

# Some challenges for the philosophy of set theory

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Thesis I: (Really a general thesis about the philosophy of mathematics) A successful philosophy of set theory should be faithful to the mathematical practices of set theorists. In particular such a philosophy will require a close reading of the mathematics produced by set theorists, an understanding of the history of set theory, and an examination of the community of set theorists and its interactions with other mathematical communities.

This talk represents an experiment in doing this kind of philosophy.

Thesis II: Set theory is a branch of mathematics. It happens to be a branch in which issues of independence and consistency lie close to the surface, and in which the tools of logic have proven very powerful, but it is not (or not only) about logic. As in any successful field of mathematics, progress in set theory comes from a dialectic between “internal” developments elaborating the central ideas, and “external” developments in which problems and ideas flow back and forth between set theory and other areas (for example topology, algebra, analysis, the theory of games).

Thesis III: Modern set theory uses a wide range of methods and discusses a wide range of objects. What is more, from a historical perspective there have been quite radical transformations in our views about set-theoretic methods and objects, and I see no reason to believe that this process of transformation has reached its end. Accordingly we should be suspicious of arguments (especially those based on philosophical views) that tend to prescribe or circumscribe the objects or methods which are legitimate. A useful criterion for a philosophical viewpoint is to ask: “Would this view, if taken seriously, have hampered or helped the development of set theory?”, and to compare it with the view that “Anything goes”.

Thesis IV: Questions about the meaning and truth value of CH have been mathematically fertile, and have given rise to a new area in set theory (sometimes called “Set theory of the continuum”) with a whole corpus of ideas and techniques. Independence results are present in this area but are by no means the whole story. This area has taken on a life of its own, with its own initiatives and insights, and continues to flourish in the absence of a solution to the Continuum Problem. In fact if CH were settled positively it would be a blow to “Set theory of the continuum”, since a major part of the subject is a rich structure theory inconsistent with CH. In brief: there is more to the continuum than the Continuum Problem.

Thesis V: It is possible (I would even say quite likely) that independence is here to stay, and that questions like CH will never be settled in a way that commands universal agreement. But we can learn to live with independence. In particular experience in combinatorial set theory suggests how to reach a modus vivendi with the independence phenomenon.

The rest of the talk consists of a series of case studies. I will discuss results by Shelah, Farah and Magidor-Shelah and use them to defend and elaborate on Theses II-V.

A *cardinal fixed point* is a cardinal  $\kappa$  such that  $\kappa = \aleph_\kappa$ . Since the function  $\kappa \mapsto \aleph_\kappa$  is continuous, there is a closed unbounded class of such cardinals. The first one is the limit of the increasing sequence

$$\aleph_0, \aleph_{\aleph_0}, \aleph_{\aleph_{\aleph_0}}, \dots$$

The following theorem appears in a paper by Saharon Shelah (“On power of singular cardinals”, Notre Dame Journal of Formal Logic 27 (1986) 263–299).

If  $\aleph_\delta$  is the  $\omega_1$ -th cardinal fixed point then  $\aleph_\delta^{\aleph_1}$  is less than the  $(2^{2^{\aleph_1}})^+$ -th cardinal fixed point.



The theorem would have made perfect sense to Cantor or Hausdorff.

Ingredients in the proof include

- Ramsey cardinals.
- Core models.
- The covering lemma.
- Generic elementary embeddings.
- Infinite games.

Ramsey cardinals: A *Ramsey cardinal* is a cardinal  $\kappa$  such that for every function  $F : [\kappa]^{<\omega} \rightarrow 2$ , there is a set  $A \in [\kappa]^\kappa$  such that  $F \upharpoonright [A]^n$  is constant for every  $n$ .

As the name suggests, Ramsey cardinals have their roots in *Ramsey theory* or *partition calculus*. A typical result in Ramsey theory has the general form “any colouring of tuples from a large set looks very regular on a large subset”. Ramsey cardinals can also be viewed in a model-theoretic light in terms of *sets of indiscernibles*.

