IS THE CONTINUUM HYPOTHESIS A DEFINITE MATHEMATICAL PROBLEM?

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"What is Cantor's Continuum Problem?" Gödel 1947

"Cantor's continuum problem is simply the question: How many points are there on a straight line in Euclidean space... In other terms: How many different sets of integers do there exist?

"The analysis of the phrase 'how many' leads unambiguously to a definite meaning for the question...

Gödel 1947 (cont'd)

"Cantor conjectured that any infinite subset of the continuum has the power either of the set of integers or of the whole continuum. This is Cantor's continuum hypothesis. …

Gödel 1947 (cont'd)

"But, although Cantor's set theory has now had a development of more than sixty years and the [continuum] problem is evidently of great importance for it, nothing has been proved so far relative to the question of what the power of the continuum is or whether its subsets satisfy the condition just stated, except that it is true for a certain infinitesimal fraction of these subsets, [namely] the analytic sets.

Gödel 1947 (cont'd)

"Not even an upper bound, however high, can be assigned for the power of the continuum. It is undecided whether this number is regular or singular, accessible or inaccessible, and (except for König's negative result) what its character of cofinality is."

“Hilbert’s first problem: the continuum hypothesis” [Martin 1976]

“Throughout the latter part of my discussion, I have been assuming a naïve and uncritical attitude toward CH. While this <u>is</u> in fact my attitude, I by no means wish to dismiss the opposite viewpoint.

Martin 1976 (cont'd)

"Those who argue that the concept of set is not sufficiently clear to fix the truth-value of CH have a position which is at present difficult to assail. As long as no new axiom is found which decides CH, their case will continue to grow stronger, and our assertion that the meaning of CH is clear will sound more and more empty."

Is the Continuum Hypothesis (CH) a Definite Mathematical Problem?

- My conjecture: No; in fact it is essentially indefinite ("inherently vague").
- That is, the concepts of arbitrary set and function as used in its formulation even at the level of $\mathcal{P}(\mathrm{N})$ are essentially indefinite.
- This comes from my general anti-platonistic view of the nature of mathematics: it is humanly based and deals with more or less clear conceptions of mathematical structures; I call that view conceptual structuralism.

Is CH absolutely undecidable?

- A proposition is *absolutely undecidable* if it is "undecidable relative to any set of axioms that are justified" [Koellner 2010]
- Prefer not to use that terminology: the idea of absolute undecidability seems to presume that the statement in question has a definite mathematical meaning and hence a definite truth value.
- But part of my critique also supports the absolute undecidability of CH for those who take it to be a definite statement.

How can CH *not* have a definite mathematical meaning?

- There is no disputing that CH is a definite statement in the language of set theory, whether considered formally or informally; it just concerns $\mathcal{P}(\mathcal{P}(\mathrm{N}))$.
- And there is no doubt that that language involves concepts that have become an established, robust part of mathematical practice.
- But that may be because mathematical practice uses relatively little from those concepts.

How can CH *not* have a definite mathematical meaning? (cont'd)

I shall examine this from three directions:

1. A thought experiment related to the Millennium Prize Problems.
2. From the point of view of Conceptual Structuralism.
3. Via a proposed logical framework for distinguishing definite from indefinite concepts.

The Millennium Prize List: A Thought Experiment

- The Millennium Prize List: 7 famous unsolved problems, including the Riemann Hypothesis, Poincaré Conjecture, P vs NP, etc. [cf. Jaffe 2006]
- The prize: $1,000,000 each.
- Scientific Advisory Board (SAB) criteria for the problems on the list: Should be historic, central, important, and difficult.

Millennium -2-

- CH a *prima facie* candidate. Was it considered for the list? (Jaffe: No excuses for why 'Problem A' is not on the list.)
- A new situation: Pereleman solved the Poincaré Conjecture but declined the prize, thus freeing up $1,000,000.
- A possible scenario: one new problem is to be added to the list; expert advice is solicited anew on its choice.
- EST, an Expert on Set Theory.

