorical, but (b) this knowledge does not *per se* tell us that the concept is first-order complete, and (c) we know the concept of the subsets of $V_{\omega+1}$ to be categorical, but we do not know whether it is first-order complete. Justifying each of these perhaps surprising assertions will take me some time.

The Peano Axioms

In arguing for categoricity of the concept of an ω-sequence, the first thing want to note is that the concept implies a version of the Peano Axioms, what I will call the *Informal Peano Axioms*. These axioms apply to structures with a unary operation S and a distinguished object 0. Nothing significant would be affected if we included binary operations $+$ and $\cdot$ and axioms for them, as in the usual first order Peano Axioms.) The axioms of Informal Peano Arithmetic are:

(1) 0 is not a value of S.

(2) S is one-one.

(3) For any property P, if 0 has P and if $S(x)$ has P whenever x has P, then everything has P.

Axiom (3), the Induction Axiom, is framed in terms of the notion of a property. (Peano framed his Induction Axiom in terms of classes.) I have followed Bertrand Russell in using the word "any" and not the word "all" in stating Induction. Russell's distinction between *any* and *all* is—if I understand it—at heart a distinction between *schematic* universal quantification and genuine universal quantification. In the way I intend (3) to be taken, it is equivalent with the following *schema*.

(3′) If 1 has property P and if $S(x)$ has P whenever x has P, then everything has P.

where there is no restriction on what may be substituted for "P" to get an instance of the schema—i.e., no restriction to any particular language. In the future, I will speak of the Induction Axiom as the "Induction Schema" or—to distinguish it from first-order induction schemas—as the "Informal Induction Schema."

One might worry that the general notion of property is vague, unclear, or even incoherent, and so that we do not have a precise notion of what counts as an instance of the Induction Schema. Perhaps this is so. But as far as *using* the schema is concerned, all the worry necessitates is making sure that the instances one uses all involve clear cases of properties.

Understanding the open-ended Induction Schema does not involve treating properties as objects. In particular, it does not involve an assumption that the notion of property is definite enough to support genuine quantification over properties. Contrast this with the Second Order Induction Axiom, the induction axiom of the Second Order Peano Axioms, i.e., the Peano Axioms as usually formulated in the formal language of full second-order logic (with non-logical symbols "1" and "S"). The language of full second order logic allows one to define properties by quantification—including nested quantification—over properties (or sets or whatever else one might take the second-order quantifiers to range over).

Of course, if one is working in a background set theory and if one is considering only structures with domains that are sets, then quantifiers over properties can be replaced by quantifiers over subsets of the domain. In this situation, the Informal Peano Axioms and the Second Order Peano Axioms are essentially the same. But that is not our situation. In arguing for categoricity, the only objects whose existence I want to assume are those belonging to the domains of the two given structures satisfying the axioms. I do not even want to treat the two structures as objects. Rather I will assume that are determined their objects, properties and relations.

Do the Informal Peano Axioms fully axiomatize the concept of an ω-sequence? Would any structure satisfying the axioms have to instantiate the concept? In so far as the questions is a definite question, the answer is yes. Consider a possible structure $\mathbf{M}$ satisfying the axioms. Let P be the property of being an object of $\mathbf{M}$ that comes from the 0 of $\mathbf{M}$ by finitely many applications of the S function of $\mathbf{M}$ By the instance of the Induction Schema given by P, every object of $\mathbf{M}$ has P. Hence $\mathbf{M}$ is an ω-sequence. Since I think that P is a clear example of a property, I think this object is valid.

Of course, the axioms are not an axiomatization of the concept the way one normally talks about axiomatization. The are not first-order axioms. It is not precisely specified exactly what the axioms are: what would count as an instance of the Informal Induction Schema. As a tool for proving theorems about the concept, they don't seem to go much beyond the first-order axioms.

In any case, what will be used in proving categoricity of the concept is only that the Informal Peano Axioms are implied by the concept, not the converse.

Someone with doubts about the objectivity of the concept of an ω-sequence might just replace that concept by the concept of being a structure satisfying the axioms. One can doubt the clarity of that concept, but it seems hard to argue that it is not objective.

Categoricity.