Core models: Gödel's model  $L$  (the constructible universe) is the least transitive model of ZFC containing all the ordinals, and enjoys many pleasant structural properties: for example the Generalised Continuum Hypothesis (GCH) holds in  $L$ . The model  $L$  can accommodate small large cardinals, for example if  $\kappa$  is inaccessible in  $V$  then  $\kappa$  is inaccessible in  $L$ .

On the other hand measurable cardinals (or even Ramsey cardinals) can not exist in  $L$ . Given a set of ordinals  $A$ , one can relativise the construction of  $L$  and obtain  $L[A]$ , the least transitive model which contains all ordinals and has  $A$  as a member:  $L[A]$  is “ $L$ -like above  $A$ ”, for example above  $\sup(A)$  GCH holds and there are no Ramsey cardinals.

A major program in set theory (the “inner model program”) has been the construction of  $L$ -like models which can accommodate larger cardinals. An important early step was the construction by Dodd and Jensen of  $K$  (technically the “Dodd-Jensen core model” or  $K_{DJ}$ , but this is the only version of  $K$  needed here).  $K$  is (roughly speaking) the largest  $L$ -like model of “ZFC plus there is no measurable cardinal”. The model  $K$  shares many of the good properties of  $L$ , for example it is a model of GCH: but  $K$  is more hospitable to large cardinals, for example Mitchell showed that if  $\kappa$  is Ramsey then  $\kappa$  is Ramsey in  $K$ . The construction can be relativised over a set of ordinals  $A$  to produce  $K[A]$ , which has the same relation to  $K$  that  $L[A]$  has to  $L$ .

The covering lemma: Jensen made a very detailed study of the “fine structure” of  $L$  and related models like  $L[A]$ . The model  $L$  is the union of levels  $L_\alpha$ , and the fine structure program (for  $L$ ) involves a careful study of definability over the models of form  $L_\alpha$ . Although fine structure appears quite formal and syntactic, there is a large mathematical payoff: the theory of core models like the model  $K$  discussed in the last paragraph, proofs of new structural facts (squares, diamonds, morasses) which hold in  $L$  and  $K$ , and (most relevant for us) the *covering lemma*.

The original form of the covering lemma states that either zero-sharp exists or *covering* holds between  $V$  and  $L$ , that is to say for every uncountable set of ordinals  $X \in V$  there is  $Y \in L$  such that  $X \subseteq Y$  and  $V \models |X| = |Y|$ . One can view this as a dichotomy theorem, stating that  $V$  is either “far from  $L$ ” or “close to  $L$ ”. In the covering case, the combinatorics of  $V$  (especially at singular cardinals) is quite  $L$ -like.

We need a generalisation involving a set of ordinals  $A$  and a stronger principle than zero-sharp: if there is no inner model of “ZFC + there exists a measurable cardinal” containing  $A$  as an element, then covering holds between  $V$  and  $K[A]$  above  $\sup(A)$ .

Generic embeddings: Recall that a cardinal  $\kappa$  is measurable if and only if it is the *critical point* of an elementary embedding  $j: V \rightarrow M \subseteq V$ , that is to say  $j \upharpoonright \kappa = id$  and  $j(\kappa) > \kappa$ . A *generic embedding* is an embedding  $j$  defined in a generic extension  $V[G]$  of  $V$ , with  $j: V \rightarrow M \subseteq V[G]$ ; the critical point of such an embedding can be a small regular uncountable cardinal such as  $\omega_1^V$ , and the critical point can have reflection properties reminiscent of those which follow from measurability; the exact nature of these properties depends on the forcing extension  $V[G]$  and the target model  $M$ .

The existence of generic embeddings can be expressed in terms of objects called *precipitous ideals*, and the existence of precipitous ideals can be formulated in terms of infinite games.