Millennium -3-

- SAB: Thanks for joining us today. Why is CH important and what efforts have been made to solve it?
- EST: Set theory is the foundation of all mathematics and this is one of its most basic unsettled problems. Hilbert put it #1 on his famous list.

  There's been lots of work on CH, a long history of efforts from Cantor and König to Sierpinski and Luzin in the mid 1930s. [cf. Moore 2011]

  And lots of modern work too.

Millennium -4-

- SAB: But Gödel says nothing was learned beyond uncountability of the continuum and König's thm.
- EST: Well, he didn't mention work on the Perfect Set Property (PSP) which if it holds of a set X implies that X has the same power as the continuum.
- Best result of Luzin and Suslin--the uncountable analytic sets have the PSP. Then Gödel (1938) showed there exist uncountable co-analytic sets without the PSP in L, the constructible sets.

Millennium -5-

- SAB: So does that settle the extent of PSP?
- EST: No, it could be consistent with ZFC that all uncountable co-analytic sets, and even all uncountable sets in the projective hierarchy have the PSP.
- In fact, that's been shown using Projective Determinacy (PD), which is a restriction of the so-called Axiom of Determinacy (AD).

Millennium -6-

- SAB: How so? And what are AD and PD?
- EST: For each subset X of the continuum, G(X) is a two-person infinite game which ends with an infinite sequence σ of 0s and 1s. Player 1 wins if σ is in X, otherwise Player 2 wins.

  AD for X says that there is a winning strategy for one of the players. But AD contradicts the Axiom of Choice (AC).

Millennium -7-

- EST (cont'd) We won't give up AC but we do like AD because of its many nice consequences (all sets Lebesgue measurable, have PSP, etc.)

  And PD has the same consequences as AD for sets in the projective hierarchy.

  The great result was by Martin and Steel, "A proof of projective determinacy" (1989).

Millennium -8-

- SAB: That sounds pretty impressive and as real progress. So what you're telling me is that not only is it consistent but it's true, though it can't be true in L by Gödel's result.
- EST: Yes, it's true if there exist infinitely many Woodin cardinals with a measurable cardinal above all of them.

Millennium -9-

- SAB: Oh... And wait a minute. I know what a measurable cardinal is and that its existence is not true in L, but what are Woodin cardinals?
- EST: The definition is pretty technical; they're among the "large" large cardinals. Their existence is stronger than measurables but not as strong as supercompacts.

Millennium -10-

- SAB: Martin and Steel didn't mention the need of Woodin cardinals in the title of their paper. Is it intuitively clear that their existence should be accepted?
- EST: Yes and No. [Continues with an explanation of the linear consistency hierarchy among "natural" extensions of ZF, and the empirically observed phenomenon that the Large Cardinal Axioms (LCAs) have been needed to mediate between equiconsistent theories. Also emphasizes the related ubiquity of restricted versions of AD.]

Millennium -11-

- SAB: That doesn't sound very convincing to me as an argument to accept the existence of such LCAs. But let's get back to CH itself. How does this new work help?
- EST: Well, now we're getting into more speculative territory. Levy and Solovay showed that CH is independent of all LCAs that have been considered, provided they are consistent. So something more is needed to deal with CH.

Millennium -12-

- SAB: Like what?
- EST: Some of the experts think that one of the most promising avenues is that being pursued by Woodin with his strong Ω−conjecture, which if true implies that the power of the continuum is aleph−2. But that would take much longer to explain.

Millennium -13-

- SAB: Hmm. We've run out of time, and I can't ask you to explain that, or why if established, we should believe in its truth, if even LCAs are not enough.

  But much thanks for your information and advice.

- Next!