The categoricity of the ω-sequence concept has indeed been proved in more than one way, and I will not be presenting a new way to prove it. But I do want to be careful about what I assume. In particular, I want to avoid non-necessary existential assumptions. Dedekind's proof (see [5]), is done in terms of sets (which he calls "systems"), and uses various existence principles for sets.

Let $\mathbf{M}$ an $\mathbf{N}$ be ω-sequences. Then both $\mathbf{M}$ and $\mathbf{N}$ satisfy the Infomral Peano

Axioms. We specify a function f sending objects of $\mathbf{M}$ to objects of $\mathbf{N}$ as follows:

$$\begin{aligned} f(0)_{\mathbf{M}} &= 0_{\mathbf{N}}; \\ f(S_{\mathbf{M}}(a)) &= S_{\mathbf{N}}(f(a)). \end{aligned}$$

Using the Informal Induction Schema, we can show that these clauses determine a unique value of $f(a)$ for every object a of $\mathbf{M}$. By more uses of Informal Induction, we can show that the f is one-one, onto, and an isomorphism.

If we had included the symbols and axioms for addition and multiplication, then we could use more instances of Informal Induction to prove that f preserves the operations these symbols stand for in the two given structures.

Note that categoricity of the Informal Peano Axioms does not by itself imply the first-order completeness of the axioms or the ω-sequence concept, for the trivial reason that categoricity implies nothing if there is no structure satisfying the axioms. Dedekind[12] was well aware that categoricity by itself was worthless, and that led him to his often maligned existence proof. What I am suggesting is that the real reason for confidence in first-order completeness is our confidence in the full determinateness of the concept of the natural numbers. Indeed, I believe that full determinateness of the concept is the only legitimate justification for the assertion that the concept is instantiate or that the natural numbers exist.

3 The Concept of Sets

The modern, iterative concept has four important components:

(1) the concept of the natural numbers;

(2) the concept of sets of x's;

(3) the concept of transfinite iteration;

(4) the concept of absolute infinity.

Perhaps we should include a the concept of Extensionality as Component (0). Component (1) might be thought of as subsumed under the other three, but I will treat it separately. In the way I am thinking of the concept of sets, it is a concept of a kind of structure—of a structuralist's structure—and so one does not have to add anything about what kind of objects a set is.

[12] Dedekind [5]

Roughly speaking, a set structure is what is gotten by starting with the natural numbers and transfinitely iterating the concept of sets of x's absolutely infintely many times. A set structure is also what is gotten by starting from the empty set and iterating the concept of sets of x's absolutely infinitely many times.

Sets of x's.

The phrase "set of x's" comes from Gödel. The idea is that the x's are some objects and the sets of x's are sets whose members are x's. Godel says of this concept,

> The operation "set of x's" cannot be defined satisfactorily (at least in the present state of knowledge), but only be paraphrased by other expressions involving again the concept of set, such as: "multitude of x's", "combination of any number of x's", "part of the totality of x's"; but as opposed to the concept of set in general (if considered a primitive) we have a clear notion of the operation.[13]

For Gödel (and for me), this concept is—like the concept of the natural numbers—typical of the basic concepts of mathematics. It is not definable in any straightforward sense. We can understand it and communicate it to one another, though what we literally say in communicating it by no means singles out the concept in any clear and precise way. I take it that our ability to understand and communicate such concepts is an striking and important fact about us. Gödel says that the concept is of set of x's is "clear." But Feferman says that the concept is unclear for the case when the x's are the natural numbers, so—one would presume—for any case when there are infinitely many x's. Feferman says that the concept of an arbitrary set of natural numbers is "vague." My own view is that clarity or unclarity of the concept of sets of x's is an open question, and that we cannot rule out that the answer varies with what the x's are, even when there are infinitely many.

Let us first look at the general concept of set of x's. This is a concept of structures with two sorts of objects and a relation that we call membership that can hold between objects of the first sort, the x's, and objects of the second sort, the sets of x's. Clearly the following axioms are implied by the concept.

(1) If sets α and β have the same members, then $\alpha = \beta$.

(2) For any property P, there is a set whose members are those x's that have P.

[13] Gödel [12], p. 180

Axiom (1) is, of course, the Axiom of Extensionality. Axiom (2) is a Comprehension Axiom, which I will interpret as an open-ended schema and call the Informal Comprehension Schema, analogous to the Informal Induction Schema.