Infinite games: Given a set  $X$  with  $|X| > 1$ , let  ${}^\omega X$  be the set of all  $\omega$ -sequences from  $X$ , and let  $A \subseteq {}^\omega X$ . We can view  $A$  as the “payoff set” in an infinite game of perfect information  $G(X, A)$  where two players (I and II) alternately play element of  $X$  for  $\omega$  steps to generate an  $\omega$ -sequence  $\vec{x}$ , and then I wins if  $\vec{x} \in A$  while II wins if  $\vec{x} \notin A$ . The notions of “winning strategy” for players I and II are defined in the obvious way, and the set  $A$  is said to be *determined* if one player has a winning strategy.

The axiom of choice implies that there are non-determined games, but the payoff sets constructed using choice tend not to have very simple definitions. In fact it is a theorem of ZFC (due to Martin) that games with Borel payoff sets are determined, and assuming the existence of large cardinals gives determinacy for various kinds of more complex payoff sets.



After this long preamble we can finally outline the proof of Shelah's result. Let  $\mu = (2^{2^{\aleph_1}})^+$ . For every  $A \subseteq \mu$  we ask whether the following statement  $\phi_A$  holds: there is an inner model  $M$  of "ZFC + there exists a Ramsey cardinal" such that  $A \in M$ . Now there are two cases.

Case one: There is  $A$  such that  $\phi_A$  fails. In this case we may appeal to the theory of  $K[A]$  to see that covering holds above  $\mu$ , and GCH holds in  $K[A]$  above  $\mu$ , from which we may obtain the desired conclusion.

Case two: For every  $A$  the statement  $\phi_A$  holds. In this case we can adapt a determinacy result due to Martin to show that the good player always wins a certain “precipitousness game”, which in turn enables us to construct a certain generic elementary embedding whose existence implies the desired conclusion.

The proof of the theorem uses methods from a number of different areas, each with a complex history. To trace its genealogy in detail would be to write a good part of the history of set theory and neighbouring areas in the twentieth century. Here are some very brief historical remarks.

Partition calculus: This originated in work of Ramsey on decidability problems for finite and countable structures, was greatly elaborated (notably by Erdős and his collaborators) in the setting of uncountable cardinals, and plays a crucial role in the theory of “small large cardinals” such as weakly compact cardinals and Ramsey cardinals.

Infinite games of perfect information: These were first studied by Banach and Mazur in connection with problems in analysis, were rediscovered by Gale and Stewart in the setting of mathematical economics, and then became central objects in descriptive set theory and inner model theory.

Fine structure: Jensen's work on fine structure builds on Gödel's original work on  $L$ , but also on work in computability theory and the study of weak set theories such as admissible set theory.

Elementary embeddings: Work of Scott showed that the large cardinal property of measurability can be phrased in terms of elementary embeddings. The subroutine of Shelah's argument where a generic elementary embedding is used is a direct descendant of Scott's theorem that GCH does not fail for the first time at a measurable cardinal, with an intermediate step being Silver's proof (using generic elementary embeddings) that GCH does not fail for the first time at a singular cardinal of uncountable cofinality.

Generic elementary embeddings and precipitous ideals:  
Solovay introduced the study of generic embeddings in his work on the theory of saturated ideals. Jech and Prikry introduced the notion of precipitous ideal and characterised precipitousness in terms of games.



Thesis II: Set theory is a branch of mathematics.

Shelah's argument is recognisably the same kind of thing as a hard argument in number theory: Shelah combines partition calculus, infinite games, elementary embeddings and so forth to prove a purely combinatorial result in just the same way that a number theorist might combine complex analysis, algebraic geometry and representation theory in order to prove a purely arithmetic result.

Thesis III: A useful criterion for a philosophical viewpoint is to ask: “Would this view, if taken seriously, have hampered or helped the development of set theory?”, and to compare it with the view that “Anything goes”.

Shelah’s result is a useful test case here: any viewpoint which would discourage interest in large cardinals, the fine structure of  $L$ , the theory of infinite games and so forth would apparently make it harder to prove certain theorems of ZFC about classical objects like cardinals and the continuum function.