Millennium Discussion

- Should SAB add CH to the list? Usual idea of mathematical truth in its ordinary sense is no longer operative in these research programs.
- Even if experts in set theory find such assumptions compelling, likelihood of their being accepted by the mathematical community at large is practically nil. So, not a good bet to add CH to the list.
- The situation is not at all like that of the experience with the Axiom of Choice.

Does this show CH is not definite?

- No, have to dig deeper into the philosophical presuppositions of set theory within a view of the nature of mathematical truth more generally.
- What are the options?
- If not a total rejectionist of set theory: Platonism, Deflationism, "Mathematics is as mathematics does", Methodological dicta ("maximize", etc.), Structuralism.

Structuralism: Mathematics and Philosophy of Mathematics

- Modern mathematics dominated by structuralist views (abstract algebra, topology, analysis; Bourbaki, category theory, etc.)
- Explicit inception often credited to Dedekind. But mathematicians have implicitly always been structuralists.
- Many modern philosophers: Benacerraf, Hellman, Resnik, Shapiro, Chihara, Parsons, Isaacson.
  Stand on CH?

Conceptual Structuralism
Thesis 1

- The basic objects of mathematical thought exist only as mental conceptions, though the source of these conceptions lies in everyday experience in manifold ways (counting, ordering, matching, combining, separating, and locating in space and time).

Thesis 3

- The basic conceptions of mathematics are of certain kinds of relatively simple ideal-world pictures which are not of objects in isolation but of structures, i.e. coherently conceived groups of objects interconnected by a few simple relations and operations.

  They are communicated and understood prior to any axiomatics or systematic logical development.

Thesis 4

- Some significant features of these structures are elicited directly from the world-pictures which describe them, while other features may be less certain. Mathematics needs little to get started and, once started, a little bit goes a long way.

Thesis 5

- Basic conceptions differ in their degree of clarity. One may speak of what is true in a given conception, but that notion of truth may only be partial. Truth in full is applicable only to completely clear conceptions.

Thesis 10

- The objectivity of mathematics lies in its stability and coherence under repeated communication, critical scrutiny and expansion by many individuals often working independently of each other.
- Incoherent concepts, or ones which fail to withstand critical examination or lead to conflicting conclusions are eventually filtered out from mathematics.
- The objectivity of mathematics is a special case of intersubjective objectivity that is ubiquitous in social reality.

Objectivity in Social Reality

- John Searle, *The Construction of Social Reality* (1995)
- " There are portions of the real world, objective facts in the world, that are only facts by human agreement. In a sense there are things that exist only because we believe them to exist. ...
- ... things like money, property, governments, and marriages. Yet many facts regarding these things are 'objective' facts in the sense that they are not a matter of [our] preferences, evaluations, or moral attitudes." (Searle 1995, p.1)

Objectivity in Social Reality: Examples

- I am a citizen of the United States.
- I have voted in every U.S. presidential election since I became eligible by age to do that.
- I have a PhD in Mathematics from the University of California.
- My wife and I own our home in Stanford, California; we do not own the land on which it sits.

More Examples

- Rafael Nadal won the 2010 men's Wimbledon tennis finals match and the 2010 U.S. Open, but lost the 2011 U.S. Open.
- In the game of chess, it is not possible to force a checkmate with a king and two knights against a lone king.
- There are infinitely many prime numbers.

The Basic Conceptions of Mathematics as Social Constructions

- The objective reality that we ascribe to mathematics is simply the result of intersubjective objectivity about those conceptions and not about a supposed independent reality in any platonistic sense.

- This view does not require total realism about truth values. It may simply be undecided under a given conception whether a given statement has a determinate truth value.

Conceptions of Sequential Generation

- The most primitive mathematical conception is that of the positive integer sequence represented by the tallies: I, II, III, ...
- Our primitive conception is of a structure $(N^+, 1, Sc, <)$
- Certain facts about this structure are evident (if we formulate them at all): $<$ is a total ordering, 1 is the least element, and $m < n$ implies $Sc(m) < Sc(n)$.