Do these axioms fully axiomatize the concept of sets of x's? It is very plausible to say they do. Gödel seems to have thought so. In the published version of his Gibbs Lecture, he says, of the case when the x's are the integers:

> For example, the basic axiom, or rather axiom schema, for the concept of set of integers says that, given a well-defined property of integers (that is, a propositional expression $\varphi(n)$ with an integer variable n) there exists the set M of those integers which have the property φ.
>
> It is true that these axioms are valid owing to the meaning of the term "set"—one might even say that they express the very meaning of the term "set"—and therefore they might fittingly be called analytic; however, the term "tautological", that is, devoid of content, for them is entirely out of place.[14]

I have omitted a few sentences between the parts of the quotation, sentences about why the axioms of the schema are not tautologies. The quotation occurs in the midst of a section in which Gödel argues mathematical truths are analytic but are not mere tautologies.

There is a similar section in Gödel's earlier "Russell's mathematical logic." In it there is a passage similar to the one I have just quoted, except that Gödel there adds the Axiom of Choice, saying that "nothing can express better the meaning of the term 'class' than the axiom of classes... and the axiom of choice." (The "..." replace a reference to the number of an earlier page on which Russell's axiom of classes is discussed.)

Does one need to add Choice to fully axiomatize the concept of set of x's? I suppose that depends on how one construes the term "property" occurring in the Informal Comprehension Schema. I will return to this issue below.

Gödel does not mention Extensionality, but clearly it is necessary for a full axiomatization of "sets of x's."

To "fully express" the concept, do we need to specify something more, for example, *what object* the set whose only member is the planet Mars is? People who think that the natural numbers can be any ω-sequence often think that sets have to be particular objects. I do not think this is so, and I also don't think

[14]Gödel [13], p. 321.

there is any way to make the specification, but I won't argue these points here. I will simply ignore any constraints the concept might put on what counts as a set and what counts as membership other than constructural constraints such as those imposed by (1) and (2).

Axioms (1) and (2) are categorical, in the sense that there cannot be two instantiations of the concept with the same x's. Here is the proof (essentially due to Zermelo, whose Separation Axiom should, I believe, be viewed as an open ended schema).

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be structures satisfying (1) and (2) and having the same x's. Let $\in_1$ and $\in_2$ be the relations of two structures. With each α that is a set in the sense of $\mathfrak{M}_1$, we associate $\pi(\alpha)$, a set in the sense of $\mathfrak{M}_2$. To do this, let P be ther property of being an x such that $x \in_1 \alpha$. By the Informal Comprehension Axiom for $\mathfrak{M}_1$, there is a set β in the sense of $\mathfrak{M}_2$ such that, for every x of $\mathfrak{M}_2$,

$$x \in_2 \beta \leftrightarrow P(\beta).$$

By Extensionality for $\mathfrak{M}_2$, there is at most one such β. Let $\pi(\alpha) = \beta$. Using Informal Comprehension and Extensionality for $\mathfrak{M}_1$, we can show that π is one-one and onto, and so is an isomorphism.

Here are some comments on the proof.

- The properties P used in the proof were defined from the given structures, so there is no problem about the legitimacy of the instances of Informal Comprehension that were used.

- The isomorphism π was defined, so we do not have to think of it as an additional entity.

- The proof can easily be modified to get an isomorphism when $\mathfrak{M}_1$ has x's, $\mathfrak{M}_2$ has y's, and we are given a one-one correspondence between the x's and the y's.

- In particular, the x's and y's could be the objects of isomorphic structures instantiating some concept (e.g., the concept of an ω-sequence or the concept of the sets of z's for some z's.

Notice that the categoricity proof did not need the Axiom of Choice. All it needed was that the property of being a member in the sense of one of the models of a set in the sense of that model. One can also say that all the proof needed was that "property" was not being understood in any restrictive way.

As with the ω-sequence concept, categoricity does not by itself guaranteee first-order completeness. I.e., it does not guarantee that the concept determines, for any x's, a truth-value for any first-order sentence. In order for categoricity such effect, the concept has to be instantiatied.

Two key cases of the x's are:

(a) the natural numbers;

(b) the natural numbers and the sets of natural numbers.