We saw here how the entry of new methods and new objects, new views of existing objects, and exchanges of ideas with other areas of mathematics led to progress on a classical problem. Returning to Thesis II, this kind of progression is typical of the evolution of a mathematical area. We could draw an analogy with the history of algebraic geometry, where the reformulation of the subject in terms of the highly abstract theory of schemes led to progress on classical and concrete problems about curves and surfaces.

The following theorem appears in a paper by Ilijas Farah (“All automorphisms of the Calkin algebra are inner”, *Annals of Mathematics* 173 (2011) 619–661)

If Todorčević’s axiom (TA) holds then all automorphisms of the Calkin algebra are inner.

After some background I will discuss the ideas that go into this result, and then use it as an example of a piece of mathematics which is closely related to the continuum but is quite remote from the Continuum Problem.

Let  $H$  be the complex inner product space  $l^2(\mathbb{C})$  whose elements are infinite sequences  $x = (x_n)$  of complex numbers with  $\sum |x_n|^2 < \infty$  and whose inner product is given by  $x \cdot y = \sum_n x_n \overline{y_n}$ . Up to isomorphism  $H$  is the unique separable infinite dimensional Hilbert space.

$B(H)$  is the space of continuous linear maps from  $H$  to  $H$ , with a norm  $\|T\| = \sup\{\|Tx\| : \|x\| \leq 1\}$ . With this norm the space  $B(H)$  becomes a Banach space.  $B(H)$  is closed under the bilinear operation of composition, and with this operation  $B(H)$  becomes a Banach algebra.

$B(H)$  has more structure. For any  $T \in B(H)$  there is a unique  $T^* \in B(H)$  such that  $(Tx).y = x.(T^*y)$ . Axiomatizing the properties of  $B(H)$  with the “involution” map  $T \mapsto T^*$  we obtain the notion of a  $C^*$ -algebra, that is a Banach algebra equipped with a map  $a \mapsto a^*$  satisfying certain identities.

A basic theorem (the Gelfand-Naimark-Segal theorem) states that every  $C^*$ -algebra is isomorphic to a norm-closed and  $*$ -closed subalgebra of  $B(K)$  for some Hilbert space  $K$ . Another theorem by Gelfand identifies the unital commutative  $C^*$ -algebras with spaces  $C(X)$  of complex-valued functions on compact Hausdorff spaces  $X$ , and we can view the study of noncommutative  $C^*$ -algebras as a form of “noncommutative geometry”.

$T \in B(H)$  is *compact* if the image of the unit ball of  $H$  under  $T$  has compact closure. We denote the class of compact operators by  $K(H)$ ; this is a  $C^*$ -algebra in its own right and forms an ideal in  $B(H)$ , so we may form a quotient  $B(H)/K(H)$  and get the object known as the “Calkin algebra”. It is reasonable to think of the compact operators as “finitary” and so the Calkin algebra consists of “operators modulo finitary perturbation”.

In any  $C^*$  algebra with a 1, a *unitary* element is an element  $u$  such that  $uu^* = u^*u = 1$ . If  $u$  is unitary then the map  $a \mapsto uau^*$  is an automorphism, and automorphisms of this type are called *inner automorphisms*. A classical question in  $C^*$  algebras asked whether all automorphisms of the Calkin algebra are inner.

Phillips and Weaver showed that under CH there is an outer (non-inner) automorphism.

Farah closed the case by showing that under TA all automorphisms are inner.



Motivating analogies: the Boolean algebra  $P\omega$  is analogous to the Banach algebra  $B(H)$ , the ideal of finite sets  $FIN$  is analogous to the ideal  $K(H)$  of compact operators, and the quotient  $P\omega/FIN$  is analogous to the Calkin algebra  $B(H)/K(H)$ .

Say that an automorphism  $\pi$  of  $P\omega$  is *trivial* if it has the form  $A \mapsto f[A]$  where  $f$  is a permutation of  $\omega$ , and  $f[A] = \{f(n) : n \in A\}$ . Since  $P\omega$  is atomic with atoms the singleton sets  $\{n\}$ , every automorphism is trivial.