Open-ended Schematic Truths and Definite Properties

- At a further stage of reflection we may recognize the least number principle: if P(n) is any definite property of members of N^+ and there is some n such that P(n) then there is a least such n.
- The schema is open-ended. What is a definite property? This requires the mathematician's judgment.
- Is the property, "GCH holds at n" definite?

Reflective Elaboration of the Structure of Positive Integers

- Concatenation of tallies immediately leads us to the operation of addition, m + n, and that leads us to m × n as "n added to itself m times".
- The basic properties of the + and × operations such as commutativity, associativity, distributivity, and cancellation are initially recognized only implicitly.
- Soon have a wealth of expression and interesting and challenging problems.

Truth in Number Theory

- N^+ is recognized as a definite totality and the logical operation $(\forall n \in N^+)\ P(n)$ is recognized as leading from definite properties to definite statements that are true or false.
- The conception of the structure $(N^+, 1, Sc, <, +, \times)$ is so clear that there is no question as to the definite meaning of 1st order statements about it and the assertion that they are true or false.
- In other words we accept realism in truth values, and the application of classical logic in reasoning about such statements is automatically legitimized.

Conceptions of the Continuum

- There is no unique concept of the continuum but rather several related ones. (Feferman 2009)
- To clear the way as to whether CH is a genuine mathematical problem one should avoid the tendency to conflate these concepts, especially those that we use in describing physical reality.
- Geometrical (Euclid, Hilbert), The real line (Cantor, Dedekind), Set theoretical (2^N, $\mathcal{P}(N)$).

Conceptions of the Continuum (Cont'd)

- Not included are physical conceptions of the continuum, since our only way of expressing them is through one of the conceptions via geometry or the real numbers.
- Which continuum is CH about? Their identity as to cardinality assumes impredicative set theory.
- NB: Set theory erases the conceptual distinction between sets and sequences.

Conceptions of Sets

- Sets are supposed to be definite totalities, determined solely by which objects are in the membership relation (∈) to them, and independently of how they may be defined, if at all.
- A is a definite totality iff the logical operation of quantifying over A, $(\forall x \in A)\ P(x)$, has a determinate truth value for each definite property $P(x)$ of elements of A.
- Extensionality is accepted.

The Structure of "all" Sets

- (V, $\in$), where V is the universe of "all" sets.
- V itself is not a definite totality, so unbounded quantification over V is not justified on this conception. Indeed, it is essentially indefinite.
- If the operation $\mathcal{P}(\ .\ )$ is conceived to lead from sets to sets, that justifies the power set axiom (Pow).
- At most, this conception justifies KP_ω+Pow+AC, with classical logic only for bounded statements as discussed below.

The Status of CH

- But--I believe--the assumption of $\mathcal{P}$(N), $\mathcal{P}(\mathcal{P}$(N)) as definite totalities is philosophically justified only on platonistic grounds.
- From the point of view of conceptual structuralism, the conception of the totality of arbitrary subsets of any given infinite set is essentially indefinite (or inherently vague).
- For, any effort to make it definite violates the idea of what it is supposed to be about.

Is there an intermediate position?

- The concept of the continuum $\mathcal{P}$(N) in its guise as 2^{N} is particularly intuitive.
- Suppose we grant the idea of 2^{N} or $\mathcal{P}$(N) as a working apparently robust idea, but nothing higher in the cumulative hierarchy.
- That justifies Dedekind completeness of R w.r.t. all sets definable in 2nd order number theory.
- But CH requires for its formulation as a definite statement, $\mathcal{P}(\mathcal{P}$(N)) as a definite totality.

A Formal Distinction Between Definite and Indefinite Concepts

- "What's definite is the domain of classical logic, what's not is that of intuitionistic logic."
- In the case of predicativity, consider systems in which quantification over natural numbers is governed by classical logic, while quantification over sets of natural numbers (and sets more generally) is governed by intuitionistic logic.
- In the 1970s, I used such systems as intermediate tools in my work applying functional interpretation with non-constructive operators.