The case of the natural numbers is essentially equivalent with that of the hereditarily finite sets. The concept of the hereditarily finite sets and the concept of an ω-sequence are definable from one another, and structures instantiating either one can be defined from structures instantiating the other. Since the ω-sequence concept is—I will simply assume from now on—fully determinate, so is the concept of the hereditarily finite sets. Moreover, the concepts of sets of these x's are interdefinable, and one is first-order complete or fully determinate if and only it the other is. Taking advantage of this, I will not pay much attention to the difference between (a) and (a') the hereditarily finite sets or that between (b) and (b') the subsets of V_ω. With (a') and (b') as x's, the sets of x's are, respectively,

(1) the concept of the subsets of V_ω;

(2) the concept of the subsets of $V_{\omega+1}$.

Are these two concepts fully determinate? Are they first-order complete?

The continuum hypothesis is a first-order statement about the subsets of $V_{\omega+1}$. If the concept of the subsets of $V_{\omega+1}$ is first-order complete, then the continuum hypothesis has a definite truth-value. I take it to be a fact about our present state of knowledge that we do not know whether the continuum hypothesis has a truth value. Thus we do not at present know that the concept of the subsets of $V_{\omega+1}$ is first-order complete. *A fortiori*, we do not know that it is fully determinate. This, I contend, implies that we do not know that there is any structure that instantiates the concept. If we knew there were such an instatiation, then by categoricity we would know that there is a unique truth-value that the continuum hypothesis has in all (and some) instantiations of the concept. Thus we would know—or, at least, have very strong evidence—that CH has a truth-value.[15] We would have very strong evidence even that the concept is fully determinate.

[15]There is a logical possibility that the concept of the subsets of $V_{\omega+1}$ does not itself determine a truth-value for the continuum hypothesis, but that there are instantiations of the concept and they are all isomorphic to each other. I do not consider this a plausible possibility, however. Nor do I

I believe that it is also a fact about our present state of knowledge that we do not know that the concept of the subsets of $V_{\omega+1}$ is *not* first-order complete or even that it is not fully determinate. Hugh Woodin's program may succeed and give us strong evidence that CH is true (or, if program shifts again, that it is false). This would remove the most important evidence against the full determinateness of the concepts of the subsets of $V_{\omega+1}$.

The case of the concept of the subsets of V_ω (or, equivalently, the concept of the natural numbers and the sets of natural numbers.) is very different from the one I have been discussing. The standard first-order axioms for the concept are what is commonly called second-order arithmetic. By adding to these axioms the schema of projective determinacy one gets a theory that (1) is as complete for the concept of the subsets of V_ω as the first-order Peano Axioms are for the concept of the natural numbers and (2) for whose truth there is a large, diverse and—to many of us—convincing body of evidence. See Koellner [14] for a statement of projective determinacy and much material on (1) and (2). In the next section, I will discuss the character and significance of the evidence involved in (2). Is the concept of the subsets of V_ω fully determinate? If we have the sort of direct, intuitive evidence that we have for the full determinateness of the concept of the natural numbers, then it is hard to see how this evidence would not apply to the concept of sets of x's in general: why the direct, intuitive evidence would not show that if the concept of the x's is fully determinate then so is the concept of the sets of x's. There are people who believe the general concept is fully determinate in this way. If they are right, then CH has a definite truth-value. Since I think whether it has a truth-value is an open question, I am dubious.

One thing I believe that (1) and (2) above provide evidence for is that all first-order statements about the subsets of V_ω have truth-values. I believe thatt (1) and (2) also provide evidence that the concept of the subsets of V_ω is first-order complete, but I regard them as less strong evidence for this. I will explain why in the next section.

As I am construing the third important component of the concept of set, the concept of transfinite iteration, that concept is essentially the concept of ordinal numbers or simply that of wellordering. It is also intimately related to the concept of L—more specifically to the concept of a proper or non-proper intial segment of L. There seems to be nothing that creates worries about it as CH does about

consider it plausible that the concept has instantiations that are all isomorphic to each other and yet the concept is not fully determinate. I do, though, consider it not implausible that CH has a truth-value but this truth-value is not determined by the concept of the subsets of $V_{\omega+1}$. I will consider how this could happen in the next section.

the concept of sets of x's. The intuition that supports confidence in the full determinateness of the ω-sequence concept extends at least to small transfinite ordinals and to the associated in initial segments of L. Since I am leaving length of iteration out of the concept of transfinite iteration, it is not fully determinate, but it—and so the concept of L—might well be fully determinate except for length.