Similarly say that an automorphism  $\pi$  of  $P\omega/FIN$  is trivial if it has the form  $[A]_{FIN} \mapsto [f[A]]_{FIN}$ , where  $f$  is a bijection between cofinite subsets of  $\omega$  and  $[B]_{FIN}$  is the equivalence class of  $B$  modulo finite: a classical question asks whether all automorphisms of  $P\omega/FIN$  are trivial.

Rudin showed that if CH holds there is a non-trivial automorphism of  $P_\omega/FIN$ . Shelah showed by a hard forcing argument that it is consistent (modulo the consistency of ZFC) that every automorphism of  $P_\omega/FIN$  is trivial, but the argument is rather specific to  $P_\omega/FIN$ .

Shelah and Steprans showed by a different approach that the strong Proper Forcing Axiom (PFA) implies that every automorphism of  $P_\omega/FIN$  is trivial.

Reflecting on the work of Shelah and Steprans, Velickovic showed that the conjunction of MA and the “Todorcevic Axiom” TA implies that every automorphism of  $P_\omega/FIN$  is trivial.

TA is a maximality principle which isolates some Ramsey-theoretic consequences of strong forcing axioms; TA follows from PFA but is known to be consistent relative to the consistency of ZFC. Farah showed that TA implies that every automorphism of the Calkin algebra is inner.

Thesis IV: There is more to the continuum than the Continuum Problem.

Farah's result is formally an independence result (the existence of an outer automorphism in the Calkin algebra is independent of ZFC) but this is a misleading frame in which to view it. It is not proved by clever coding tricks or an elaborate forcing construction, but rather by infusing ideas from one field (the set theory of the continuum) into another (the theory of the Calkin algebra). The proof is fertile for the theory of  $C^*$ -algebras, for example it contains ideas which lead to a new and simpler proof of the Phillips-Weaver theorem. It has led to an exchange in which ideas and problems flow in both directions between areas which have been separated since the era of von Neumann.

The powerful axiom PFA (and the large cardinals and elaborate forcing technology needed to show it is consistent) play no role in the final result. However PFA was essential in the development of the circle of ideas which underlie Farah's proof.

In my last case study I will outline some work by Magidor and Shelah (“When does almost free imply free?”, *Journal of the American Mathematical Society*). I will use this as an example of how to make mathematical progress in a field (combinatorial set theory) which is beset by many issues of independence.

This work is motivated by problems in group theory involving free groups, and is set in a very general framework. I will describe a simple case involving the transversal property for families of sets.

A *transversal* for a family  $X$  of non-empty sets is a 1-1 choice function on  $X$ , that is a function  $f$  such that  $f(A) \in A$  for all  $A \in X$  and  $A \neq B \implies f(A) \neq f(B)$ . It follows rather easily from the compactness theorem in first order logic that if  $X$  is a family of *finite* sets, and every finite subset of  $X$  has a transversal, then  $X$  has a transversal. As we will see, the situation for countable sets is more complicated.



Let  $S$  be a stationary subset of  $\omega_1$ , choose for each  $\alpha \in S$  a cofinal set  $x_\alpha \subseteq \alpha$ , and let  $X = \{x_\alpha : \alpha \in S\}$ . It is not hard to see that every countable subset of  $X$  has a transversal. However Fodor's lemma implies that  $X$  itself has no transversal. For a regular  $\kappa$  let  $R(\kappa)$  be the assertion "there is a family  $X$  of  $\kappa$  many countable sets such that  $X$  has no transversal, while every subset with size less than  $\kappa$  does have a transversal". We proved  $R(\aleph_1)$ .

$R(\kappa)$  is a typical “incompactness” or “non-reflection” statement. It asserts that there is a family of sets with a property not shared by any small subset. Such statements are ubiquitous in combinatorial set theory: another example is “there exists a  $\kappa$ -Aronszajn tree”.

Incompactness properties tend to follow from  $V = L$ . Their negations (if consistent at all) tend to have large cardinal strength, and to be proved consistent by arguments involving generic elementary embeddings.