A Formal Distinction (Continued)

- In the case of set theory, where every set is conceived to be a definite totality, but the universe of sets is an indefinite totality, accept classical logic for bounded quantification while use intuitionistic logic for unbounded quantification.
- Some early case studies on relatively strong semi-intuitionistic subsystems of ZF: Friedman (1973, 1980), Wolf (1974); recent work, Feferman (2010).

A General Pattern for Studies

- Start with a system T formulated in fully classical logic, and consider an associated system SI-T formulated in a mixed, semi-intuitionistic logic.
- Ask whether there is any essential loss in proof-theoretical strength when passing from T to SI-T
- In the cases that are studied, it turns out that there is no such loss. (Feferman 2010)

A General Pattern (Continued)

- But there can be an advantage in going to such a semi-intuitionistic system SI-T
- Namely, we can beef it up to a semi-constructive system SC-T without changing the proof-theoretical strength from that of T (the original classical system), by the adjunction of certain principles that go beyond what is admitted in SI-T

The Case of Admissible Set Theory

- Start with T = KPω, the classical system of admissible set theory (including the Axiom of Infinity)
- SI-KPω has the same axioms as KPω, but is based on intuitionistic logic plus the Law of Excluded Middle for bounded formulas plus a form MP of Markov's Principle.
- (Δ_0-LEM) $\varphi \vee \neg\varphi$, for all Δ_0 formulas φ.
- SI-KPω = IKPω + (Δ_0-LEM) + MP

A Semi-Constructive System of Admissible Set Theory

- Beef up SI-KPω to a system SC-KPω that includes the Full Axiom of Choice Scheme for sets (AC_{Set}),

$$\forall x \in a \exists y\ \varphi(x,y) \rightarrow \exists r[\mathrm{Fun}(r) \wedge \mathrm{dom}(r)=a \wedge \forall x \in a\ \varphi(x,r(x))]$$

for φ an arbitrary formula,

- Then SC-KPω proves the Full Collection Axiom Scheme,

$$\forall x \in a \exists y\ \varphi(x,y) \rightarrow \exists b \forall x \in a \exists y \in b\ \varphi(x,y),$$ for φ

arbitrary, while this holds only for Σ_1 formulas in KPω.

Adding the Power Set Axiom

- Let Pow be the axiom $\forall a \exists b \forall x(x \in b \leftrightarrow x \subseteq a)$ in SC-KPω
- The axiom Pow, with a new constant symbol $\mathcal{P}$, is written $x \in \mathcal{P}(a) \leftrightarrow x \subseteq a$.
- Pow(ω) is the special case of Pow: $x \in \mathcal{P}(\omega) \leftrightarrow x \subseteq \omega$.

On the Strength of Semi-Constructive Systems of KPω

Theorem We have the following proof-theoretical equivalences:

(i) KPω ≡ SC-KPω

(ii) KPω + Pow(ω) ≡ SC-KPω + Pow(ω)

(iii) KPω + Pow ≡ SC-KPω + Pow

The same hold with 'SI' in place of 'SC'.

What Statements are Definite?

- φ is formally definite in one of our semi-constructive systems if $\varphi \vee \neg\varphi$ is provable there.
- Is the Continuum Hypothesis definite?
- CH is expressible in SC-KPω + Pow(ω)
  but I conjecture not formally definite there.
  (How prove?)
  It *is* formally definite in SC-KPω + Pow.

What more can be said about What's Definite, What's Not?

- Formal definiteness is an initial criterion of definiteness.
- Proving that CH is not definite in SC-KPω + Pow(ω) would be an interesting start.
- Need more refined notions of definiteness/indefiniteness to throw light on whether CH is a definite statement.

The End