The concept of an initial segment of L has as much claim to be (informally) axiomatized as the concepts of the natural numbers and the sets of x's. An open-ended Informal Wellfoundedness Axiom plays the role analogous to that of the Informal Induction and Informal Comprehension. These axioms are categorical except for length.

The first-order theory ZF + the Axiom of Determinacy + $V = L(\mathbb{R})$ does for the concept of element of $L(\mathbb{R})$ pretty much what second order arithmetic + Projective Determinacy does for the concept of subset of V_ω.

Cantor described the sequence of all the ordinal numbers a "absolutely infinite," so I am using the term "absolute infinity" for the concept that is the fourth component of the concept of set. One can argue that the concept is categorical, and that any two instantiations of the concept of set (of the concept of an absolutely infinite iteration of the sets of x's operation) have to be isomorphic.[16] But it is hard to see how there could be a full informal axiomatization of the concept of set. There are also worries about the coherence of the concept. People worry, e.g., that if the universe of sets can be regarded as a "completed" totality, then the cumulative set hierarchy should go even further. Such worries are one of the reasons for the currently popular doubts that it is possible to quantify over absolutely everything. I am also dubious about the notion of absolute infinity, but this does not make me question quantification over everything.

4 Extrinsic Evidence

The natural picture to go with the account I have been giving would be that we study such basic concepts as those of natural numbers and sets of natural numbers via informal axioms that we can see as implied by the concepts. In the case of the natural numbers, this would seem to demand using only the Informal Peano Axioms; in the case of the sets of natural numbers, using only the Informal Peano Axioms, Extensionality, Comprehension, and perhaps Choice; in the case of the concept of sets, using only the informal ZFC axioms plus perhaps further prin-

[16] See [16].

ciples justified on the basis of the absolute infinity of the sequence of ordinal numbers.

But this is not the only thing that happens in practice. In the arithmetical case, many assertions that can be stated in the languange of first order PA:

(i) have been proved in second or higher order number arithmetic or in first-order ZFC, so follow from the associated higher order concept;

(ii) are not provable in first order PA, and are not known to follow from the informal PA axioms.

One ubiquitous species of examples is that of consistency statements, all of which are equivalent to Π_1 sentences of the language of first-order PA. The consistency of first-order PA itself is probably not an example. While it is not provable in first-order PA, it can be plausibly argued that it follows from the concept of the natural numbers. But the consistency of fragments of second-order arithmetic provides examples, as do the consistency statements for of stronger fragments of ZFC.

This phenomenon occurs at every level. Statements about the sets of natural numbers are provable in ZFC but not in second or higher order number theory, and there is almost always no clue as to whether or how they might be deduced in second order arithmetic, and in many cases we have proofs that they cannot be so proved (unless the higher order theory used to prove them is inconsistent). The problem also appears to occur with respect to the full concept of set. For example, Woodin, in his work aimed toward deciding the continuum hypothesis, always assumes the existence of a proper class of Woodin cardinals. One might wonder whether this assuption is justified only by a concept properly extending the concept of set.

One might cling to the dream that some radically new method will be found that yields, for example, proofs (a) directly from the concept of the natural numbers and (b) whose conclusions are arithmetical statements of the kind we have been talking about, including statements whose truth is equivalent to the consistency of ZFC or even of strong large cardinal axioms. But this seems as good an example of a pipedream as one is apt to come upon.

Those of us who believe that the concept of the natural numbers is fully determinate thus have to live with the very high probability that, for example, either ZFC + "there is a measurable cardinal" is inconsistent or else there are truths of number theory that can be known only by knowing that ZFC + "there is a measurable cardinal" is consistent. Evidence for the latter alternative is arguably evidence for ZFC + "There is a measurable cardinal."

Can one justify, from the conceptualist point of view, the claim that, e.g., the consistency of second-order arithmetic is a known fact about the concept of the natural numbers, using the argument that this arithmetical fact is a theorem of, say, third order number theory. Or is this just a fact about the stronger concept of the subsets of $V_{\omega+1}$?[17]

In this special case of the natural numbers, I think we can. This because we know, or at least have strong evidence for, the following two assertions.