A stationary subset  $S$  of an uncountable cardinal  $\kappa$  is said to *reflect* if there is an ordinal  $\alpha < \kappa$  such that  $S \cap \alpha$  is stationary in  $\alpha$ , and to be *non-reflecting* if no such  $\alpha$  exists. The existence of a non-reflecting stationary set is a prototypical “incompactness” property of the sort discussed above. For every  $n < \omega$ , the set  $\aleph_{n+1} \cap \text{cof}(\aleph_n)$  is an example of a non-reflecting stationary set.

A general “stepping up” result by Milner and Shelah shows that if  $R(\kappa)$  holds and there is a non-reflecting stationary set in  $\lambda$  consisting of ordinals of cofinality  $\kappa$ , then  $R(\lambda)$  holds. Using the examples of non-reflecting sets from the last paragraph, it follows easily that  $R(\aleph_n)$  holds for every  $n$ . So now we ask what about  $\aleph_{\omega+1}$ ?

A natural first try might be to prove that there are non-reflecting stationary sets in  $\aleph_{\omega+1}$ . This is actually true if  $V = L$ , or more generally as long as there are no inner models with large cardinals. However Magidor showed it to be consistent (modulo some very large cardinals) that every stationary subset of  $\aleph_{\omega+1}$  reflects.

Remark: It is a general fact in this area that small cardinals can (sometimes) consistently exhibit the reflection properties associated with large cardinals. In the model built by Magidor each  $\aleph_n$  for  $0 < n < \omega$  is the critical point of a generic elementary embedding, and these embeddings jointly enforce the reflection property.

At this point we may begin to speculate that  $R(\aleph_{\omega+1})$  is independent of ZFC, but this speculation would be premature. Magidor and Shelah used Shelah's "PCF theory" to isolate a stationary set of points of cofinality  $\aleph_1$  (the "good set") which can be used in place of the non-reflecting set from the Milner-Shelah argument. They used this to show that  $R(\aleph_{\omega+1})$  is true.

Remark: Again there is a general phenomenon to be observed, namely that ZFC is in some ways a surprisingly powerful theory. In particular ZFC can prove weak forms of combinatorial principles (Jensen's diamond and square) which follow from  $V = L$ . The stationarity of the good set is a case in point. Jensen's principle square holds if  $V = L$ , and implies that all points are good and non-reflecting stationary sets exist. In ZFC alone we can't prove square, and there may be non-good points, but we can still establish that there are stationarily many good points and use this to our advantage.

Having reached  $\aleph_{\omega+1}$ , we can use similar ideas to proceed inductively and prove  $R(\kappa)$  for all regular  $\kappa < \omega^2$ , at which point we again become stuck. PCF theory gives some information: the induction can be pushed to  $\aleph_{\omega^2+1}$  *unless* a certain subset of  $\aleph_{\omega^2+1}$  (the “chaotic set”) is stationary. This gives the clue about how to do a hard forcing construction which shows (modulo large cardinals, by way of generic elementary embeddings) that  $R(\aleph_{\omega^2+1})$  is consistently false.



Thesis V: We can learn to live with independence.

The work I just described gives an example of how to make mathematical progress in an area which is rife with independence. We can identify some heuristics which were used in this work, and which have a good track record in combinatorial set theory.

- Cast the problem in terms of the tension between compactness and incompactness.
- Settle the problem on the basis of  $V = L$  or its consequences (GCH, the square principle, the existence of non-reflecting stationary sets).
- Search for refined arguments which rely on the ZFC-provable consequences of  $V = L$ .
- When combinatorial arguments fail, mine them for clues about how a consistency result would have to look.
- Search for a consistency result involving generic elementary embeddings.

Challenges for the philosophy of set theory. Say something which

- A) Is faithful to set-theoretic practice.
- B) Does not impose undue methodological restrictions.
- C) Takes into account the complex history and dynamic nature of the subject.
- D) Keeps the role of logic and the problem of independence in perspective.