(1) The concept of the subsets of $V_{\omega+1}$ is coherent, or at least consistent.

(2) The concept of the natural numbers is fully determinate, or at least first-order complete.

Assume that φ is a sentence of the first-order language of arithmetic, and assume that

$$\text{Third Order Arithmetic } \vdash \varphi.$$

By the first-order completeness of of the concept of the natural numbers, that concept implies whichever of φ or $\neg\varphi$ is true. The concept of the natural numbers is contained in the concept of the subsets of $V_{\omega+1}$, and so that concept implies this correct answer. We know that the axioms of third order arithmetic are implied by the concept of the subsets of $V_{\omega+1}$. Hence the concept of the subsets of $V_{\omega+1}$ implies φ. By the consistency of that concept, φ must be true. Hence it is φ, not its negation, that is implied by the concept of the natural numbers.

To apply this kind of argument at a higher level, e.g., to the subsets of V_ω in the place of the natural numbers, we need to have evidence of full determinateness—or at least the first-order completeness—of the of the higher level concept. Do we, then, have evidence for the full determinateness or the first-order completeness of higher level concepts? For the concept of subsets of $V_{\omega+1}$, the status of CH keeps me from thinking we any more evidence of first-order completeness than we do for first-order incompleteness. But what about the concept of the subsets of V_ω?

In the case of the natural numbers, the strongest evidence for the full determinateness comes from directly considering the concept. We feel we know exactly what a structure instantiating it would have to be like. Do we have such a feeling in the case of of the subsets of V_ω? I have certainly heard people say that we do—or, at least, that *they* do. A strong version of this claim would be that the concept of the sets of x's can be directly seen to be fully determinate whenever there is a

[17] There are clearly concepts weaker than that of the subsets $V_{\omega+1}$ which imply the consistency of second order number theory. Indeed, one could try to make a case that this follows from the concept of the subsets of V_ω.

fully determinate concept of the x's. The truth of this claim would imply that we know that the concept of the subsets of $V_{\omega+1}$ is fully determinate, and hence that CH has a truth value. Since I do not think we know at present whether or not the CH has a truth value, I find the claim doubtful.

Is the concept of the subsets of $V_\omega + 1$ special in a way that of the subsets of $V_{\omega+1}$ is not?

I said in the preceeding section that we have strong evidence for axioms that yield a rich first-order theory of the subsets of V_ω, a theory that goes far beyond the theory given by the first-order ZFC axioms. This fact counts as evidence (whether or not convincing evidence) for the first-order completeness of the concept of the subsets of V_ω. It is—for many of us—convincing evidence that these axioms are true. Is it convincing evidence that these axiom are implied by the concept of the subsets of V_ω?

A reason for hesitating about saying yes is that the evidence for these axioms (which is mainly the evidence presented in Koellner [14]) is not intrinsic evidence, coming directly from the concept. Instead it is *extrinsic* evidence.

One kind of extrinsic evidence is the kind I have been talking about. Suppose, for example, that φ is a statement about a concept $\mathcal{C}_1$. Suppose that φ is implied by a concept $\mathcal{C}_2$ that contains $\mathcal{C}_1$. (To keep it simple let us mean by "$\mathcal{C}_2$ contains $\mathcal{C}_1$" that there is a uniform definition that would yield, for any structure S instantiating $\mathcal{C}_2$, a substructure of a reduct of S that instantiated $\mathcal{C}_1$. This gives intrinsic evidence about $\mathcal{C}_2$ but, in general, only extrinsic evidence about $\mathcal{C}_1$. An example where φ is one of the aforementioned axioms for the subsets of V_ω is given by the fact that projective determinacy is provable in ZFC + some large cardinal axioms. Here $\mathcal{C}_1$ is the concept of the subsets of V_ω, φ is one of the instances of projective determinacy, and $\mathcal{C}_2$ is some concept that extends ZFC and implies the existence of the needed large cardinals.

There are various attitudes one might have toward such evidence.

One attitude is to believe that that most of our concepts about levels of the cumulative hierarchy and our extensions of the concepts of set involving large cardinals are not first-order complete. Answers to many questions about these concepts are not implied by the concepts themselves, but many and perhaps all these questions are implied by stronger concepts from our stock of basic concepts.

Another possible additude is to take a proofs of statements φ about $\mathcal{C}_1$ using one of our stronger concepts, $\mathcal{C}_2$, to imply (or yield strong evidence) that $\mathcal{C}_1$ also implies the statement φ. We might admit that our much of our epistemic access to $\mathcal{C}_1$ comes via stronger concepts like $\mathcal{C}_2$, but we would nevetheless assume that what we can prove about $\mathcal{C}_2$ is implied by $\mathcal{C}_1$. Gödel [13] seems to suggest such a view.[18]

Here a way we might try to justify such a position. (It's not Gödel's way.) We might make a methodological defeasible assumption that our basic concepts are fully determinate (or even that they are instantiated). This assumption would allow us to conclude that proof of φ from $\mathcal{C}_2$ always shows that $\mathcal{C}_1$ implies φ, as long as we have reason to believe $\mathcal{C}_2$ consistent.[19]

This still leaves us with the problem of explaining the validity of other kinds of extrinsic evidence, for example, extrinsic evidence for large cardinal axioms. Are we free choose to strengthen the basic iterative concept of sets to a concept that implies the existence of large cardinals? On the methodological viewpoint just sketched, our ability to make such a choice is severely limited. If our methodology involves the assumption that the iterative concept of set is itself fully determinate, then we are assuming that this concept must either imply the existence of large cardinals or their non-existence. In order for an extension of the concept that includes large cardinals to be consistent, their existence must be implied by the basic set concept. Thus our methodology requires that we have strong evidence for the truth of large cardinals axioms in order to justify the adoption of a concept that implies their existence.

I have not discussed any of the evidence for adopting determinacy axioms or large cardinal axioms. The question of whether this evidence justifies belief that these axioms are implied by, respectively, the concept of the subsets of V_ω and the basic iterative concept of set is a difficult one. One thing I would like to point out, though, is that a lot of the evidence for large cardinals and determinacy (and also much of the evidence Woodin has cited in his endeavor to solve the continuum problem) does indeed feel like evidence for *truth* and not just for satisfying methodological desiderata. Examples of evidence of this kind are diverse. One example that I am particularly fond of involves prediction and confirmation. This is the example of the Wadge degrees. Wadge proved that determinacy for a class of subsets of Baire space implies that the sets in that class are essentially linearly ordered by the relation "is a continuous preimage of." This ordering was later shown to be a wellordering. Wadge's proof from determinacy is about one line long. Wadge's theorem for the special case of Borel sets is a statement about subsets of V_ω. Several years after Wadge's proof, that special case was proved from the ZFC axioms, by a fairly complex proof. Several years after that, the Borel

[18]In the discussion of axiomatizing the concept of set that begins near the bottom of page 305 and runs through page 307.

[19]The possibility such a methodological position was suggested to me by Peter Koellner.

case was proved in second order arithmetic, by a very long and complex proof. These facts seem to me clear evidence for the truth of general determinacy hypotheses (and projective determinacy in particular), evidence more solid than any view about what such truth consists in.

The methodology suggested above fits nicely with regarding such evidence as evidence for truth. If we follow the methodology, we are assuming that the concept of the subsets of V_ω is fully determinate. This means that all (mathematical) statements about the concept are either true or false. We are endeavoring to determine which ones are true. Whether or not our assumption is correct, it is clear what, on the assumption, we are looking for evidence for.

At the beginning of [7], Feferman quotes a statement I made in 1976:

> Throughout the latter part of my discussion, I have been assuming a naive and uncritical attitude toward CH. While this *is* in fact my attitude, I by no means wish to dismiss the opposite viewpoint. Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty. [20]

I rarely understand my earlier self, but in this case I think it natural to interpret the first part of what he says ("this is in fact my viewpoint") as a declaration that he is following the methodology I have now been discussing. I don't disagree with what the young man says in the last sentence, with one qualification: what I and others regard as the success in finding very probably true axioms that answer questions about the subsets of V_ω and about $L(\mathbb{R})$ gives some reason to be optimistic about finding probably true axioms that settle CH. Moreover, ideas and results like Woodin's give direct reason for hope about CH.

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